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Department of Mathematics



New Results of Fuzzy Soft Banach Algebra

A Thesis

**Submitted to Council of the College of Science, University of Al-
Qadisiyah as a partial Fulfillment of the Requirements for the Degree of
Master of Science in Mathematics**

By

Qasim Ali Hatif Al- Araji

Supervised by

Prof. Dr. Noori Farhan Al- Mayahi

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بِسْمِ اللَّهِ الرَّحْمَنِ الرَّحِيمِ

﴿ عَلَّمَ الْإِنْسَانَ مَا لَمْ يَعْلَمْ ﴾

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(سورة العلق / آية : 5)

DEDICATO

*I would like to dedicate this thesis to
my dear mother and dear father, to my dear
brothers and sisters, and to my supervisor*

(Prof. Dr. Noorí F. Al- Mayahi)

*For his enthusiasm and inspiration.
to my dear friends, to everyone who spared
effort in helping me, and to everyone who
contributed to receiving me, even with a letter,
in my academic life.*

Qasim

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Supervisor Certification

I certify that the preparation for this thesis, entitled " **New Results of Fuzzy Soft Banach Algebra** " was prepared by " **Qasim Ali Hatif** " under my Supervision at the Mathematics Department, College of Science, AI-Qadisiyah University, in partial fulfillment of the requirements for the degree of Master of Science in Mathematics

Supervisor

Signature:

Date: / / 2024

Name: Prof. Dr. Noori Farhan Al- Mayahi

Dept. of Mathematics - College of Science

In view of the available recommendation, we forward this thesis for debate by the examining committee.

Head of the Mathematics Department

Signature:

Date: / / 2024

Name: Prof. Dr. Waggas Galib Atshan

Dept. of Mathematics - College of Science.

Language Expert

This is to certify that the thesis entitled " **New Results of Fuzzy Soft Banach Algebra**" by " **Qasim Ali Hatif** " was edited by me, therefore the thesis is qualified for debate.

Signature:

Name:

Date: / / 2024

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We certify that we have read this thesis " **New Results of Fuzzy Soft Banach Algebra**" and as an examining committee, we examined the study in its content with, and that in our opinion it meets the standards of a thesis for the degree of Master of Science in Mathematics.

Signature:

Name:

Date: / / 2024

(Chairman)

Signature:

Name:

The Title

Date: / / 2024.

(Member)

Signature:

Name: Prof. Dr. **Noori Farhan Al- Mayahi**

The Title: University of AI-Qadisiyah.

Date: / / 2024.

(Supervisor)

Approved by the university committee on graduate

Signature:

Prof. Dr. **Mohammed Abdalwahab AL-Askeri**

Dean of College of Science -University of AI-Qadisiyah.

Date: / / 2024.

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Abstract

The main goal of this thesis is to study the convergences and types of convergences of the type s-convergence and st-convergence in the Banach algebra space on a large scale and the dual theory in the Banach algebra space and Definition of the Banach fuzzy algebra space over the fuzzy number, the Banach fuzzy algebra space, and Fuzzy soft unit space and some applications of fuzzy soft sets in Banach algebra. These results are presented in four axes, firstly, defining fuzzy sets, fuzzy numbers, and fuzzy soft sets and defining Banach's fuzzy soft algebra. Secondly, Banach's fuzzy soft algebra was defined for a study on fuzzy soft numbers, convergences, and types of convergences such as s-convergence and st-convergence, Third, the fuzzy soft Banach algebra space is defined by defining a fuzzy soft base on (real numbers) which is called the fuzzy soft algebra Banach real space. Using this standard space, we defined the fuzzy soft linear function and studied some of its properties. We presented the fuzzy soft double space and the fuzzy soft double operator and studied their properties. State and prove the theorem about representing a fuzzy soft linear function by the inner product in fuzzy soft Hilbert spaces. Fourth, some applications of Banach algebra fuzzy soft were presented, such as fuzzy soft linear space and some definitions and examples, fuzzy soft balanced sets and some of their properties, and fuzzy soft convex set.

List of Symbols

Symbols	Meaning
I	The Closed Interval $[0, 1]$
I^X	The Family Of All Fuzzy Sets In X
(X, N)	Banach algebra
$\mathcal{R}^*(I)$	The Set Of All Fuzzy Numbers Without Zero
$\emptyset$	The Empty Set
$\mathbb{R}$	The Set Of Real Numbers
$\mathbb{F}$	Field (real or complex numbers)
$\mathbb{Z}^+$	The Set Of All Positive Integral Numbers
$\bar{A}$	The Closure Of A Fuzzy Set A
$\forall$	For all
$(\oplus, \ominus, \odot)$	The addition, minus, and scalar multiplication of fuzzy Sets.
d	Metric function from X^2 to R
sup	Supremum (least upper bound)
$(\tilde{X}, \tilde{d}, \Gamma_A)$	a fuzzy soft metric space
inf	Infimum (greatest lower bound)
$B(x, \bar{r})$	Fuzzy Open Ball
$\bar{B}(x, \bar{r})$	Fuzzy Closed Ball
L_x	Fuzzy Normed Algebra Function
$\otimes$	Scaler Multiplication Of Banach Algebra.
$\odot$	Scaler Multiplication Of Fuzzy Banach Algebra And Fuzzy Normed Algebra.

E	The Set Of Parameters
$P(X)$	The Power Set Of X
F_E	The collection of set all soft sets over a universe X .
$\mathcal{R}^*(E)$	The set of all positive real fuzzy soft number
$\bar{x}_e$	soft point
$(SV\tilde{X})$	The set of all soft vector over $\tilde{X}$
$SP(\tilde{X})$	The set of all soft points over $\tilde{X}$
$\emptyset$	null soft set
$\tilde{\in}$	Soft belong to.
$\tilde{\subseteq}$	The subset of soft sets.
$\tilde{\cap}, \tilde{\cup}$	The Intersection And Union Of Soft Sets.
$(\boxplus, \boxminus, \boxtimes)$	The Addition, Minus And Scaler Multiplication Of Soft Sets.
$\tilde{\setminus}$	The Difference Between Soft Sets.
$(F_A)^c$	The Complement Of A Soft Set (F_A) .
$B(x_{e_1}, r)$	soft Open ball
$\bar{B}(x_{e_1}, r)$	soft Closed ball
$SV(\tilde{X})$	soft Vector Space
$(\tilde{X}, \Gamma, *)$	Fuzzy soft Banach algebra
$(SSV(\tilde{X}), \langle . \rangle, A)$	fuzzy soft inner product space

Introduction

Functional analysis is one of the branches of mathematics that is concerned with the study of the spaces of functions. The roots of functional analysis lie in the study of transformations, such as the Fourier transform, and in the study of differential and integral equations. The concept of fuzzy sets was introduced by Zadeh [31] in 1965 and thereafter several authors applied it to different branches of pure and applied mathematics.

