



Differential subordination and superordination results for p -valent analytic functions associated with (r, k) -Srivastava fractional integral calculus[☆]

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ABSTRACT

The object of the present paper is to investigate generalizations of the hypergeometric function and Srivastava fractional integral calculus by using a general version of gamma function (namely (r, k) -gamma function).

- Some fundamental results for these new concepts are provided.
- We introduced differential subordination and superordination results associated with the defined new fractional integral operator.
- Also, we establish sandwich results for p -valent analytic functions involving this operator.
- Finally, an application to fluid mechanics is discussed.

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Background

This paper aims to generalize Srivastava fractional integral calculus through the (r, k) -hypergeometric function. By establishing fundamental results related to this function, we enhance the theoretical framework of fractional calculus.

The motivation for providing this methodology stems from the need to advance the existing understanding of fractional calculus and its applications. We introduce a new differential subordination and superordination results associated with a newly defined fractional integral operator can be used in many areas, such as fluid mechanics, by showing new results for fractional subordination. Additionally, we present sandwich results for p -valent analytic functions, highlighting the utility of the new operator in formulating relevant theorems.

By employing the (r, k) -gamma function, we reformulate the fractional integral operator, broadening the scope of fractional calculus and its applications in complex analysis. This research lays the groundwork for future studies on the (r, k) -hypergeometric function, contributing to advancements in mathematical analysis within fractional calculus and related areas.

Method details

Introduction and preliminaries

Analytic function theory is an ancient subject that originated around the beginning of the century. Moreover, in recent years, it has undergone swift and ongoing advancements. Among its prominent topics is the study of the geometric and analytic properties of certain common types of analytic functions, known as univalent and multivalent functions within unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. One of the important topics of analytic function theory is the establishment of differential subordination and superordination, which examine how one analytic function affects another.

The journey of development in differential subordination began in 1974 with the study of properties of univalent functions by Miller et al. [1], where he established

$$\operatorname{Re}\left(f(z) + \alpha \frac{zf'(z)}{f(z)}\right) > 0, \text{ implies } \operatorname{Re}(f(z)) > 0.$$

Subsequent studies have yielded new insights in analytic function theory, with ongoing active research in this field as evidenced by recent literature [2–12]. Essentially, the concepts of subordination and superordination are very important, serving as basic tools for complex field $\mathbb{C}$. Given the challenges of using traditional comparisons $\geq$ (or $\leq$) in this context, these concepts provide an alternative that enables functions to adapt to imposed constraints. Subordination bounds one function by another, allowing us to describe its behavior and properties, as well as calculate its Taylor-Maclaurin coefficients. Additionally, these concepts give Sandwich theorem, which provides a framework for bounding the properties and behavior of a function within range defined by two other functions.

On the other hand, fractional calculus studies generalizations of the basic operators of ordinary integration and differentiation, which is defined by allowing their order from integer domain to complex domain. Many researchers have proposed numerous generalizations for fractional differentiation and integration, such as [13–17].

After mentioning the theory of analytic functions, geometric functions, and differential subordination, as well as fractional calculus. We now highlight studies that have extended these topics by using the concepts of fractional differentiation and integration; for instance, see [6, 18–28]. In our work, we generalized the hypergeometric function pFq (see [29]) and subsequently utilized it to generalize the Srivastava fractional integral calculus $I_z^{\lambda, \delta, \sigma}$ detailed in [23]. Therefore, we recall these definitions pFq and $I_z^{\lambda, \delta, \sigma}$ as follows: for $a_j \in \mathbb{C}$, $j = 1, 2, \dots, p$ and $b_k \in \mathbb{C} \setminus \mathbb{Z}_0^-, k = 1, 2, \dots, q$, the extended hypergeometric function is given by

$$pFq(a_1, \dots, a_p; b_1, \dots, b_q; z) = \sum_{n=0}^{\infty} \frac{(a_1)_n \dots (a_p)_n}{(b_1)_n \dots (b_q)_n} \frac{z^n}{n!}, \quad (p \leq q+1; p, q \in \mathbb{N}_0), \quad (1.1)$$

where $(a)_n$ is the Pochhammer symbol given by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1, & n=0 \\ a(a+1)(a+2)\dots(a+n-1), & n \in \mathbb{N} \end{cases}.$$

For $\lambda \in \mathbb{R}^+$, $\delta, \sigma \in \mathbb{R}$, the fractional integral operator $I_z^{\lambda, \delta, \sigma}$ is given by

$$I_z^{\lambda, \delta, \sigma} f(z) = \frac{z^{-\lambda-\delta}}{\Gamma(\lambda)} \int_0^z (z-\zeta)^{\lambda-1} \times {}_2F_1\left(\lambda+\delta, -\sigma; \lambda; 1-\frac{\zeta}{z}\right) f(\zeta) d\zeta, \quad (1.2)$$

where $f(z)$ is an analytic function in a simple connected region of $\mathbb{C}$ containing 0 with order $f(z) = o(|z|^\epsilon)$, $z \rightarrow 0$, and $\epsilon > \max\{\delta - \sigma, 0\} - 1$.

Refer to [30] for all the bellow mentioned concepts. Let $\mathcal{H}(U)$ denote the class of analytic function in U and $\mathcal{H}[a, n]$ the subclass of $\mathcal{H}(U)$ represented by

$$f(z) = a + \sum_{n=p}^{\infty} a_n z^n \quad (1.3)$$

The p -valent (normalized) analytic functions are defined $\mathcal{A}_p = \mathcal{H}[0, p]$ ($\mathcal{A} = \mathcal{H}[0, 1]$) with $a_p = 1$ ($a_1 = 1$), respectively. Denote by S , the subclass of $\mathcal{A}$ which contains all univalent functions (analytic and one-to-one functions) in U .

For a function f is purported to be subordinate to g (g is superordinate to f) and denoted by $f < g$ if there is an analytic function $w : U \rightarrow U$ with $w(0) = 0$ and $|w(z)| < 1$ ($z \in U$) implies $f(z) = g(w(z))$. Equivalently, $f(0) = g(0)$ and $f(U) \subset g(U)$ (see [31,32]).

Let $\psi(r, s, t; z) : \mathbb{C}^3 \times U \rightarrow \mathbb{C}$ and $h \in S$. If ξ is analytic in U and satisfies

$$\psi(\xi(z), z\xi'(z), z^2\xi''(z); z) < h(z), \quad (1.4)$$

hence ξ is referred to as a solution of (1.4). Let $\chi \in S$ with $\xi < \chi$ for all ξ satisfying (1.4); then the function χ is called a dominant of the inequality. The best dominant is χ that it corresponds to $\chi < \chi$ for all dominant χ of (1.4).

Let $\psi(r, s, t; z) : \mathbb{C}^3 \times U \rightarrow \mathbb{C}$ and h are analytic functions. If ξ and $\psi(\xi(z), z\xi'(z), z^2\xi''(z); z)$ are belong to S and satisfy

$$h(z) < \psi(\xi(z), z\xi'(z), z^2\xi''(z); z), \quad (1.5)$$

hence ξ is referred to as a solution of (1.5). Let χ an analytic function with $\chi < \xi$ for all ξ satisfying (1.5); then the function χ is called a subordinate of (1.5). The best subordinate is χ that corresponds to $\chi < \chi$ for all subordinate χ of (1.5).

Following Miller and Mocanu [30,33], Bulboacă [34] investigated certain classes of first order differential subordinations as well as superordinations preserving integral operators [35]. Ali et al. [36] use the results obtained by Bulboacă [34] and give the sufficient conditions of certain normalized analytic functions $f(z)$ to satisfy

$$\chi_1(z) < \frac{f(z)}{zf'(z)} < \chi_2(z),$$

where χ_1 and χ_2 are univalent with $\chi_1(0) = \chi_2(0) = 1$.

The following definitions and results are required in our work:

Definition 1.1 [30]. Denote $\mathcal{Q}$ by the subclass of $\mathcal{H}(U)$ and any f in $\mathcal{Q}$ is injective on $\bar{U} \setminus E(f)$, such that $E(f) = \{\zeta \in \partial U : \lim_{Z \rightarrow \zeta} f(z) = \infty\}$ and $f'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(f)$. Further, let $\mathcal{Q}_a$ denote the subclass of $\mathcal{Q}$ consisting of functions f for which $f(0) = a$.

Lemma 1.1 [30]. Let χ be a univalent function from U into domain D , Ξ , and Y be analytic in a domain D with Y as non omitted value at $\omega \in \chi(U)$. Choose $Q(z) = z\chi'(z)Y(\chi(z))$ and $\mathfrak{h}(z) = \Xi(\chi(z)) + Q(z)$. Consider hypotheticalal circumstances under which

(1) $Q(z)$ is starlike and $Re\left(\frac{z\mathfrak{h}'(z)}{Q(z)}\right) > 0$ for $z \in U$.

(2) ξ is analytic in U , $\xi(U) \subset D$, $\xi(0) = \chi(0)$, and

$$\Xi(\xi(z)) + z\xi'(z)Y(\xi(z)) < \Xi(\chi(z)) + z\chi'(z)Y(\chi(z)). \quad (1.6)$$

Then $\xi < \chi$ and the dominant χ of (1.6) is besting.

Lemma 1.2 [34]. Let χ be a convex univalent function from U into domain D , Ξ and Y be analytic in a domain D . Consider hypotheticalal circumstances under with

(1) $Q(z) = z\chi'(z)Y(\chi(z))$ is starlike and $Re\left(\frac{\Xi'(\chi(z))}{Y(\chi(z))}\right) > 0$, $z \in U$.

(2) $\xi \in H[\chi(0), 1] \cap \mathcal{Q}$, $\xi(U) \subset D$, where $\Xi(\xi(z)) + z\xi'(z)Y(\xi(z))$ is univalent in U and satisfying

$$\Xi(\chi(z)) + z\chi'(z)Y(\chi(z)) < \Xi(\xi(z)) + z\xi'(z)Y(\xi(z)). \quad (1.7)$$

Then $\chi < \xi$ and the dominant χ of (1.7) is besting.