In 1975, Kramosil and Michalek introduced the fuzzy metric space as a generalization of a metric space. In 1994, George and Veeramani modified the notion of fuzzy metric space with the help of continuous t -norm and It has several unequal definitions and many applications, then presented the definition of fuzzy normed space by cheng and mordeson in 1994,

In 2005 Saadati and Vaezpour [23] proved some results on complete fuzzy normed spaces. Sadeqi and Amiripour in 2007 [24] introduced the definition of "fuzzy Banach algebra" and proved some results of this space. In 2009 Sadeqi and Kia studied fuzzy normed linear space and its topological structure. In 2010 Si, and ef al. studied continuity in an intuitionistic fuzzy normed space. Dinda et al. in 2010 introduced some "properties of intuitionistic fuzzy complete fuzzy normed algebra". In 2015 Nadaban studied the properties of fuzzy continuous mapping on fuzzy normed linear spaces. A condition for "continuous product in a fuzzy normed algebra" is introduced in 2016 by Binzar et al. also proved some "properties of fuzzy Banach algebra"

In 1999, Molodtsov [18] addressed this problem and initiated the theory of soft sets by introducing a concept of the parametrized family of sets.

Thus, P.K. Maji et al [14, 15] defined and studied several basic notions of soft set theory. In 2011, Shabir and Naz introduced soft topological spaces. Later Hazra et al [9], and Gagman et al also studied soft topology in different ways. This thesis consists of four chapters.

The first chapter consists of several sections, including the definition of the fuzzy set, the soft set, and fuzzy soft sets. The second chapter consists of the definitions of fuzzy soft sets. We created a fuzzy soft Banach algebra space, studied convergence, and obtained the types of convergences, s-convergence and st-convergence. In the third chapter, we studied the fuzzy soft Banach algebra space, presented some properties of Banach algebra, and introduced the fuzzy soft dual space. Fuzziness and the fuzzy soft double operator and a study of their properties. We stated and proved the theorem about representing a fuzzy soft linear function by the inner product in fuzzy soft Hilbert spaces. In the fourth chapter, we presented some applications of Banach fuzzy soft algebra, such as fuzzy soft linear space, balanced fuzzy soft sets, and fuzzy soft convex sets.

Chapter One

Fundamental Concepts

Chapter One

Fundamental Concepts

1.1 Introduction:

This chapter. It consists of important definitions and basic concepts. It consists of five sections. The first three section deal with the concepts of fuzzy sets, fuzzy numbers, and fuzzy soft sets and some of their properties. Section four discusses the concept of Banach algebra space and some properties.

1.2 Fuzzy Sets

We start this section by the following definition .

Definition (1.2.1)[28]: Let X be a non-empty set, and I denote for the closed interval $[0,1]$ of real numbers, that is

$I = [0,1] = \{x \in \mathbb{R} : 0 \leq x \leq 1\}$ and I^X denote all functions from X to I , that is $I^X = \{A : A \text{ is a function from } X \text{ to } I\}$

Definition (1.2.2) [28]: A fuzzy set A in X (or a fuzzy subset from X) is a function from X into I , that is, $A \in I^X$. If A is a fuzzy set in X then A is described as a characteristic function that connects every $x \in X$ by a real number $A(x)$ in the interval I . $A(x)$ is the grade of membership function to X in A . A can be described completely as:

$A = \{(x, A(x)) : x \in X, 0 \leq A(x) \leq 1\}$, where $A(x)$ is called the membership function for the fuzzy set A . The fuzzy set A may also be termed as: $A = \left\{ \frac{A(x)}{x} : x \in X \right\}$.

Example (1.2.3)[3]: Let $X = \{a, b, c\}$, and let the function $A : X \rightarrow I$ which is defined as: $A(a) = \frac{1}{5}$, $A(b) = \frac{2}{7}$, $A(c) = \frac{2}{3}$ represents a fuzzy set in X , while the function $B : X \rightarrow \mathbb{R}$ which is defined as: $B(a) = \frac{1}{8}$, $B(b) = \frac{5}{2}$, $B(c) = 5$ does not represent a fuzzy set in X because $B(b), B(c) > 1$.

Definition (1.2.4)[9]: Let $\otimes$ be a binary operation on the set that is $\otimes : I \times I \rightarrow I$ is a function, then $\otimes$ is said to be t-norm (triangular-norm) on the set I if the following axioms are satisfied:

- (1) $a \otimes 1 = a$ for all $a \in I$.
- (2) $\otimes$ is commutative (i.e. $a \otimes b = b \otimes a$ for all $a, b \in I$).
- (3) $\otimes$ is monotone (i.e. if $b, c \in I$ such that $b \leq c$, then $a \otimes b \leq a \otimes c$ for all $a \in I$).
- (4) $\otimes$ is associative (i.e. $a \otimes (b \otimes c) = (a \otimes b) \otimes c$ for all $a, b, c \in I$).

Theorem (1.2.5)[9]: Let $\otimes$ be a t-norm on the set I , then

- (1) $1 \otimes 1 = 1$.
- (2) $0 \otimes 1 = 0$.
- (3) $1 \otimes 0 = 0$.
- (4) $0 \otimes 0 = 0$.
- (5) $a \otimes a \leq a$ for all $a \in I$.
- (6) If $a \leq c, b \leq d$ then $a \otimes b \leq c \otimes d$ for all $a, b, c, d \in I$.

Example (1.2.6)[9]: Let $\mathbb{R}$ be the set of real numbers, and A be a fuzzy set in $\mathbb{R}$. Then one can define the grade of membership function mathematical as:

$$A(x) = \begin{cases} \frac{x-1}{x} & , x > 1 \\ 0 & , x \leq 1 \end{cases}$$

Remark (1.2.7) [9]: If one wants to know the difference between fuzzy sets and ordinary sets, one notes if A is an ordinary set, the definition of the grade of membership takes only two valuables 0,1. that is

$$A(x) = \begin{cases} 1 & , x \in x \\ 0 & , x \notin x \end{cases}$$

Therefore $A(x) \in \{0,1\}$. While if A is a fuzzy set in X , then $0 \leq A(x) \leq 1$ for all $x \in X$, and thus the regular set becomes a special case for the fuzzy sets.

Definition (1.2.8) [7]: Let A be a fuzzy set in X :

(1)The support of the fuzzy set A is denoted by A^* or $Supp(A)$ and is defined as:

$$A^* = Supp(A) = \{x \in X : A(x) > 0\}.$$

(2)The point $x \in X$ is the crossover point for the fuzzy set A if $A(x) = \frac{1}{2}$.

(3)A fuzzy set A is called normal if there exists $x_0 \in X$ such that $A(x_0) = 1$,

that is. $\{x \in X : A(x) = 1\} \neq \emptyset$.

(4)The height of the fuzzy set A is denoted by $ht(A)$ and defined as:

$$ht(A) = \sup \{A(x) : x \in X\}.$$

As a special case, if A is normal, then $ht(A) = 1$, and so A is said to be finite if A^* is a finite set, inverse that A is said to be infinite.

Example (1.2.9)[19]: Let $X = \{a, b, c\}$ and let A be a fuzzy set in X defined as:

$$A(a) = \frac{1}{2}, A(b) = \frac{1}{3}, A(c) = 0, \quad A^* = \{x \in X : A(x) > 0\} = \{a, b\}.$$

The point a is a crossover point for the fuzzy set u because $A(a) = \frac{1}{2}$ and u is not normal because there is no element whose image equal to 1 and

$$ht(A) = \sup \{A(x) : x \in X\} = \sup \left\{ \frac{1}{2}, \frac{1}{3}, 0 \right\} = \frac{1}{2}.$$

Definition (1.2.10)[7] : Let X be a non-empty set. A fuzzy set $A: X \rightarrow I$ is defined by $A(x) = 0$, for all $x \in X$, is called an empty fuzzy set and is denoted by $\emptyset$ or 0 . The fuzzy set $A: X \rightarrow I$ is called a non-empty if there exists at least $x \in X$ such that $A(x) \neq 0$.

Definition (1.2.11)[7] : A fuzzy set $A: X \rightarrow I$ which is defined as $A(x) = 1$, for all $x \in X$ is called a universal fuzzy set and denoted by X or 1 .

Definition (1.2.12) [19]: Let A, B be two fuzzy sets in X . They denote the union of two sets A, B by $A \cup B$ and defined as: $(A \cup B)(x) = \max \{A(x), B(x)\}$ for all $x \in X$, and so denote the intersection of two sets A, B by $A \cap B$ and defined as:

$$(A \cap B)(x) = \min \{A(x), B(x)\}.$$

Remark (1.2.13)[28]: If A, B are two fuzzy sets in X , then $A \cup B, A \cap B$ are two fuzzy sets in X .

Example (1.2.14)[2]: Let $X = \{a, b, c\}$ and A, B be two fuzzy sets in X such that

$A(a) = \frac{1}{6}$, $A(b) = \frac{1}{2}$, $A(c) = \frac{1}{10}$, $B(a) = \frac{1}{5}$, $B(b) = \frac{1}{7}$, $B(c) = \frac{1}{3}$. Then : $(A \cup B)(a) = \max \{A(a), B(a)\} = \max\{\frac{1}{6}, \frac{1}{5}\} = \frac{1}{5}$

$$(A \cup B)(b) = \max \{A(b), B(b)\} = \max\{\frac{1}{2}, \frac{1}{7}\} = \frac{1}{2}$$

$$(A \cup B)(c) = \max \{A(c), B(c)\} = \max\{\frac{1}{10}, \frac{1}{3}\} = \frac{1}{3}$$

$$(A \cap B)(a) = \min \{A(a), B(a)\} = \min\{\frac{1}{6}, \frac{1}{5}\} = \frac{1}{6}$$

$$(A \cap B)(b) = \min \{A(b), B(b)\} = \min\{\frac{1}{2}, \frac{1}{7}\} = \frac{1}{7}$$

$$(A \cap B)(c) = \min \{A(c), B(c)\} = \min\{\frac{1}{10}, \frac{1}{3}\} = \frac{1}{10}.$$

1.3 Fuzzy Numbers

Definition (1.3.1)[11]: A fuzzy set A in $\mathbb{R}$ is called a fuzzy number if:

A is normal that is. $\exists x \in \mathbb{R}$ such that $A(x) = 1$.

1. A is convex, that is.

$$A(\gamma x + (1 - \gamma)y) \geq \min\{A(x), A(y)\} \text{ for all } x, y \in A.$$

$$0 \leq \gamma \leq 1.$$

2. A is upper semi-continuous, that is. The $\{x \in \mathbb{R} : A(x) \leq \gamma\}$ is closed in $\mathbb{R}$ for all $\gamma \in \mathbb{R}$.

3. $\overline{A^*}$ is compact in $\mathbb{R}$, where $A^* = \text{supp}(A) = \{x \in X : A(x) > 0\}$.

Remark (1.3.2)[36,8]: The set of all fuzzy numbers is denoted by $\mathcal{R}(I)$ and the set of all fuzzy numbers without zero is denoted by $\mathcal{R}^*(I)$.

Definition (1.3.3)[4]: Let $(G, \cdot)$ be a group and A, B fuzzy set in G . for all $A, B \in I^G$ define $*$: $I^G \times I^G \rightarrow I^G$ such that

$$(A * B)(x) = \sup\{\min\{A(y), B(z)\} : x = y \cdot z, y, z \in G\}$$

$x \in \text{Im}(\cdot)$ for all $A, B \in I^G$

Definition (1.3.4):[28]: Let $A, B \in I^X$. Define A^{-1} as $A^{-1}(x) = A(x^{-1})$.

Remark (1.3.5)[27]: For all $A * B$ and A^{-1} are fuzzy set in G .

$A * B, A^{-1} \in I^G$ and $A * B$ is called the beating process and A^{-1} is called inverse A .