Definition 1.2 [37,38]. Let Ω be a set in $\mathbb{C}$, $\chi \in \mathcal{Q}$ and $n \in \mathbb{Z}^+$. The class of admissible functions $\Psi_n[\Omega, \chi]$, consists of those functions $\psi : \mathbb{C}^3 \times U \rightarrow \mathbb{C}$ that satisfy the admissibility condition:

$$\psi(r, s, t; z) \notin \Omega \quad (1.8)$$

whenever

$$r = \chi(\mathcal{G}), \quad s = m\mathcal{G}\chi'(\mathcal{G}), \quad \text{and}$$

$$Re\left(\frac{t}{s} + 1\right) \geq mRe\left(\frac{\mathcal{G}\chi''(\mathcal{G})}{\chi'(\mathcal{G})} + 1\right)$$

$$z \in U, \quad \mathcal{G} \in \partial U/E(\chi), \quad \text{and} \quad m \geq n.$$

Theorem 1.1 [30]. Let $\psi \in \Psi_n[\Omega, \chi]$ with $\chi(0) = a$, if $\xi \in \mathcal{H}[a, n]$ satisfies

$$\psi(\xi(z), z\xi'(z), z^2\xi''(z); z) \in \Omega, \quad (1.9)$$

then

$$\xi(z) < \chi(z).$$

Finally, our motivation is inspired by the significant contributions in the literature, particularly in works such as [6,8,18,36,39–45], which have explored the powerful potential of fractional calculus in various contexts. Building on these studies, we aim to leverage our new fractional operator to derive new and meaningful results in the areas of differential subordination and superordination. In this regard, we present several sandwich results that highlight the intricate interplay between fractional operators and analytic functions. Moreover, we introduce a new class of admissible functions that further extends the applicability of these concepts. Finally, motivated by the practical implications of fractional calculus, we explore its potential applications in fluid mechanics, demonstrating how our theoretical findings can contribute to the modeling and analysis of complex fluid flow behavior.

(r, k) -Hypergeometric function and (r, k) -Srivastava fractional integral calculus

Firstly, it is pertinent to review the (r, k) -gamma function $\Gamma_{r,k}$, a generalization of ordinary gamma function, which was defined in [22] as follows, before introducing an extension of the hypergeometric function: for $z \in \mathbb{C}$, $\operatorname{Re}(z) > 1 - \frac{1}{r}$, $r \in \mathbb{N}$, $k > 0$, then (r, k) -gamma function $\Gamma_{r,k}$ is defined by

$$\Gamma_{r,k}(\delta) = \int_0^\infty t^{r(\delta-1)} e^{-\frac{t^k}{k}} dt. \quad (2.1)$$

Moreover,

$$\Gamma_{r,k}(\delta) = k^{\frac{r(\delta-1)+1-k}{k}} \Gamma\left(\frac{r(\delta-1)+1}{k}\right). \quad (2.2)$$

In this section, we establish general results for well-known properties of hypergeometric functions ${}_pF_q$, including the concept of (r, k) -gamma function and Hadamard product. These results subsequently lead to the definition of a new fractional integral, which induces a new integral operator from $\mathcal{A}_p$ into itself.

Definition 2.1. Let $a_i \in \mathbb{C}$, $i = 1, 2, \dots, p$, $b_j \in \mathbb{C} \setminus \mathbb{Z}_0^-$, $j = 1, 2, \dots, q$, $p \leq q + 1$, $k > 0$, $r \in \mathbb{N}$. The (r, k) -hypergeometric function is defined by

$${}_pF_{r,k}q(a_1, \dots, a_p; b_1, \dots, b_q; r, k; z) = \sum_{n=0}^{\infty} \frac{(a_1)_{n,r,k} \cdots (a_p)_{n,r,k}}{(b_1)_{n,r,k} \cdots (b_q)_{n,r,k}} \frac{z^n}{n!}, \quad (2.3)$$

where $(a)_{n,r,k}$ is the (r, k) -Pochhammer is defined as below

$$(a)_{n,r,k} = \frac{\Gamma_{r,k}\left(a + \frac{k}{r}n\right)}{\Gamma_{r,k}(a)},$$

equivalently,

$$(a)_{n,r,k} = k^n \left(\frac{r(a-1)+1}{k} \right)_n.$$

Example 2.1. The following are some examples of Definition 2.1 (see Fig.1):

$$(a) \quad {}_2F_{1,1}1(0.5, 0.5; 0.75; 1, 1; z^2) = {}_2F_1(0.5, 0.5; 0.75; z^2) = \frac{\sin^{-1}(z)}{z}.$$

$$(b) \quad {}_2F_{1,0.5}1(0.5, 0.5; 0.75; 1, 0.5; z^2) = \sum_{n=0}^{\infty} \frac{(0.5)_{n,1,0.5}(0.5)_{n,1,0.5}}{(0.75)_{n,1,0.5}} \frac{z^{2n}}{n!}.$$

$$(c) \quad {}_2F_{2,0.5}1(0.75, 0.75; 0.9; 2, 0.5; z^2) = \sum_{n=0}^{\infty} \frac{(0.75)_{n,2,0.5}(0.75)_{n,2,0.5}}{(0.9)_{n,2,0.5}} \frac{z^{2n}}{n!}.$$

Theorem 2.1. If $\operatorname{Re}(c-b) > 1 - \frac{1}{r}$, $k \geq 1$, then

$${}_2F_{r,k}1\left(a, b; c + \frac{1}{r} - 1; r, k; z\right) * \left(kI\left(\frac{z}{k}\right)\right) = \frac{\Gamma_{r,k}\left(c + \frac{1}{r} - 1\right)}{\Gamma_{r,k}(b)\Gamma_{r,k}(c-b)} \int_0^1 t^{\frac{r(b-1)+1}{k}-1} (1-t)^{\frac{r(c-b-1)+1}{k}-1} (1-tz)^{-\frac{r(a-1)+1}{k}} dt, \quad (2.4)$$

where $I(z) = \frac{1}{1-z}$, $\arg(t) = \arg(1-t) = 0$, and $(1-tz)^{-\frac{r(a-1)+1}{k}}$ has principle value.

Proof. By taking the right side of (2.4)

$$\frac{\Gamma_{r,k}\left(c + \frac{1}{r} - 1\right)}{\Gamma_{r,k}(b)\Gamma_{r,k}(c-b)} \int_0^1 t^{\frac{r(b-1)+1}{k}-1} (1-t)^{\frac{r(c-b-1)+1}{k}-1} (1-tz)^{-\frac{r(a-1)+1}{k}} dt$$

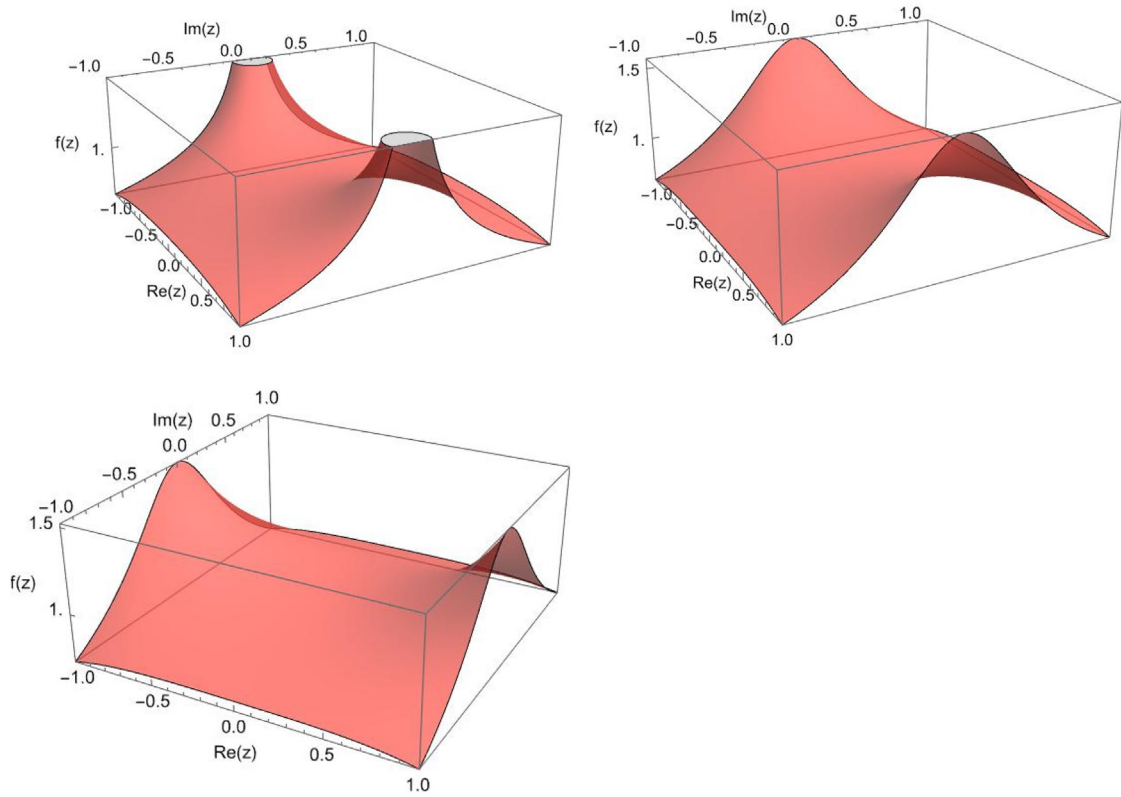


Fig. 1. Complex plot of a, b, and c in Example 2.1 from $-1 - i$ to $1 + i$, resp.