Let A, B be fuzzy sets in a linear space X over $\mathbb{F}$ and $\gamma \in \mathbb{F}$. we defined $A \oplus B$ and

$\gamma \odot A$ as:

1. $A \oplus B = F(A \times B)$, where $(A \times B)(x, y) = \min\{A(x), B(y)\}$ and $F: X \times X \rightarrow X$ is function defined by $F(x, y) = x \oplus y$ for all $x, y \in X$.

2. $\gamma \odot A = g(A)$ where $g: x \rightarrow x$ is function defined by $g(x) = \gamma \odot x$ for all $x \in X$.

Theorem (1.3.6)[11]: Let A, B be fuzzy sets in a linear space X over $\mathbb{F}$ and $\gamma \in \mathbb{F}$. then:

$$1. (A \oplus B)(z) = \sup\{\min\{A(y), B(z)\} : x = y + z, x, y \in X\}.$$

$$2. (A \oplus B)(z) = \sup\{\min(u(x), v(z - x)) : x \in X\}$$

$$(A \oplus B)(z) = \sup\{\min(u(z - x), v(x)) : x \in X\}.$$

$$3. (\gamma \odot A)(x) = \begin{cases} A\left(\frac{1}{\gamma} x\right) & , \gamma \neq 0 \\ 0 & \gamma = 0, x = 0 \\ \sup\{A(y) : y \in X\} & \gamma = 0, x = 0 \end{cases}.$$

$$4. (\gamma \odot A)(x) > A(x) \text{ for all } x \in X.$$

$$5. A \oplus B \text{ and } \gamma \odot A \text{ are fuzzy sets in } X.$$

1.4 Fuzzy Soft Sets

This section introduce the concept of soft set and fuzzy soft set. The basic definition and properties of fuzzy soft set discuss.

Definition (1.4.1) [16]: A pair (F, E) is said to be a soft set over X , where F is a function given by $F : E \rightarrow P(X)$.

Example (1.4.2) [13]: Suppose the following :

X is the set of houses

E is the set of parameters. Each parameter is a word or a sentence.

$E = \{\text{expensive, beautifully, wooden, cheap, in the green surroundings}\}.$

Then $X = \{h_1, h_2, h_3, h_4, h_5, h_6\}$ and $E = \{e_1, e_2, e_3, e_4, e_5\}.$

Where

e_1 stands for the parameter 'expensive '

e_2 stands for the parameter 'beautiful '

e_3 stands for the parameter 'wooden '

e_4 stands for the parameter 'cheap '

e_5 stands for the parameter 'in the green surroundings '

Suppose that $F(e_1) = \{h_2, h_4\}$, $F(e_2) = \{h_1, h_3\}$, $F(e_3) = \{h_3, h_4, h_5\}$,

$F(e_4) = \{h_1, h_3, h_5\}$

$F(e_5) = \{h_1\}.$

Then the soft set (F, E) is a parameterized family of X and gives us a collection of approximate descriptions of an object.

Thus , we can view the soft set as $(F, E) = \{ \{h_2, h_4\}, \{h_1, h_3\}, \{h_3, h_4, h_5\}, \{h_1, h_3, h_5\}, \{h_1\} \}$

Definition (1.4.3) [8]: A soft set (F, E) over X is said to be a null soft set denoted by $\check{\emptyset}$, if for all $e \in E$, $F(e) = \check{\emptyset}$.

Definition (1.4.4) [8]: A soft set (F, E) over X is said to be an absolute soft set denoted by $\tilde{X}$, if for all $e \in E$, $F(e) = X$.

Definition (1.4.5)[8]: Let (F, A) and (G, B) be two soft sets over universe X , we say (F, A) is a soft subset of (G, B) and denoted by $(F, A) \subseteq (G, B)$ if:

- (i) $A \subseteq B$.
- (ii) $F(e) \subseteq G(e)$, for all $e \in A$.

also, one says that (F, A) and (G, B) are soft equal denoted by $(F, A) = (G, B)$, if $(F, A) \subseteq (G, B)$ and $(G, B) \subseteq (F, A)$.

Example (1.4.6) [13]: Suppose the following

Let $A = \{e_1, e_3, e_5\} \subset E$ and $B = \{e_1, e_2, e_3, e_5\} \subset E$. Clearly $A \subset B$.

Let (F, A) and (G, B) be two soft sets over the same universe $X = \{h_1, h_2, h_3, h_4, h_5, h_6\}$

and $E = \{e_1, e_2, e_3, e_4, e_5\}$ such that .

$G(e_1) = \{h_2, h_4\}$, $G(e_2) = \{h_1, h_3\}$, $G(e_3) = \{h_3, h_4, h_5\}$, $G(e_5) = \{h_1\}$, and

$F(e_1) = \{h_2, h_4\}$, $F(e_3) = \{h_3, h_4, h_5\}$, $F(e_5) = \{h_1\}$.

Therefore, $(F, A) \subseteq (G, B)$.

Definition (1.4.7) [5]: The soft complement of a soft set F_E over a universe X is denoted by $(F_E)^c$ and it is defined by $(F_E)^c = (F^c, E)$, where F^c is function given by $F^c: E \rightarrow P(X)$, $F^c(e) = X \setminus F(e)$, for all $e \in E$.

That is. $(F_E)^c = \{(e, X \setminus F(e)): \forall e \in E\}$. It is clear that:

- (i) $\emptyset^c = \tilde{X}_E$; $\tilde{X}_E^c = \emptyset$.
- (ii) If $\tilde{x}_{e1} \in F_E$, then $\tilde{x}_{e1} \notin (F_E)^c$.

Remark (1.4.8) [8]: $\emptyset$ is a soft subset of any soft set F_E .

- (i) any soft set F_E is a soft subset of $\tilde{X}_E$.

Definition (1.3.9) [5]: The intersection of the two soft sets (F, A) and (G, B) over the common universe X is the soft set (H, C) , where $C = A \cap B$ and for all $e \in C$, $H(e) = F(e) \cap G(e)$. We write $(F, A) \cap (G, B) = (H, C)$.