$$\begin{aligned}
 &= \frac{\Gamma_{r,k}\left(c + \frac{1}{r} - 1\right)}{\Gamma_{r,k}(b)\Gamma_{r,k}(c-b)} \sum_{n=0}^{\infty} \frac{\left(\frac{r(a-1)+1}{k}\right)_n}{n!} z^n \int_0^1 t^{\frac{r\left(\frac{kn}{r}+b-1\right)+1}{k}-1} (1-t)^{\frac{r(c-b-1)+1}{k}-1} dt \\
 &= \frac{\Gamma_{r,k}\left(c + \frac{1}{r} - 1\right)}{\Gamma_{r,k}(b)\Gamma_{r,k}(c-b)} \sum_{n=0}^{\infty} \frac{\left(\frac{r(a-1)+1}{k}\right)_n}{n!} z^n \left\{ \frac{\Gamma\left(\frac{r\left(\frac{kn}{r}+b-1\right)+1}{k}\right) \Gamma\left(\frac{r(c-b-1)+1}{k}\right)}{\Gamma\left(\frac{r\left(\frac{kn}{r}+c-2\right)+2}{k}\right)} \right\} \\
 &= \frac{k^{\frac{r(c-2)+2}{k}-1} \Gamma\left(\frac{r(c-2)+2}{k}\right)}{k^{\frac{r(b-1)+1}{k}-1} \Gamma\left(\frac{r(b-1)+1}{k}\right) k^{\frac{r(c-b-1)+1}{k}-1}} \sum_{n=0}^{\infty} \frac{(a)_{n,r,k}}{k^n n!} z^n \left\{ \frac{\Gamma\left(\frac{r\left(\frac{kn}{r}+b-1\right)+1}{k}\right)}{\Gamma\left(\frac{r\left(\frac{kn}{r}+c-2\right)+2}{k}\right)} \right\} \\
 &= k \sum_{n=0}^{\infty} \frac{(a)_{n,r,k} (b)_{n,r,k}}{k^n \left(c + \frac{1}{r} - 1\right)_{n,r,k}} \frac{z^n}{n!} \\
 &= {}_2F_{r,k}1\left(a, b; c + \frac{1}{r} - 1; r, k; z\right) * \left(kI\left(\frac{z}{k}\right)\right).
 \end{aligned}$$

Theorem 2.2. Let $\operatorname{Re}(c-b-a) > 0$, $\operatorname{Re}(c-b) > 1 - \frac{1}{r}$, $k \geq 1$. Then

$${}_2F_{r,k}1\left(a, b; c + \frac{1}{r} - 1; r, k; 1\right) * \left(kI\left(\frac{1}{k}\right)\right) = \frac{\Gamma_{r,k}\left(c + \frac{1}{r} - 1\right) \Gamma_{r,k}\left(c-b-a+1-\frac{1}{r}\right)}{\Gamma_{r,k}(c-b) \Gamma_{r,k}(c-a)} \quad (2.5)$$

Proof. From (2.4)

$$\begin{aligned}
 2F_{r,k}1\left(a, b; c + \frac{1}{r} - 1; r, k; 1\right) * \left(kI\left(\frac{1}{k}\right)\right) &= \frac{\Gamma_{r,k}\left(c + \frac{1}{r} - 1\right)}{\Gamma_{r,k}(b)\Gamma_{r,k}(c-b)} \int_0^1 t^{\frac{r(b-1)+1}{k}-1} (1-t)^{\frac{r(c-b-a)}{k}-1} dt \\
 &= \frac{\Gamma_{r,k}\left(c + \frac{1}{r} - 1\right)}{\Gamma_{r,k}(b)\Gamma_{r,k}(c-b)} \left\{ \frac{\Gamma\left(\frac{r(b-1)+1}{k}\right) \Gamma\left(\frac{r\left[(c-b-a+1-\frac{1}{r})-1\right]+1}{k}\right)}{\Gamma\left(\frac{r(c-a-1)+1}{k}\right)} \right\} \\
 &= \frac{\Gamma_{r,k}\left(c + \frac{1}{r} - 1\right)}{k^{\frac{r(b-1)+1}{k}} \Gamma_{r,k}(c-b)} \times \frac{\Gamma\left(\frac{r\left[(c-b-a+1-\frac{1}{r})-1\right]+1}{k}\right)}{\Gamma\left(\frac{r(c-a-1)+1}{k}\right)} \\
 &= \frac{\Gamma_{r,k}\left(c + \frac{1}{r} - 1\right) \Gamma_{r,k}\left(c-b-a+1-\frac{1}{r}\right)}{\Gamma_{r,k}(c-b) \Gamma_{r,k}(c-a)} ..
 \end{aligned}$$

Theorem 2.3. If $Re(c-e) > 1 - \frac{1}{r}$, $k \geq 1$, and $|\arg(1-z)| < \pi$, then

$$\begin{aligned}
 2F_{r,k}1\left(a, b; c + \frac{1}{r} - 1; r, k; z\right) * \left(kI\left(\frac{z}{k}\right)\right) &= \frac{\Gamma_{r,k}\left(c + \frac{1}{r} - 1\right)}{\Gamma_{r,k}(e)\Gamma_{r,k}(c-e)} \int_0^1 t^{\frac{r(e-1)+1}{k}-1} (1-t)^{\frac{r(c-e-1)+1}{k}-1} \\
 &\quad \left[2F_{r,k}1(a, b; e; r, k; tz) * I\left(\frac{z}{k}\right) \right] dt.
 \end{aligned} \tag{2.6}$$

Proof. From Theorem 2.1, integrating $1F_{r,k}0(a; -; r, k; tz) * I\left(\frac{z}{k}\right)$ after multiplying by distribution beta $t^{\frac{r(b-1)+1}{k}-1} (1-t)^{\frac{r(c-b-1)+1}{k}-1}$ gives $2F_{r,k}1(a, b; c + \frac{1}{r} - 1; r, k; z) * (kI\left(\frac{z}{k}\right))$.

More generally, from proof of Theorem 2.1, we obtain

$$\begin{aligned}
 3F_{r,k}2\left(a, b, d; e, c + \frac{1}{r} - 1; r, k; z\right) * \left(kI\left(\frac{z}{k}\right)\right) &= \frac{\Gamma_{r,k}\left(c + \frac{1}{r} - 1\right)}{\Gamma_{r,k}(d)\Gamma_{r,k}(c-d)} \int_0^1 t^{\frac{r(d-1)+1}{k}-1} (1-t)^{\frac{r(c-d-1)+1}{k}-1} \\
 &\quad \left[2F_{r,k}1(a, b; e; r, k; tz) * I\left(\frac{z}{k}\right) \right] dt.
 \end{aligned} \tag{2.7}$$

Now, take $d = e$ into (2.6), we conclude that

$$\begin{aligned}
 2F_{r,k}1\left(a, b; c + \frac{1}{r} - 1; r, k; z\right) * \left(kI\left(\frac{z}{k}\right)\right) &= \frac{\Gamma_{r,k}\left(c + \frac{1}{r} - 1\right)}{\Gamma_{r,k}(e)\Gamma_{r,k}(c-e)} \int_0^1 t^{\frac{r(e-1)+1}{k}-1} (1-t)^{\frac{r(c-e-1)+1}{k}-1} \\
 &\quad \left[2F_{r,k}1(a, b; e; r, k; tz) * I\left(\frac{z}{k}\right) \right] dt.
 \end{aligned}$$

Definition 2.2. Let $\lambda > 1 - \frac{1}{r}$, $\delta, \sigma \in \mathbb{R}$. (r, k) -Srivastava fractional integral denoted by $I_{r,k}^{\lambda, \delta, \sigma}$ and defined as follows:

$$I_{r,k}^{\lambda, \delta, \sigma} f(z) = \frac{z^{-\frac{r(\lambda+\delta-2)+2}{k}}}{\Gamma_{r,k}(\lambda)} \int_0^z (z-\zeta)^{\frac{r(\lambda-1)+1}{k}-1} \left[2F_{r,k}1\left(\lambda + \delta, -\sigma; \lambda; 1 - \frac{\zeta}{z}\right) * I\left(\frac{1}{k} - \frac{\zeta}{kz}\right) \right] f(\zeta) d\zeta, \tag{2.8}$$

where $f(z)$ is an analytic function in a simple connected region of $\mathbb{C}$ containing 0 with order $f(z) = o(|z|^\epsilon)$, $z \rightarrow 0$, and $\epsilon > \max\{\delta - \sigma, 0\} - 1$.

The definition here given of a generalized fractional integral operator is more general than that of Srivastava at el. $I_z^{\lambda, \delta, \sigma}$ given in [23]. Also in [6] Morais and Zayed introduced the fractional integro-differential operator $I_z^{\lambda, \delta, \sigma}(m)$ as a generalization of Srivastava integral operator. We note that

- (1) $I_{1,1}^{\lambda, \delta, \sigma} f(z) = I_z^{\lambda, \delta, \sigma} f(z)$.
- (2) $I_{1,1}^{\lambda, \delta, \sigma} f^{(m)}(z) = I_z^{\lambda, \delta, \sigma}(m) f(z)$.

It is significant to not that many authors defined the operators having hypergeometric under integration and $D_z^{-\lambda}$ associated with a real-valued function (see [46–50]).

Lemma 2.1. Let $\lambda > 1 - \frac{1}{r}$, $\frac{k\mu+k-1}{r} > \delta - \sigma - 2 + \frac{1}{r}$. Then

$$I_{r,k}^{\lambda,\delta,\sigma} z^\mu = \frac{\Gamma_{r,k}\left(\frac{k\mu+k-1}{r} + 1\right) \Gamma_{r,k}\left(\frac{k\mu+k-1}{r} - \delta + \sigma + 2 - \frac{1}{r}\right)}{\Gamma_{r,k}\left(\frac{k\mu+k-1}{r} - \delta + 1\right) \Gamma_{r,k}\left(\frac{k\mu+k-1}{r} + \lambda + \sigma + 1\right)} z^{\mu - \frac{r(\delta-1)+1}{k}}. \quad (2.9)$$

Proof. In view (2.8)

$$I_{r,k}^{\lambda,\delta,\sigma} z^\mu = \frac{z^{-\frac{r(\lambda+\delta-2)+2}{k}}}{\Gamma_{r,k}(\lambda)} \int_0^z (z-\zeta)^{\frac{r(\lambda-1)+1}{k}-1} \left[2F_{r,k}1\left(\lambda + \delta, -\sigma; \lambda; 1 - \frac{\zeta}{z}\right) * I\left(\frac{1}{k} - \frac{\zeta}{kz}\right) \right] \zeta^\mu d\zeta.$$

Set $t = 1 - \frac{\zeta}{z}$, we obtain

$$I_{r,k}^{\lambda,\delta,\sigma} z^\mu = \frac{z^{\mu - \frac{r(\delta-1)+1}{k}}}{\Gamma_{r,k}(\lambda)} \int_0^1 t^{\frac{r(\lambda-1)+1}{k}-1} (1-t)^\mu \left[2F_{r,k}1(\lambda + \delta, -\sigma; \lambda; t) * I\left(\frac{t}{k}\right) \right] \zeta^\mu d\zeta.$$

Applying Theorem 2.3 with $e = \lambda$, $c = \frac{k\mu+k-1}{r} + \lambda + 1$, $a = \lambda + \delta$, and $b = -\sigma$, we obtain

$$I_{r,k}^{\lambda,\delta,\sigma} z^\mu = \frac{z^{\mu - \frac{r(\delta-1)+1}{k}}}{\Gamma_{r,k}(\lambda)} \times \frac{\Gamma_{r,k}(\lambda) \Gamma_{r,k}\left(\frac{k\mu+k-1}{r} + 1\right)}{\Gamma_{r,k}\left\{\left(\frac{k\mu+k-1}{r} + \lambda + 1\right) + \frac{1}{r} - 1\right\}} \left[2F_{r,k}1(\lambda + \delta, -\sigma; \lambda; 1) * \left(kI\left(\frac{1}{k}\right)\right) \right].$$

Theorem 2.2 follows

$$I_{r,k}^{\lambda,\delta,\sigma} z^\mu = \frac{\Gamma_{r,k}\left(\frac{k\mu+k-1}{r} + 1\right) \Gamma_{r,k}\left(\frac{k\mu+k-1}{r} - \delta + \sigma + 2 - \frac{1}{r}\right)}{\Gamma_{r,k}\left(\frac{k\mu+k-1}{r} - \delta + 1\right) \Gamma_{r,k}\left(\frac{k\mu+k-1}{r} + \lambda + \sigma + 1\right)} z^{\mu - \frac{r(\delta-1)+1}{k}}.$$

Finally, we use above lemma to define $\mathfrak{I}_{r,k}^{\lambda,\delta,\sigma} : \mathcal{A}_p \rightarrow \mathcal{A}_p$ by

$$\begin{aligned} \mathfrak{I}_{r,k}^{\lambda,\delta,\sigma} f(z) &= \frac{\Gamma_{r,k}\left(\frac{kp+k-1}{r} - \delta + 1\right) \Gamma_{r,k}\left(\frac{kp+k-1}{r} + \lambda + \sigma + 1\right)}{\Gamma_{r,k}\left(\frac{kp+k-1}{r} + 1\right) \Gamma_{r,k}\left(\frac{kp+k-1}{r} - \delta + \sigma + 2 - \frac{1}{r}\right)} z^{\frac{r(\delta-1)+1}{k}} I_{r,k}^{\lambda,\delta,\sigma} f(z) \\ &= z^p + \sum_{n=p+1}^{\infty} \frac{\Gamma_{r,k}\left(\frac{kp+k-1}{r} - \delta + 1\right) \Gamma_{r,k}\left(\frac{kp+k-1}{r} + \lambda + \sigma + 1\right) \Gamma_{r,k}\left(\frac{kn+k-1}{r} + 1\right) \Gamma_{r,k}\left(\frac{kn+k-1}{r} - \delta + \sigma + 2 - \frac{1}{r}\right)}{\Gamma_{r,k}\left(\frac{kp+k-1}{r} + 1\right) \Gamma_{r,k}\left(\frac{kp+k-1}{r} - \delta + \sigma + 2 - \frac{1}{r}\right) \Gamma_{r,k}\left(\frac{kn+k-1}{r} - \delta + 1\right) \Gamma_{r,k}\left(\frac{kn+k-1}{r} + \lambda + \sigma + 1\right)} z^n. \end{aligned}$$

Easily to check that

$$z\left(\mathfrak{I}_{r,k}^{\lambda,\delta,\sigma} f(z)\right)' = \left[p + \frac{r}{k}(\lambda + \sigma)\right] \mathfrak{I}_{r,k}^{\lambda - \frac{k}{r}, \delta, \sigma} f(z) - \frac{r}{k}(\lambda + \sigma) \mathfrak{I}_{r,k}^{\lambda, \delta, \sigma} f(z), \quad (2.10)$$

where $r \in \mathbb{N}$, $k \in \mathbb{R}^+$, $\delta, \sigma \in \mathbb{R}$, and $\lambda > 1 - \frac{1-k}{r}$.

In special case

$$\mathfrak{I}_z^{\lambda,\delta,\sigma} f(z) = \mathfrak{I}_{1,1}^{\lambda,\delta,\sigma} f(z) = z^p + \sum_{n=p+1}^{\infty} \frac{\Gamma(p - \delta + 1) \Gamma(p + \lambda + \sigma + 1) \Gamma(n + 1) \Gamma(n - \delta + \sigma + 1)}{\Gamma(p + 1) \Gamma(p - \delta + \sigma + 1) \Gamma(n - \delta + 1) \Gamma(n + \lambda + \sigma + 1)} z^n,$$

and

$$z\left(\mathfrak{I}_z^{\lambda,\delta,\sigma} f(z)\right)' = (p + \lambda + \sigma) \mathfrak{I}_z^{\lambda-1, \delta, \sigma} f(z) - (\lambda + \sigma) \mathfrak{I}_z^{\lambda, \delta, \sigma} f(z). \quad (2.11)$$

Differential subordination results

Unless indicated otherwise, we will assume throughout the rest of this paper that $r \in \mathbb{N}$, $k \in \mathbb{R}^+$, $\delta, \sigma \in \mathbb{R}$, and $\lambda > 1 - \frac{1-2k}{r}$. In this section, we present applications of differential subordination and superordination, incorporating our fractional operator. The applications lead to the presentation of corollaries and remarks, some of which generalize well-known results, such as Marx-Strohhaker theorem.

Theorem 3.1. Let χ be univalent in U with $\chi(0) = 1$, $\varphi, \rho \in \mathcal{A}_p$, $\theta, \omega \in \mathbb{R}^+$ and $\gamma \in \mathbb{R}^*$. Also, suppose that $\frac{z\chi'(z)}{\chi(z)}$ is starlike and $\operatorname{Re}(\chi(z)) > 0$. If $f \in \mathcal{A}_p$ satisfies the below inequality:

$$\Omega_1(z) < 1 + \theta \chi(z) + \frac{z\omega \chi'(z)}{\chi(z)}, \quad (3.1)$$

where

$$\Omega_1(z) = 1 + \theta \left(\frac{\mathfrak{I}_{r,k}^{\lambda-\frac{k}{r},\delta,\sigma}(f * \varphi)(z)}{\mathfrak{I}_{r,k}^{\lambda,\delta,\sigma}(f * \varphi)(z)} \right)^\gamma + \gamma \omega \left[\left(p + \frac{r}{k}(\lambda + \sigma) - 1 \right) \frac{\mathfrak{I}_{r,k}^{\lambda-\frac{2k}{r},\delta,\sigma}(f * \varphi)(z)}{\mathfrak{I}_{r,k}^{\lambda-\frac{k}{r},\delta,\sigma}(f * \varphi)(z)} - \left(p + \frac{r}{k}(\lambda + \sigma) \right) \frac{\mathfrak{I}_{r,k}^{\lambda-\frac{k}{r},\delta,\sigma}(f * \varphi)(z)}{\mathfrak{I}_{r,k}^{\lambda,\delta,\sigma}(f * \varphi)(z)} + 1 \right], \quad (3.2)$$

then

$$\left(\frac{\mathfrak{I}_{r,k}^{\lambda-\frac{k}{r},\delta,\sigma}(f * \varphi)(z)}{\mathfrak{I}_{r,k}^{\lambda,\delta,\sigma}(f * \varphi)(z)} \right)^\gamma < \chi(z),$$

and $\chi(z)$ is the best dominant of (3.1).

Proof. Define

$$\xi(z) = \left(\frac{\mathfrak{I}_{r,k}^{\lambda-\frac{k}{r},\delta,\sigma}(f * \varphi)(z)}{\mathfrak{I}_{r,k}^{\lambda,\delta,\sigma}(f * \varphi)(z)} \right)^\gamma, \quad z \in U. \quad (3.3)$$

Then ξ is an analytic and $\xi(0) = 1$. After some computation, we get

$$\frac{z\xi'(z)}{\xi(z)} = \gamma \left(\frac{z \left(\mathfrak{I}_{r,k}^{\lambda-\frac{k}{r},\delta,\sigma}(f * \varphi)(z) \right)'}{\mathfrak{I}_{r,k}^{\lambda-\frac{k}{r},\delta,\sigma}(f * \varphi)(z)} - \frac{z \left(\mathfrak{I}_{r,k}^{\lambda,\delta,\sigma}(f * \varphi)(z) \right)'}{\mathfrak{I}_{r,k}^{\lambda,\delta,\sigma}(f * \varphi)(z)} \right). \quad (3.4)$$

Substituting (2.10) into (3.4), we obtain

$$\frac{z\xi'(z)}{\xi(z)} = \gamma \left[\left(p + \frac{r}{k}(\lambda + \sigma) - 1 \right) \frac{\mathfrak{I}_{r,k}^{\lambda-\frac{2k}{r},\delta,\sigma}(f * \varphi)(z)}{\mathfrak{I}_{r,k}^{\lambda-\frac{k}{r},\delta,\sigma}(f * \varphi)(z)} - \left(p + \frac{r}{k}(\lambda + \sigma) \right) \frac{\mathfrak{I}_{r,k}^{\lambda-\frac{k}{r},\delta,\sigma}(f * \varphi)(z)}{\mathfrak{I}_{r,k}^{\lambda,\delta,\sigma}(f * \varphi)(z)} + 1 \right].$$

Consequently,

$$1 + \theta \xi(z) + \frac{z\omega \xi'(z)}{\xi(z)} = \Omega_1(z), \quad (3.5)$$

where Ω_1 is given by (3.2).