1. The union of two soft sets (F, A) and (G, B) over the common universe X is the soft set

(H, C) , where $C = A \cup B$ and for all $e \in C$,

$C = A \cup B$ and one writes $H_C = F_A \cup G_B$ such that

$$H(e) = \begin{cases} F(e) & , \text{ if } e \in A \setminus B \\ G(e) & , \text{ if } e \in B \setminus A \\ F(e) \cup G(e) & , \text{ if } e \in A \cap B \end{cases}.$$

We express it as $(F, A) \cup (G, B) = (H, C)$.

2. The difference (H, E) of two soft sets (F, E) and (G, B) over X , denoted by

$(F, A) \setminus (G, B)$, Is defined by $H(e) = F(e) \setminus G(e)$, for all $e \in E$.

Theorem (1.4.10) [5]: Let (F, E) be soft set then the following hold:

- (i) $(F, E)^c \cup (F, E) = (\tilde{X}, E)$.
- (ii) $(F, E) \cap (F, E)^c = \tilde{\emptyset}_E$.
- (iii) $(F, E) \cap (\tilde{X}, E) = (F, E)$.
- (iv) $(F, E) \cap \tilde{\emptyset}_E = \tilde{\emptyset}_E$.
- (v) $(F, E) \cup \tilde{\emptyset}_E = (F, E)$.

Remark (1.4.11)[18]: 1. A soft set F_E for which $F(e)$ is a singleton set, for all $e \in E$, is called singleton soft set.

2. In sense, when $|E| = 1$ a soft set F_E behaves similarly to a set. In this case the soft set is the same as the set $F(e)$, where $E = \{e\}$.

Example(1.4.12)[15]: Let $X = \{1,2,3\}$, $E = \{e_1, e_2\}$ then $F_E = \{(e_1, \{1,2\}), (e_2, \{2,3\})\}$ is not a singleton soft set. But $G_E = \{(e_1, \{1\}), (e_2, \{2\})\}$ is a singleton soft set.

Definition (1.4.13) [5]: Let $\mathbb{R}$ be the set of real numbers, $B(\mathbb{R})$ be the collection of all non-empty bounded subsets of $\mathbb{R}$ and E taken as a set of parameters. Then a function $F: E \rightarrow B(\mathbb{R})$ is called a soft real set. If a soft real set is a singleton soft set, it will be said a soft real number and denoted by $\tilde{r}, \tilde{s}, \tilde{t}$ etc. $\check{0}$ and $\check{1}$ are the soft real numbers where $\check{0}(e) = 0$, $\check{1}(e) = 1$ for all $e \in E$.

Remark (1.4.14) [14]: The set of all soft real numbers is denoted by $\mathcal{R}(E)$ and the set of all non-negative soft real numbers by $\mathcal{R}^*(E)$.

Definition (1.4.15) [5]: Let $\tilde{r}, \tilde{s}$ two soft real numbers, then, the following statements:

1. $\tilde{r} \preceq \tilde{s}$ if $\tilde{r}(e) \leq \tilde{s}(e)$, for all $e \in E$.
2. $\tilde{r} \succeq \tilde{s}$ if $\tilde{r}(e) \geq \tilde{s}(e)$, for all $e \in E$.
3. $\tilde{r} \prec \tilde{s}$ if $\tilde{r}(e) < \tilde{s}(e)$, for all $e \in E$.
4. $\tilde{r} \succ \tilde{s}$ if $\tilde{r}(e) > \tilde{s}(e)$, for all $e \in E$.

Definition (1.4.16) [22]: The soft set F_E is said to be (finite soft set) if the set of parameters E is finite and $F(e)$ is finite set, $\forall e \in E$.

Remark (1.4.17) [25]: Let $\tilde{r}, \tilde{s}, \tilde{t} \in \mathcal{R}(E)$. Then the soft addition $\tilde{r} \boxplus \tilde{s}$ of $\tilde{r}, \tilde{s}$ and soft scalar multiplication $\tilde{t} \boxtimes \tilde{r}$ of $\tilde{t}$ and $\tilde{r}$ are defined by:

- (i) $(\tilde{r} \boxplus \tilde{s})(e) = \tilde{r}(e) \boxplus \tilde{s}(e)$, for all $e \in E$.
- (ii) $(\tilde{r} \boxminus \tilde{s})(e) = \tilde{r}(e) \boxminus \tilde{s}(e)$, for all $e \in E$.
- (iii) $(\tilde{r} \boxdot \tilde{s})(e) = \tilde{r}(e) \boxdot \tilde{s}(e)$, for all $e \in E$.
- (iv) $(\tilde{r}/\tilde{s})(e) = \tilde{r}(e)/\tilde{s}(e)$, and $\tilde{s}(e) \neq 0$ for all $e \in E$.

Remark (1.4.18)[14]: For two soft real numbers $\tilde{r}, \tilde{s}$, we have:

- (i) If $\tilde{r} \lesssim \tilde{s}$, then $\tilde{r} \boxplus \tilde{t} \lesssim \tilde{s} \boxplus \tilde{t}$; for all $\tilde{t} \in \mathcal{R}(A)$.
- (ii) If $\tilde{r} \lesssim \tilde{s}$, then $\tilde{r} \boxdot \tilde{t} \lesssim \tilde{s} \boxdot \tilde{t}$; for all $\tilde{t} \in \mathcal{R}(A)^*$.

Theorem (1.4.19)[5]: Let F_E, G_E, H_E are soft sets in (F, E) , $\check{x}_e \neq \check{\emptyset}$. Then the following hold:

- (i) $\forall e \in E, (e, \emptyset) \in F_E$.
- (ii) $\check{x}_e \in [F_E \cup G_E]$ iff $\check{x}_e \in F_E \vee \check{x}_e \in G_E$.
- (iii) $\check{x}_e \in [F_E \cap G_E]$ iff $\check{x}_e \in F_E \wedge \check{x}_e \in G_E$.
- (iv) $\check{x}_e \in [F_E \setminus G_E]$ iff $\check{x}_e \in F_E \wedge \check{x}_e \notin G_E$.

Definition (1.4.20) [6]: A soft point (F, E) over X is said to be a soft point if there is exactly one $e \in E$, such that

$F(e) = \{x\}$ for some $x \in X$ and $F(e) = \emptyset$, for all $e \in E \setminus \{e\}$. It will be denoted by $\check{x}_e$.

Theorem (1.4.21) [6]: Every soft set can be expressed as a union of all soft points belonging to it. iff, any set of soft points can be considered a soft set

Definition (1.4.22) [25]: Let (F, E) be a soft set over $\check{X}$. The set soft (F, E) is said to be a soft vector and denoted by $\check{x}_e$ if there is exactly one $e \in E$. Such that $F(e) = \{x\}$ for some $x \in X$ and $F(ei) = \check{\emptyset}$,

$\forall e \in E \setminus \{e\}$ set. The set of all soft vector over $\check{X}$ will be denoted by $SV(\check{X})$.

Definition (1.4.23) [25]: The set $SV(\check{X})$ is called soft vector space.

Theorem (1.4.24)[6]: Every soft set can be expressed as union of all soft points belonging to it. Conversely, any set of soft points can be considering as a soft set.

Definition (1.4.25): [14]: A pair (F, E) is called a fuzzy soft set over (X, E) , if $F: E \rightarrow I^X$ is a function from E into I^X . The collection of all fuzzy soft sets over (X, E) is denoted by $F(X, E)$.

Definition (1.4.26):[12]: A Fuzzy soft set (F, A) over (X, E) is said to be an absolute fuzzy soft set, if for all $e \in E$, $F(e)$ is a fuzzy universal set $\tilde{1}$ over X and is denoted by $\tilde{E}$.

Definition (1.4.27):[12] : A fuzzy soft set (F, E) over (X, E) is said to be a null fuzzy soft set, if for all $e \in E$, $F(e)$ is the null fuzzy set $\tilde{0}$ over X . It is denoted by $\tilde{\emptyset}$.

Definition (1.4.28): [12]: For two fuzzy soft sets (F, A) and (G, B) in $F(X, E)$ we say that $(F, A) \preceq (G, B)$ if $A \subseteq B$ and $F(e)(x) \leq G(e)(x)$.

Definition (1.4.29): [12]: Two fuzzy soft sets (F, A) and (G, B) in $F(X, E)$ are equal if $F \preceq G$ and $G \preceq F$.

Definition (1.4.30) [12]: The different between two fuzzy soft sets (F, E) and (G, E) in $F(X, E)$ is a fuzzy soft set $(F/G, E)$ (say) defined by $(F/G)(e) = F(e)/G(e)$ for each $e \in E$.

$$(F/G)(e): X \rightarrow I, (F/G)(e)(x) = F(e)(x) \check{\neg} (G(e)(x))^c = \min\{F(e)(x), G^c(e)(x)\} \text{ for all } x \in X.$$

Definition (1.4.31) [12]: The complement of a fuzzy soft set (F, E) is a fuzzy soft set (F^c, E) defined by $F^c(e) = 1/F(e)$ for each $e \in E$,

$$(F^c(e))(x) = 1 \ominus F(e)(x) \text{ for all } x \in X.$$

Definition (1.4.32):[10] : Let (F, A) and (G, B) be two fuzzy soft sets in $F(X, E)$ with $A \check{\cap} B \neq \Phi$,

a) An intersection $(F \check{\cap} G, C)$ is a fuzzy soft set, where $C = A \check{\cap} B$ and $(F \check{\cap} G)(e) = F(e) \check{\cap} G(e)$ for each $e \in C$, $(F \check{\cap} G)(e)(x) = \min\{(e, F_e(x)), (e, G_e(x))\}$

b) An union $(F \check{\cup} G, C)$ is a fuzzy soft set, where

$C = A \check{\cup} B$ and $(F \check{\cup} G)e = F(e) \check{\cup} G(e)$ for each $e \in C$

$(F \check{\cup} G)(e)(x) = \max\{F(e)(x), G(e)(x)\}.$

1.5 Banach algebra

A complete normed space is called a Banach space.

Definition(1.5.1): A complete normed algebra is called a Banach algebra.

That is. A nonempty set X is called a Banach algebra over a field if

1. X is an algebra
2. $(X, \|\cdot\|)$ is an algebra norm on X
3. $\|\mathbf{x} \cdot \mathbf{y}\| \leq \|\mathbf{x}\| \|\mathbf{y}\|, x, y \in X$
4. if X has identity, then $\|\mathbf{e}\| = 1$
5. X is complete metric space induced by its norm.

Definition(1.5.2)[7] : A soft normed space $(X, \|\cdot\|)$ is said that X is complete if every Cauchy sequence $\{\check{x}_N\}$ in X converges to a soft element in X . A complete soft normed space is called a soft Banach space.

Definition (1.5.3): Let X be a vector space over a field $K(K = R)$ and the parameter set E be the real number set R . Let (F, E) be a soft set over X . The soft set (F, E) is said to be a soft vector and denoted by $\check{x}_e$ if there is exactly one $e \in E$, such that $F(e) = \{x\}$ for some $x \in X$ and $F(e') = \varphi, \forall e' \in E/\{e\}.$

The set of all soft Banach algebra over $\tilde{X}$ will be denoted by $SV(\tilde{X})$.

The set $SV(\tilde{X})$ is called a soft Banach algebra.

Definition (1.5.4)[7]: Let $SV(\tilde{X})$ be a soft Vector space.

The mapping $\|\cdot\|: SV(\tilde{X}) \rightarrow R^+(E) \cup \{\tilde{0}\}$

is called a soft norm $(\tilde{X})$, if it satisfies the following conditions

- (i) $\|\tilde{x}_e\| \geq \tilde{0}$ for all $\tilde{x}_e \in SV(x)$ and $\|\tilde{x}_e\| = \tilde{0} \leftrightarrow \tilde{x}_e = \tilde{\theta}_0$.
- ii) $\|\tilde{r} \cdot \tilde{x}_e\| = |\tilde{r}| \|\tilde{x}_e\|$ for all $\tilde{x}_e \in SV(\tilde{X})$ for every soft scalar $\tilde{r}$.
- iii) $\|\tilde{x}_e + \tilde{y}_e\| \leq \|\tilde{x}_e\| + \|\tilde{y}_e\|$ for all $\tilde{x}_e, \tilde{y}_e \in SV(\tilde{X})$.

The soft vector space $\tilde{x}_e \in SV(\tilde{X})$ with a soft norm $\|\cdot\|$ on $\tilde{X}$ is said to be a soft normed linear space and is denoted by $(\tilde{X}, \|\cdot\|)$.