Using (3.1) and (3.5), we obtain

$$1 + \theta \xi(z) + \frac{z\omega \xi'(z)}{\xi(z)} < 1 + \theta \chi(z) + \frac{z\omega \chi'(z)}{\chi(z)}.$$

By putting $\Xi(w) = 1 + \theta w$ and $Y(w) = \frac{\omega}{w}$. Obviously, check that $\Xi(w)$ and $Y(w)$ are analytic in $\mathbb{C} \setminus \{0\}$ and that $Y(w) \neq 0$, $w \in \mathbb{C} \setminus \{0\}$. We obtain

$$Q(z) = z\chi'(z)Y(\chi(z)) = \frac{z\omega \chi'(z)}{\chi(z)},$$

and

$$\mathfrak{h}(z) = \Xi(\chi(z)) + Q(z) = 1 + \theta \chi(z) + \frac{z\omega \chi'(z)}{\chi(z)}.$$

In light of the hypothesis, it is observed that $Q(z)$ is starlike and

$$\operatorname{Re} \left\{ \frac{z\mathfrak{h}'(z)}{Q(z)} \right\} = \operatorname{Re} \left\{ \frac{z \left(\theta \chi'(z) + Q'(z) \right)}{Q(z)} \right\} = \operatorname{Re} \left\{ \frac{\theta}{\omega} \chi(z) \right\} + \operatorname{Re} \left\{ \frac{zQ'(z)}{Q(z)} \right\} > 0.$$

Thus Lemma 1.1 leads to requirement .

Choose $\varphi(z) = \rho(z) = z^p/1 - z$ and $\chi(z) = 1 + z/1 - z$, then $\operatorname{Re}(\chi(z)) > 0$ and $Q(z) = z\chi'(z)/\chi(z)$ is starlike. Taking $\omega = p = r = k = 1$ into Theorem 3.1, we have the following corollary:

Corollary 3.1. Let $f \in \mathcal{A}$, $\theta > 0$, and $\gamma \in \mathbb{R}^*$. If

$$1 + \theta \left(\frac{\mathfrak{I}_z^{\lambda-1,\delta,\sigma} f(z)}{\mathfrak{I}_z^{\lambda,\delta,\sigma} f(z)} \right)^\gamma + \gamma \left[(\lambda + \sigma) \frac{\mathfrak{I}_z^{\lambda-2,\delta,\sigma} f(z)}{\mathfrak{I}_z^{\lambda-1,\delta,\sigma} f(z)} - (1 + \lambda + \sigma) \frac{\mathfrak{I}_z^{\lambda-1,\delta,\sigma} f(z)}{\mathfrak{I}_z^{\lambda,\delta,\sigma} f(z)} + 1 \right] < 1 + \theta \frac{1+z}{1-z} + \frac{2z}{1-z^2},$$

then

$$\left(\frac{\mathfrak{I}_z^{\lambda-1,\delta,\sigma} f(z)}{\mathfrak{I}_z^{\lambda,\delta,\sigma} f(z)} \right)^\gamma < \frac{1+z}{1-z},$$

or equivalently,

$$\operatorname{Re} \left(\left(\frac{\mathfrak{T}_z^{\lambda-1,\delta,\sigma} f(z)}{\mathfrak{T}_z^{\lambda,\delta,\sigma} f(z)} \right)^\gamma \right) > 0.$$

Choose $\varphi(z) = \rho(z) = z^p/1 - z$ and $\chi(z) = e^z$, then $\operatorname{Re}(\chi(z)) > 0$ and $Q(z) = z\chi'(z)/\chi(z) = z$ is starlike. Taking $\omega = p = r = k = 1$ into [Theorem 3.1](#), we have the following corollary:

Corollary 3.2. Let $f \in \mathcal{A}$, $\theta > 0$, and $\gamma \in \mathbb{R}^*$. If

$$1 + \theta \left(\frac{\mathfrak{T}_z^{\lambda-1,\delta,\sigma} f(z)}{\mathfrak{T}_z^{\lambda,\delta,\sigma} f(z)} \right)^\gamma + \gamma \left[(\lambda + \sigma) \frac{\mathfrak{T}_z^{\lambda-2,\delta,\sigma} f(z)}{\mathfrak{T}_z^{\lambda-1,\delta,\sigma} f(z)} - (1 + \lambda + \sigma) \frac{\mathfrak{T}_z^{\lambda-1,\delta,\sigma} f(z)}{\mathfrak{T}_z^{\lambda,\delta,\sigma} f(z)} + 1 \right] < 1 + z + \theta e^z,$$

then

$$\left(\frac{\mathfrak{T}_z^{\lambda-1,\delta,\sigma} f(z)}{\mathfrak{T}_z^{\lambda,\delta,\sigma} f(z)} \right)^\gamma < e^z.$$

Choose $\varphi(z) = \rho(z) = z^p/1 - z$ and $\chi(z) = 1/1 - z$, then $\operatorname{Re}(\chi(z)) > 0$ and $Q(z) = z\chi'(z)/\chi(z) = z/1 - z$ is starlike. Taking $\omega = p = r = k = 1$ and $\theta = 0$ into [Theorem 3.1](#), we have the following corollary:

Corollary 3.3. Let $f \in \mathcal{A}$, and $\gamma \in \mathbb{R}^*$. If

$$\operatorname{Re} \left(1 + \gamma \left[(\lambda + \sigma) \frac{\mathfrak{T}_z^{\lambda-2,\delta,\sigma} f(z)}{\mathfrak{T}_z^{\lambda-1,\delta,\sigma} f(z)} - (1 + \lambda + \sigma) \frac{\mathfrak{T}_z^{\lambda-1,\delta,\sigma} f(z)}{\mathfrak{T}_z^{\lambda,\delta,\sigma} f(z)} + 1 \right] \right) > \frac{1}{2},$$

then

$$\operatorname{Re} \left(\left(\frac{\mathfrak{T}_z^{\lambda-1,\delta,\sigma} f(z)}{\mathfrak{T}_z^{\lambda,\delta,\sigma} f(z)} \right)^\gamma \right) > \frac{1}{2}.$$

Theorem 3.2. Let χ be univalent in U with $\chi(0) = 1$, $\varphi, \rho \in \mathcal{A}_p$, $\theta, \omega \in \mathbb{R}^+$, $\ell_1, \ell_2 \in \mathbb{C}$ with $\ell_1 + \ell_2 \neq 0$, and $\gamma \in \mathbb{R}^*$. Also, suppose that $\frac{z\chi'(z)}{\chi(z)}$ is starlike and $\operatorname{Re}(\chi(z)) > 0$. If $f \in \mathcal{A}_p$ satisfies the below inequality:

$$\Omega_2(z) < 1 + \theta\chi(z) + \frac{z\omega\chi'(z)}{\chi(z)}, \quad (3.6)$$

where

$$\begin{aligned} \Omega_2(z) = 1 + \theta & \left(\frac{\ell_1 \mathfrak{T}_{r,k}^{\lambda-\frac{k}{r},\delta,\sigma}(f * \varphi)(z) + \ell_2 \mathfrak{T}_{r,k}^{\lambda,\delta,\sigma}(f * \rho)(z)}{(\ell_1 + \ell_2)z^p} \right)^\gamma \\ & + \gamma\omega \left[\frac{\ell_1 \left(p + \frac{r}{k}(\lambda + \sigma) - 1 \right) \left(\mathfrak{T}_{r,k}^{\lambda-\frac{2k}{r},\delta,\sigma}(f * \varphi)(z) - \mathfrak{T}_{r,k}^{\lambda-\frac{k}{r},\delta,\sigma}(f * \varphi)(z) \right) + \ell_2 \left(p + \frac{r}{k}(\lambda + \sigma) \right) \left(\mathfrak{T}_{r,k}^{\lambda-\frac{k}{r},\delta,\sigma}(f * \rho)(z) - \mathfrak{T}_{r,k}^{\lambda,\delta,\sigma}(f * \rho)(z) \right)}{\ell_1 \mathfrak{T}_{r,k}^{\lambda-\frac{k}{r},\delta,\sigma}(f * \varphi)(z) + \ell_2 \mathfrak{T}_{r,k}^{\lambda,\delta,\sigma}(f * \rho)(z)} \right], \end{aligned} \quad (3.7)$$

then

$$\left(\frac{\ell_1 \mathfrak{T}_{r,k}^{\lambda-\frac{k}{r},\delta,\sigma}(f * \varphi)(z) + \ell_2 \mathfrak{T}_{r,k}^{\lambda,\delta,\sigma}(f * \rho)(z)}{(\ell_1 + \ell_2)z^p} \right)^\gamma < \chi(z),$$

and $\chi(z)$ is the best dominant of (3.9).

Proof. Define

$$\xi(z) = \left(\frac{\ell_1 \mathfrak{T}_{r,k}^{\lambda-\frac{k}{r},\delta,\sigma}(f * \varphi)(z) + \ell_2 \mathfrak{T}_{r,k}^{\lambda,\delta,\sigma}(f * \rho)(z)}{(\ell_1 + \ell_2)z^p} \right)^\gamma, \quad (z \in U). \quad (3.8)$$

Then the function $\xi(z)$ is analytic and $\xi(0) = 1$.