Definition(1.5.5)[27] : $(X, N, *)$ is called fuzzy Banach algebra space if satisfies:

- (i) X is an algebra.
- (ii) $(X, N, *)$ is fuzzy linear space.
- (iii) $N(ab, tk) \geq N(a, t) * N(b, k)$, for all $a, b \in X$ and $t, k \geq 0$. If $(X, N, *)$ is fuzzy complete space, then $(X, N, *)$ is fuzzy Banach algebra.

Definition (1.5.6): Let X be an algebra over F , Γ_A on X is called Fuzzy Soft Banach Algebra if satisfies that:

- (i) Fuzzy soft Banach space.
- (ii) Fuzzy soft algebra.

(iii) $N(a, b, t, k) \geq \min\{N(a, t), N(b, k)\}$ for all $a, b \in FS$, for all $t, k \in R \setminus \{0\}$ $E^* \wedge^* (E)$.

(iv) If $e \in \Gamma_x$ then $N(e, t) = 1$.

Definition (1.5.7): Let $\tilde{X}$ be an absolute soft Banach algebra over the scalar field K . Suppose $*$ is a continuous t-norm, $R(A^*)$ is the set of all non-negative soft real numbers and $SSP(\tilde{X})$ denote the set of all soft points on $\tilde{X}$. A fuzzy subset Γ on $SSP(\tilde{X}) \times R(A^*)$ is called a fuzzy soft norm $\tilde{X}$ on if and only if for $\tilde{x}_e, \tilde{y}_e \in SSP(\tilde{X})$ and $\tilde{k} \in \tilde{K}$ (where $\tilde{k}$ is a soft scalar) the following conditions hold

1) $\Gamma(\tilde{x}_e, \tilde{t}) = 0$, with $\tilde{t} \leq \tilde{0}$.

2) $\Gamma(\tilde{x}_e, \tilde{t}) = 1, \forall \tilde{t} \in R(A^*)$ and only if $\tilde{x}_e = \tilde{\theta}_0$.

3) $\Gamma(\tilde{k}\tilde{x}_e, \tilde{t}) = \Gamma(\tilde{x}_e, \frac{\tilde{t}}{|\tilde{k}|})$ if $\tilde{k} \neq \tilde{0} \forall \tilde{t} \in R(A^*)$.

4) $\Gamma(\tilde{x}_e \oplus \tilde{y}_e, \tilde{t} \oplus \tilde{s}) > \Gamma(\tilde{x}_e, \tilde{t}) * \Gamma(\tilde{y}_e, \tilde{s}), \forall \tilde{t}, \tilde{s} \in R(A^*), \tilde{x}_e, \tilde{y}_e \in SSP(\tilde{X})$.

5) $\Gamma(\tilde{x}_e, \cdot)$ is a continuous non decreasing function of $R(A^*)$ and $\lim_{\tilde{x} \rightarrow \infty} \Gamma(\tilde{x}_e, \tilde{t}) = 1$.

The triple $(\tilde{X}, \Gamma, *)$ will be referred to as a fuzzy soft Banach algebra.

Definition (1.5.8)[17]: Let $SV(\tilde{X})$ be a soft Vector space.

The mapping $\|\cdot\|: SV(\tilde{X}) \rightarrow R^+(E)$

is called a soft norm $(\tilde{X})$, if it satisfies the following conditions

(i) $\|\tilde{x}_e\| \geq \tilde{0}$ for all $\tilde{x}_e \in SV(x)$ and $\|\tilde{x}_e\| = \tilde{0} \leftrightarrow \tilde{x}_e = \tilde{\theta}_0$.

ii) $\|\tilde{r} \cdot \tilde{x}_e\| = |\tilde{r}| \|\tilde{x}_e\|$ for all $\tilde{x}_e \in SV(\tilde{X})$ for every soft scalar $\tilde{r}$.

iii) $\|\tilde{x}_e + \tilde{y}_e\| \leq \|\tilde{x}_e\| + \|\tilde{y}_e\|$ for all $\tilde{x}_e, \tilde{y}_e \in SV(\tilde{X})$.

The soft vector space $\tilde{x}_e \in SV(\tilde{X})$ with a soft norm $\|\cdot\|$ on $\tilde{X}$ is said to be a soft normed linear space and is denoted by $(\tilde{X}, \|\cdot\|)$.

Chapter Two

The Convergences In Fuzzy Soft Banach algebra

Chapter Two

On Convergences In Fuzzy Soft Banach algebra

2.1 Introduction:

In this chapter, we presented the definition of the soft fuzzy set and some of its properties, and we defined the soft fuzzy Banach algebra, and we studied the convergence in the soft fuzzy Banach algebra and some types of convergences in this space, such as s-convergence and st-convergence, and results appeared in the convergence.

2.2 Fuzzy Soft Banach algebra

Definition (2.2.1): Let X be a Banach algebra over a field $K(K = R)$ and the parameter set E be the real number set R . Let (F, E) be a soft set over X . The soft set (F, E) is said to be a soft vector and denoted by $\tilde{x}_e$ if there is exactly one $e \in E$, such that $F(e) = \{x\}$ for some $x \in X$ and $F(e') = \varphi, \forall e' \in E/\{e\}$.

The set of all soft Banach algebra over $\tilde{X}$ will be denoted by $SV(\tilde{X})$.

The set $SV(\tilde{X})$ is called a soft Banach algebra.

Definition (2.2.2): Let X be Vector space over a field $F(F = R \text{ or } C)$. Let N be a fuzzy subset of $X \times R$. Then N is called a fuzzy norm on X if for all

$x, y \in X$ and $c \in F$,

(N1) for all $t \leq 0, N(x, t) = 0$,

(N2) for all $t > 0$, $N(x, t) = 1$ if and only if $x=0$

(N3) for all $t > 0$ and $k \neq 0$, $N(kx, t) = N(x, \frac{t}{|k|})$

(N4) for all $t, s \in R$, $N(x + y, s + t) \geq \min\{N(x, s), N(y, t)\}$,

(N5) $N(x, t)$ is a non-decreasing function of R such that $\lim_{t \rightarrow \infty} N(x, t) = 1$

The pair $(X, N, \min)$ will be referred to as a fuzzy Banach algebra.