After direct computation, we obtain

$$\frac{z\xi'(z)}{\xi(z)} = \gamma \left[\frac{\ell_1 \left(p + \frac{r}{k}(\lambda + \sigma) - 1 \right) \left(\mathfrak{I}_{r,k}^{\lambda - \frac{2k}{r}, \delta, \sigma} (f * \varphi)(z) - \mathfrak{I}_{r,k}^{\lambda - \frac{k}{r}, \delta, \sigma} (f * \varphi)(z) \right) + \ell_2 \left(p + \frac{r}{k}(\lambda + \sigma) \right) \left(\mathfrak{I}_{r,k}^{\lambda - \frac{k}{r}, \delta, \sigma} (f * \varphi)(z) - \mathfrak{I}_{r,k}^{\lambda, \delta, \sigma} (f * \varphi)(z) \right)}{\ell_1 \mathfrak{I}_{r,k}^{\lambda - \frac{k}{r}, \delta, \sigma} (f * \varphi)(z) + \ell_2 \mathfrak{I}_{r,k}^{\lambda, \delta, \sigma} (f * \varphi)(z)} \right]. \quad (3.9)$$

Consequently,

$$\Omega_2(z) = 1 + \theta \xi(z) + \frac{z\omega \xi'(z)}{\xi(z)}. \quad (3.10)$$

Substituting (3.10) into (3.6), we have

$$1 + \theta \xi(z) + \frac{z\omega \xi'(z)}{\xi(z)} < 1 + \theta \chi(z) + \frac{z\omega \chi'(z)}{\chi(z)}.$$

By choose $\Xi(w) = 1 + \theta w$, and $Y(w) = \frac{\omega}{w}$. Obviously, check that $\Xi(w)$ and $Y(w)$ are analytic in $\mathbb{C} \setminus \{0\}$ and that $Y(w) \neq 0$, $w \in \mathbb{C} \setminus \{0\}$. We obtain

$$Q(z) = z\chi'(z)Y(\chi(z)) = \frac{z\omega \chi'(z)}{\chi(z)},$$

and

$$\mathfrak{h}(z) = \Xi(\chi(z)) + Q(z) = 1 + \theta \chi(z) + \frac{z\omega \chi'(z)}{\chi(z)}.$$

In light of the hypothesis, it is observed that $Q(z)$ is starlike and

$$Re \left\{ \frac{z\mathfrak{h}'(z)}{Q(z)} \right\} = Re \left\{ \frac{z \left(\theta \chi'(z) + Q'(z) \right)}{Q(z)} \right\} = Re \left\{ \frac{\theta}{\omega} \chi'(z) \right\} + Re \left\{ \frac{zQ'(z)}{Q(z)} \right\} > 0.$$

Hence, the proof is complete from applying Lemma 1.1.

Using the same substitutions mentioned in Corollary 3.1, 3.2, and 3.3, respectively, also select $\ell_1 = 0$, and without loss of generality we take $\lambda > 1$. Consequently, we can directly obtain the following three corollaries from Theorem 3.2:

Corollary 3.4. Let $f \in \mathcal{A}$, $\theta > 0$, and $\gamma \in \mathbb{R}^*$. If

$$1 + \theta \left(\frac{\mathfrak{I}_z^{\lambda, \delta, \sigma} f(z)}{z} \right)^\gamma + \gamma \left[\frac{(1 + \lambda + \sigma) \mathfrak{I}_z^{\lambda-1, \delta, \sigma} f(z)}{\mathfrak{I}_{r,k}^{\lambda, \delta, \sigma} f(z)} - (1 + \lambda + \sigma) \right] < 1 + \theta \frac{1+z}{1-z} + \frac{2z}{1-z^2},$$

then

$$\left(\frac{\mathfrak{I}_z^{\lambda, \delta, \sigma} f(z)}{z} \right)^\gamma < \frac{1+z}{1-z},$$

or equivalently,

$$Re \left(\left(\frac{\mathfrak{I}_z^{\lambda, \delta, \sigma} f(z)}{z} \right)^\gamma \right) > 0.$$

Corollary 3.5. Let $f \in \mathcal{A}$, $\theta > 0$, and $\gamma \in \mathbb{R}^*$. If

$$1 + \theta \left(\frac{\mathfrak{I}_z^{\lambda, \delta, \sigma} f(z)}{z} \right)^\gamma + \gamma \left[\frac{(1 + \lambda + \sigma) \mathfrak{I}_z^{\lambda-1, \delta, \sigma} f(z)}{\mathfrak{I}_{r,k}^{\lambda, \delta, \sigma} f(z)} - (1 + \lambda + \sigma) \right] < 1 + z + \theta e^z,$$

then

$$\left(\frac{\mathfrak{I}_z^{\lambda, \delta, \sigma} f(z)}{z} \right)^\gamma < e^z.$$

Corollary 3.6. Let $f \in \mathcal{A}$, and $\gamma \in \mathbb{R}^*$. If

$$Re \left(1 + \gamma \left[\frac{(1 + \lambda + \sigma) \mathfrak{I}_z^{\lambda-1, \delta, \sigma} f(z)}{\mathfrak{I}_{r,k}^{\lambda, \delta, \sigma} f(z)} - (1 + \lambda + \sigma) \right] \right) > \frac{1}{2},$$

then

$$\operatorname{Re}\left(\left(\frac{\mathfrak{F}_z^{\lambda,\delta,\sigma} f(z)}{z}\right)^\gamma\right) > \frac{1}{2}.$$

Remark 3.1. Let $f \in \mathcal{A}$. Put $zf'(z) = g(z)$, then $g \in \mathcal{A}$ and

$$f(z) = \int_0^z \frac{g(w)}{w} dw.$$

Choose, $\lambda - 1 \rightarrow 0$, $\delta = 1 - \lambda$, $\sigma = -1$, $\gamma = 1$, then

$$\mathfrak{F}_z^{\lambda-1,\delta,\sigma} g(z) = g(z). \quad (3.11)$$

On the other hand,

$$\mathfrak{F}_z^{\lambda,\delta,\sigma} g(z) = \int_0^z \frac{g(w)}{w} dw. \quad (3.12)$$

Consequently, it can be concluded that [Corollary 3.6](#) generalizes one of the conditions for Marx-Strohhacker theorem (see [\[51,52\]](#))

Theorem 3.3. Let χ be convex in U with $\chi(0) = 1$, $\varphi, \rho \in \mathcal{A}_p$, $\theta, \omega \in \mathbb{R}^+$ and $\gamma \in \mathbb{R}^*$. Also, suppose that $\frac{z\chi'(z)}{\chi(z)}$ is starlike and $(\chi^2)'(z)$ has positive real part in U , respectively. If $f \in \mathcal{A}_p$ and

$$\left(\frac{\mathfrak{F}_{r,k}^{\lambda-\frac{k}{r},\delta,\sigma}(f * \varphi)(z)}{\mathfrak{F}_{r,k}^{\lambda,\delta,\sigma}(f * \rho)(z)}\right)^\gamma \in H[1, 1] \cap Q,$$

$\Omega_1(z)$ is given by [\(3.2\)](#), and Ω_1 is univalent in U . If

$$\rho + \theta\chi(z) + \frac{\omega z\chi'(z)}{\chi(z)} < \Omega_1(z), \quad (3.13)$$

then

$$\chi(z) < \left(\frac{\mathfrak{F}_{r,k}^{\lambda-\frac{k}{r},\delta,\sigma}(f * \varphi)(z)}{\mathfrak{F}_{r,k}^{\lambda,\delta,\sigma}(f * \rho)(z)}\right)^\gamma,$$

and $\chi(z)$ is the best dominant of [\(3.13\)](#).

Proof. Define

$$\xi(z) = \left(\frac{\mathfrak{F}_{r,k}^{\lambda-\frac{k}{r},\delta,\sigma}(f * \varphi)(z)}{\mathfrak{F}_{r,k}^{\lambda,\delta,\sigma}(f * \rho)(z)}\right)^\gamma, \quad z \in U.$$

Then ξ is analytic and $\xi(0) = 1$. It follows from [\(2.10\)](#), [\(3.2\)](#), and [\(3.13\)](#) that

$$\rho + \theta\chi(z) + \frac{\omega z\chi'(z)}{\chi(z)} < \rho + \theta\xi(z) + \frac{\omega z\xi'(z)}{\xi(z)}.$$

By choose $\Xi(w) = 1 + \theta w$ and $Y(w) = \frac{\omega}{w}$. Obviously, the check that $\Xi(w)$ and $Y(w)$ are analytic in $\mathbb{C} \setminus \{0\}$ and that $Y(w) \neq 0$, $w \in \mathbb{C} \setminus \{0\}$. Since $(\chi^2)'(z)$ has positive real part, then

$$\operatorname{Re}\left\{\frac{\Xi'(\chi(z))}{Y(\chi(z))}\right\} = \operatorname{Re}\left\{\frac{\theta}{\omega}\chi(z)\chi'(z)\right\} > 0,$$

also from hypothesis $Q(z)$ is starlike The application of [Lemma 1.2](#) leads to the requirement.

Let χ be convex in U with $\chi(0) = 1$, $\varphi, \rho \in \mathcal{A}_p$, $\theta, \omega \in \mathbb{R}^+$ and $\gamma \in \mathbb{R}^*$. Also, suppose that $\frac{z\chi'(z)}{\chi(z)}$ and $(\chi^2)'(z)$ are starlike and univalent close-to-convex in U , respectively. If $f \in \mathcal{A}_p$ and

$$\left(\frac{\mathfrak{F}_{r,k}^{\lambda-\frac{k}{r},\delta,\sigma}(f * \varphi)(z)}{\mathfrak{F}_{r,k}^{\lambda,\delta,\sigma}(f * \rho)(z)}\right)^\gamma \in H[1, 1] \cap Q,$$

$\Omega_1(z)$ is given by [\(3.3\)](#), and Ω_1 is univalent in U . If

$$\rho + \theta\chi(z) + \frac{\omega z\chi'(z)}{\chi(z)} < \Omega_1(z), \quad (3.14)$$

then

$$\chi(z) \prec \left(\frac{\mathfrak{T}_{r,k}^{\lambda-\frac{k}{r},\delta,\sigma}(f * \varphi)(z)}{\mathfrak{T}_{r,k}^{\lambda,\delta,\sigma}(f * \varphi)(z)} \right)^\gamma,$$

and $\chi(z)$ is the best dominant of (3.13).

Theorem 3.4. Let χ be convex in U with $\chi(0) = 1$, $\varphi, \varrho \in \mathcal{A}_p$, $\theta, \omega \in \mathbb{R}^+$ and $\gamma \in \mathbb{R}^*$. Also, suppose that $\frac{z\chi'(z)}{\chi(z)}$ is starlike and $(\chi^2)'(z)$ has positive real part in U , respectively. If $f \in \mathcal{A}_p$ and

$$\left(\frac{\ell_1 \mathfrak{T}_{r,k}^{\lambda-\frac{k}{r},\delta,\sigma}(f * \varphi)(z) + \ell_2 \mathfrak{T}_{r,k}^{\lambda,\delta,\sigma}(f * \varrho)(z)}{(\ell_1 + \ell_2)z^p} \right)^\gamma \in H[1, 1] \cap Q,$$

where $\Omega_2(z)$ is given by (3.7), and Ω_2 is univalent in U . If

$$\rho + \theta \chi(z) + \frac{\omega z \chi'(z)}{\chi(z)} \prec \Omega_2(z), \quad (3.15)$$

then

$$\chi(z) \prec \left(\frac{\ell_1 \mathfrak{T}_{r,k}^{\lambda-\frac{k}{r},\delta,\sigma}(f * \varphi)(z) + \ell_2 \mathfrak{T}_{r,k}^{\lambda,\delta,\sigma}(f * \varrho)(z)}{(\ell_1 + \ell_2)z^p} \right)^\gamma,$$

and $\chi(z)$ is the best dominant of (3.15).

Proof. The result is followed by applying the same method as Theorem 3.3, setting

$$\xi(z) = \left(\frac{\ell_1 \mathfrak{T}_{r,k}^{\lambda-\frac{k}{r},\delta,\sigma}(f * \varphi)(z) + \ell_2 \mathfrak{T}_{r,k}^{\lambda,\delta,\sigma}(f * \varrho)(z)}{(\ell_1 + \ell_2)z^p} \right)^\gamma.$$

Sandwich results

Combining Theorems 3.1, 3.2, 3.3, and 3.4, we conclude the following sandwich results for p -valent functions involving $\mathfrak{T}_{r,k}^{\lambda,\delta,\sigma}$:

Theorem 4.1. Let χ_1 and χ_2 be convex univalent functions in U with $\chi_1(0) = \chi_2(0) = 1$, $\varphi, \varrho \in \mathcal{A}_p$, $\theta, \omega \in \mathbb{R}^+$ and $\gamma \in \mathbb{R}^*$. Moreover, $z\chi_1'(z)/\chi_1(z)$ and $z\chi_2'(z)/\chi_2(z)$ are starlike in U , also suppose the real part of $\chi_2(z)$ and $(\chi_1^2)'(z)$ are positive. If $f \in \mathcal{A}_p$ and

$$\left(\frac{\mathfrak{T}_{r,k}^{\lambda-\frac{k}{r},\delta,\sigma}(f * \varphi)(z)}{\mathfrak{T}_{r,k}^{\lambda,\delta,\sigma}(f * \varrho)(z)} \right)^\gamma \in H[1, 1] \cap Q,$$

Ω_1 is given by (3.2), and Ω_1 is univalent in U . If

$$1 + \theta \chi_1(z) + \frac{\omega z \chi_1'(z)}{\chi_1(z)} \prec \Omega_1(z) \prec 1 + \theta \chi_2(z) + \frac{\omega z \chi_2'(z)}{\chi_2(z)},$$

then

$$\chi_1(z) \prec \left(\frac{\mathfrak{T}_{r,k}^{\lambda-\frac{k}{r},\delta,\sigma}(f * \varphi)(z)}{\mathfrak{T}_{r,k}^{\lambda,\delta,\sigma}(f * \varrho)(z)} \right)^\gamma \prec \chi_2(z),$$

and the dominant χ_2 (the subordinant χ_1) is besting.

Theorem 3.2. Let χ_1 and χ_2 be convex univalent functions in U with $\chi_1(0) = \chi_2(0) = 1$, $\varphi, \varrho \in \mathcal{A}_p$, $\theta, \omega \in \mathbb{R}^+$, $\ell_1, \ell_2 \in \mathbb{C}$ with $\ell_1 + \ell_2 \neq 0$, and $\gamma \in \mathbb{R}^*$. Moreover, $\frac{z\chi_1'(z)}{\chi_1(z)}$ and $\frac{z\chi_2'(z)}{\chi_2(z)}$ are starlike in U , also suppose the real parts of $\chi_2(z)$ and $(\chi_1^2)'(z)$ are positive. If $f \in \mathcal{A}_p$ and

$$\left(\frac{\ell_1 \mathfrak{T}_{r,k}^{\lambda-\frac{k}{r},\delta,\sigma}(f * \varphi)(z) + \ell_2 \mathfrak{T}_{r,k}^{\lambda,\delta,\sigma}(f * \varrho)(z)}{(\ell_1 + \ell_2)z^p} \right)^\gamma \in H[1, 1] \cap Q,$$

and Ω_2 is given by (3.7). If Ω_2 is an univalent in U and satisfy the bellow inequality:

$$1 + \theta \chi_1(z) + \frac{\omega z \chi_1'(z)}{\chi_1(z)} < \Omega_1(z) < 1 + \theta \chi_2(z) + \frac{\omega z \chi_2'(z)}{\chi_2(z)},$$

then

$$\chi_1(z) < \left(\frac{\ell_1 \mathfrak{F}_{r,k}^{\lambda-\frac{k}{r}, \delta, \sigma}(f * \varphi)(z) + \ell_2 \mathfrak{F}_{r,k}^{\lambda, \delta, \sigma}(f * \varphi)(z)}{(\ell_1 + \ell_2) z^p} \right)^\gamma < \chi_2(z),$$

and the dominant χ_2 (the subordinant χ_1) is besting.

Class of admissible functions involving $\mathfrak{F}_z^{\lambda, \delta, \sigma}$

Below, we provide a definition of a class of admissible functions that includes the operator $\mathfrak{F}_z^{\lambda, \delta, \sigma}$, we also present some results related to this definition.

Definition 5.1. Let $\Omega \subset \mathbb{C}$, $\chi \in Q$, and $n \in \mathbb{Z}^+$. The class of admissible functions $L_p[\Omega, \chi]$ consists of those functions $\phi : \mathbb{C}^3 \times U \rightarrow \mathbb{C}$ that satisfy the admissibility condition:

$$\phi(a, b, c; z) \notin \Omega \quad (5.1)$$

whenever

$$a = \chi(\mathcal{G}), \quad b = \frac{m \mathcal{G} \chi'(\mathcal{G}) + (\lambda + \sigma) \chi(\mathcal{G})}{p + \lambda + \sigma},$$

and

$$\operatorname{Re} \left(\frac{c(p + \lambda + \sigma - 1)(p + \lambda + \sigma) - b(p + \lambda + \sigma)(2\lambda + 2\sigma - 1) - 3a(\lambda + \sigma)^2}{b(p + \lambda + \sigma) - a(\lambda + \sigma)} \right) \geq m \operatorname{Re} \left(\frac{\mathcal{G} \chi''(\mathcal{G})}{\chi'(\mathcal{G})} + 1 \right),$$

$$z \in U, \quad \mathcal{G} \in \partial U \setminus E(\chi), \quad \text{and } m \geq p.$$

Theorem 5.1. Let $f \in \mathcal{A}_p$ and $\chi \in Q_0 \cap \mathcal{H}[0, p]$. If $\phi \in L_p[\Omega, q]$ and

$$\phi(\mathfrak{F}_z^{\lambda, \delta, \sigma} f(z), \mathfrak{F}_z^{\lambda-1, \delta, \sigma} f(z), \mathfrak{F}_z^{\lambda-2, \delta, \sigma} f(z); z) \in \Omega, \quad (5.2)$$

then

$$\mathfrak{F}_z^{\lambda, \delta, \sigma} f(z) < \chi(z).$$

Proof. Define

$$k(z) = \mathfrak{F}_z^{\lambda, \delta, \sigma} f(z), \quad (z \in U). \quad (5.3)$$

Differentiating 5.3 with respect to z and use (2.11), we obtain

$$\mathfrak{F}_z^{\lambda-1, \delta, \sigma} f(z) = \frac{zk'(z) + (\lambda + \sigma)k(z)}{p + \lambda + \sigma}. \quad (5.4)$$

By using the same argument to calculation of 5.4, we have

$$\mathfrak{F}_z^{\lambda-2, \delta, \sigma} f(z) = \frac{z^2 k''(z) + 2(\lambda + \sigma)zk'(z) + (\lambda + \sigma - 1)(\lambda + \sigma)k(z)}{(p + \lambda + \sigma - 1)(p + \lambda + \sigma)}. \quad (5.5)$$

Define the following transformations from $\mathbb{C}^3$ into $\mathbb{C}$

$$\begin{aligned} a(r, s, t) &= r, \quad b(r, s, t) = \frac{s + (\lambda + \sigma)r}{p + \lambda + \sigma}, \quad \text{and} \\ c(r, s, t) &= \frac{t + 2(\lambda + \sigma)s + (\lambda + \sigma - 1)(\lambda + \sigma)r}{(p + \lambda + \sigma - 1)(p + \lambda + \sigma)}. \end{aligned} \quad (5.6)$$

Let $\psi : \mathbb{C}^3 \times U \rightarrow \mathbb{C}$ is given by

$$\begin{aligned} \psi(a, b, c; z) &= \phi(a, b, c; z) \\ &= \phi \left(r, \frac{s + (\lambda + \sigma)r}{p + \lambda + \sigma}, \frac{t + 2(\lambda + \sigma)s + (\lambda + \sigma - 1)(\lambda + \sigma)r}{(p + \lambda + \sigma - 1)(p + \lambda + \sigma)}; z \right). \end{aligned} \quad (5.7)$$

In view of 5.3–5.6, the Eq. (5.7) leads to

$$\psi(k(z), zk'(z), z^2 k''(z); z) = \phi(\mathfrak{T}_z^{\lambda, \delta, \sigma} f(z), \mathfrak{T}_z^{\lambda-1, \delta, \sigma} f(z), \mathfrak{T}_z^{\lambda-2, \delta, \sigma} f(z); z),$$

so it follows from 5.2 that

$$\psi(k(z), zk'(z), z^2 k''(z); z) \in \Omega.$$

Moreover,

$$\begin{aligned} \operatorname{Re}\left(\frac{t}{s} + 1\right) &= \operatorname{Re}\left(\frac{c(p + \lambda + \sigma - 1)(p + \lambda + \sigma) - b(p + \lambda + \sigma)(2\lambda + 2\sigma - 1) - 3a(\lambda + \sigma)^2}{b(p + \lambda + \sigma) - a(\lambda + \sigma)}\right) \\ &\geq m \operatorname{Re}\left(\frac{\mathcal{G}\chi''(\mathcal{G})}{\chi'(\mathcal{G})} + 1\right). \end{aligned}$$

Consequently, the classes $\Psi_p[\Omega, \chi]$ and $L_p[\Omega, \chi]$ are equivalent (from Definition 1.2 and Definition 5.1). According to 5.2, from Theorem 1.1, we obtain

$$\mathfrak{T}_z^{\lambda, \delta, \sigma} f(z) < \chi(z).$$

Therefore, our theorem is proven.

Taking $\chi(z) = \frac{1+z}{1-z}$ in the Definition 5.1, the conditions of the class $L_p[\Omega, \chi]$ become

$$\phi\left(\rho i, \frac{\tau + (\lambda + \sigma)\rho i}{p + \lambda + \sigma}, \frac{(\mu + i\nu) + 2(\lambda + \sigma)\tau + (\lambda + \sigma - 1)(\lambda + \sigma)\rho i}{(p + \lambda + \sigma - 1)(p + \lambda + \sigma)}; z\right) \notin \Omega,$$

where $\rho, \tau, \mu, \nu \in \mathbb{R}$, $\tau + \mu \leq 0$, $\tau \leq -\frac{\rho}{2}(1 + \rho^2)$, and $z \in U$. In this particular case Theorem 5.1 become

Corollary 5.1. Let $f \in \mathcal{A}_p$. If $\phi \in L_p[\Omega, q]$ and

$$\phi(\mathfrak{T}_z^{\lambda, \delta, \sigma} f(z), \mathfrak{T}_z^{\lambda-1, \delta, \sigma} f(z), \mathfrak{T}_z^{\lambda-2, \delta, \sigma} f(z); z) \in \Omega,$$

then

$$\operatorname{Re}(\mathfrak{T}_z^{\lambda, \delta, \sigma} f(z)) > 0.$$

Theorem 5.2. Let $f \in \mathcal{A}_p$. Then

$$\operatorname{Re}\left(\frac{\mathfrak{T}_z^{\lambda, \delta, \sigma} f(z) + p}{2} + \frac{z(\mathfrak{T}_z^{\lambda, \delta, \sigma} f(z))'}{\mathfrak{T}_z^{\lambda, \delta, \sigma} f(z) + 1}\right) > 0,$$

implies that

$$\operatorname{Re}(\mathfrak{T}_z^{\lambda, \delta, \sigma} f(z)) > 0.$$

Proof. It follows from (2.11) that

$$\frac{\mathfrak{T}_z^{\lambda, \delta, \sigma} f(z) + p}{2} + \frac{z(\mathfrak{T}_z^{\lambda, \delta, \sigma} f(z))'}{\mathfrak{T}_z^{\lambda, \delta, \sigma} f(z) + 1} = \frac{\mathfrak{T}_z^{\lambda, \delta, \sigma} f(z) + p}{2} + \frac{(p + \lambda + \sigma)\mathfrak{T}_z^{\lambda-1, \delta, \sigma} f(z) - (\lambda + \sigma)\mathfrak{T}_z^{\lambda, \delta, \sigma} f(z)}{\mathfrak{T}_z^{\lambda, \delta, \sigma} f(z) + 1}.$$

Put $\Omega = \{w : \operatorname{Re}(w) > 0\}$, and

$$\phi(\mathfrak{T}_z^{\lambda, \delta, \sigma} f(z), \mathfrak{T}_z^{\lambda, \delta, \sigma} f(z); z) = \frac{\mathfrak{T}_z^{\lambda, \delta, \sigma} f(z) + p}{2} + \frac{(p + \lambda + \sigma)\mathfrak{T}_z^{\lambda-1, \delta, \sigma} f(z) - (\lambda + \sigma)\mathfrak{T}_z^{\lambda, \delta, \sigma} f(z)}{\mathfrak{T}_z^{\lambda, \delta, \sigma} f(z) + 1}.$$

Hence,

$$\phi(\mathfrak{T}_z^{\lambda, \delta, \sigma} f(z), \mathfrak{T}_z^{\lambda, \delta, \sigma} f(z); z) \in \Omega.$$

Therefore, we need only show that $\phi \in L_p[\Omega, q]$ and apply Corollary 5.1. Indeed, this is sufficient to show that

$$\phi\left(\rho i, \frac{\tau + (\lambda + \sigma)\rho i}{p + \lambda + \sigma}; z\right) \notin \Omega.$$

This follows immediately since

$$\begin{aligned} \operatorname{Re}\left(\phi\left(\rho i, \frac{\tau + (\lambda + \sigma)\rho i}{p + \lambda + \sigma}; z\right)\right) &= \operatorname{Re}\left(\frac{\rho i + p}{2} + \frac{(p + \lambda + \sigma)\frac{\tau + (\lambda + \sigma)\rho i}{(p + \lambda + \sigma)} - (\lambda + \sigma)\rho i}{\rho i + 1}\right) \\ &= \frac{p}{2} + \frac{\tau}{1 + \rho^2} \leq 0 \end{aligned}$$

as required.

Applications to fluid mechanics

The phenomenon of vortex motion in fluid dynamics occurs when gases or liquids rotate around a specific center, and it is considered significant in numerous applications. It has a notable impact on atmospheric systems and marine engineering. Vortices are present in various forms, from natural hurricanes to turbines in engineering applications.

Zayed et al. studied vortex dynamics inside a cylindrical container with a radius of ρ in [6,13], where they clarified that the vortex can be represented as $\xi(z)$ and the cylinder surface as $\chi(z)$, then it is required to show that $\xi(z) < \chi(z)$, ($z \in U$). In vortex motion, the object and cloak correspond to a vortex within the circular cylinder and the cylinder's surface, respectively, with the latter rendering the object invisible to an observer. This paper examines the vortex within an infinite circular cylinder of radius ρ , with the flow described by the following equations:

- (1) The velocity of a source of radius r with a flow rate of strength κ is expressed as $v_\rho = 0$.
- (2) The tangential velocity is zero, specifically, $v_\theta = -\kappa/2\pi\rho$.

Considering the Cauchy-Riemann equations, we get

$$v_\rho = \frac{\partial A}{\partial \rho} = \frac{1}{\rho} \frac{\partial B}{\partial \theta}, \quad v_\theta = \frac{1}{\rho} \frac{\partial A}{\partial \theta} = -\frac{\partial B}{\partial \rho}.$$

Consequently, the potential, stream, and complex potential functions for the source are as follows:

$$A = -\frac{\kappa}{2\pi} \theta \text{ and } B = \frac{\kappa}{2\pi} \log r,$$

Therefore,

$$T = A + iB = \frac{i\kappa}{2\pi\rho} \log z.$$

It is obvious that

$$\frac{dT}{dz} = \frac{i\kappa}{2\pi(z - z(0))} \quad (z(0) \text{ is a situated source}).$$

Both the vortices within the circular cylinder and the surface of the cylinder are simply connected regions in the complex plane; thus, both regions are identical to conformal mappings on the open unit disk U . In our work, we have presented several results confirming $\xi(z) < \chi(z)$, ($z \in U$) as in Theorem 3.1, 3.2, 3.3, 3.4, and 5.1.

Conclusion. In summary, we have effectively extended Srivastava's fractional integral calculus by using the (r, k) -gamma function and the (r, k) -hypergeometric function. Fundamental results pertaining to the (r, k) -hypergeometric function were established, providing new insights and expanding the existing structure of fractional calculus. Furthermore, the sandwich results for p -valent analytic functions have been presented in the context of subordination and superordination. The process of generalization involved the employment of the (r, k) -gamma function to reformulate the fractional integral operator, including it into the structure of the (r, k) -hypergeometric function. The scope of fractional calculus and its applications have been broadened in complex analysis by this novel approach. The results derived from the theories of subordination and superordination further demonstrate the efficacy of the new fractional operator in formulating sandwich theorems. Finally, our work paves the way for subsequent studies to explore additional characteristics and applications of the (r, k) -hypergeometric function, thereby contributing to the advancement of mathematical analysis in fractional calculus and its related fields. Future research will concentrate on investigating new classes of analytic functions that incorporate conditions pertaining to our fractional integral operator. Consequently, these will be closely linked to the results articulated in our work. The research scope will be broadened to encompass results related to partial derivatives in the context of multi-variable complex functions. We also urge scholars to utilize our fractional integral operator in the examination of specific subclasses of bi-univalent functions. Additionally, we advocate for the definition and derivation of novel conclusions pertaining to the fractional differential operator, which is dual to $\mathfrak{F}_z^{\lambda, \delta, \sigma}$.

Method validation

Not applicable.

Limitations

None.

Ethics statements

Not applicable.

CRediT author statement

Adel Salim Tayyab: contributed to the design and implementation of the research, conducted the experiments, and drafted the initial manuscript. Meanwhile, **Waggas Galib Atshan:** provided supervision for the research project, offered critical guidance throughout the study, and assisted in reviewing the final manuscript.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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