(N6) If $N(x, 0) > 0$ for all $t > 0$, then $x = 0$.

For a fuzzy Banach algebra (X, N) with (N6), define a function $\|\cdot\|_\alpha$ on X by

$$\|x\|_\alpha = \wedge\{t : N(x, t) \geq \alpha\}$$

for $\alpha \in (0, 1)$. Then $\|\cdot\|_\alpha$ is a norm on X [1]. We call these norms as α -norms on X corresponding to the fuzzy norm N on X .

(N7) For $\alpha \in (0, 1)$. Then $\|\cdot\|_\alpha$ is a continuous function of R and strictly increasing on $\{t : 0 < N(x, t) < 1\}$. $t \in R$.

2.3 Relations of Convergences in Fuzzy Soft Banach algebra Introduced by Bag and Samanta

In Definition 2.2.2, it is stated that $N(x, \cdot)$ is non-decreasing on R . But the following inequality from (N4) shows that $N(x, \cdot)$ is non-decreasing:

$$N(x, s) > \min\{N(0, s - t), N(x, t)\} = \min\{1, N(x, t)\} = N(x, t), \text{ for } s > t.$$

We now introduce the following definitions:

Definition (2.3.1): Let $\{x_n\}$ be a sequence in a fuzzy soft Banach algebra $(\tilde{X}, \Gamma, *)$ and $x \in X$. Then

(1) $\{x_n\}$ is said to be convergent to x if, for all $\varepsilon \in (0,1)$ and $t > 0$, there exists $n_{\varepsilon,t} \in N$ depending on ε and t , such that $N(x_n - x, t) > 1 - \varepsilon$, for $n \geq n_{\varepsilon,t}$.

(2) $\{x_n\}$ is said to be s-convergent to x if, for all $\varepsilon \in (0,1)$, there exists $n_\varepsilon \in N$ such that $N\left(\left|x_n - x, \frac{1}{n}\right|\right) > 1 - \varepsilon$, for $n \geq n_\varepsilon$.

(3) $\{x_n\}$ is said to be st-convergent to x if, for all $\varepsilon \in (0,1)$, there exists $n_\varepsilon \in N$ such that $N(|x_n - x, t|) > 1 - \varepsilon$, for $n \geq n_\varepsilon$, and for all $t > 0$.

We now investigate relations of convergence, s-convergence and st-convergence in a fuzzy soft Banach algebra introduced by Bag and Samanta.

Proposition (2.3.2): Let $(\tilde{X}, \Gamma, *)$ be a fuzzy soft Banach algebra. Then

(1) st-convergence in $(\tilde{X}, \Gamma, *)$ implies s-convergence.

(2) s-convergence in $(\tilde{X}, \Gamma, *)$ implies convergence.

Proof. It is easy to prove (1).

(2) Suppose that $\{x_n\}$ is s-convergent sequence to x in $(\tilde{X}, \Gamma, *)$. Let $\varepsilon \in (0,1)$ and $t > 0$. Then there exists $n_\varepsilon \in N$ such that if $n \geq n_\varepsilon$, $N\left(\left|x_n - x, \frac{1}{n}\right|\right) > 1 - \varepsilon$. Choose $n_t \in \cdot$ such that $\frac{1}{n_t} < t$. Let $n_1 = \max\{n_t, n_\varepsilon\}$. Then if $n \geq n_1$, $N(|x_n - x, t|) > N\left(\left|x_n - x, \frac{1}{n}\right|\right) > 1 - \varepsilon$, since $n \geq n_1 \geq n_\varepsilon$.

The converses of Proposition 2.3.2 do not hold. The following example is the norm form of [Previous Example].

Example (2.3.3): Let $X = R$ and we define a fuzzy soft Banach algebra $N(x, t)$ on X by

$$N(x, t) = \begin{cases} \frac{t}{t + |x|} & \text{if } t > 0 \\ 0 & \text{if } t \leq 0. \end{cases}$$

It is easy to see that $(\tilde{X}, \Gamma, *)$ is a fuzzy soft Banach algebra

Let $x_n = \frac{1}{n}$. Then

$$N\left(\left|x_n, \frac{1}{n}\right|\right) = \frac{\frac{1}{n}}{\frac{1}{n} + \frac{1}{n}} = \frac{1}{2}.$$

This implies that $\{x_n\}$ is not s-convergent to 0. We now show that $\{x_n\}$ is a convergent sequence in $(\tilde{X}, \Gamma, *)$. Let $\varepsilon \in (0, 1)$ and $t > 0$. Then there exists $n_{\varepsilon, t} \in \mathbb{N}$ such that $n_{\varepsilon, t} > \frac{1-\varepsilon}{t\varepsilon}$. If $n \geq n_{\varepsilon, t}$, $N(x, t) = \frac{t}{t + \frac{1}{n}} \geq \frac{t}{t + \frac{1}{n_{\varepsilon, t}}} > 1 - \varepsilon$.

This means that $\{x_n\}$ is convergent to 0. Therefore, $\{x_n\}$ is a convergent and non-s-convergent sequence in a fuzzy soft Banach algebra.

The converse of (2) in Proposition 2. 3.2 does not hold.

Let $y_n = \frac{1}{n^2}$. Then

$$N\left(\left|y_n, \frac{1}{n}\right|\right) = \frac{\frac{1}{n}}{\frac{1}{n} + \frac{1}{n^2}} = \frac{1}{1 + \frac{1}{n}}.$$

This implies that $\{y_n\}$ is s-convergent to 0. We now show that $\{y_n\}$ is not st-convergent. Suppose that $\{y_n\}$ is st-convergent. Let $\varepsilon \in (0, 1)$. Then there exists $n_\varepsilon \in \mathbb{N}$ such that if $n \geq n_\varepsilon$, $N\left(\left|y_n, \frac{1}{n}\right|\right) = \frac{t}{t + \frac{1}{n^2}} > 1 - \varepsilon$ for all $t > 0$.

This implies that for all $t > 0$, $t > \frac{1-\varepsilon}{\varepsilon n_\varepsilon^2}$. We get the contradiction.

Therefore, $\{y_n\}$ is the s-convergent sequence which is not st-convergent.

The converses of (1) in Proposition 3.2 do not hold.

By Proposition 2.3.2 and Example 2.3.3, we get the following strict implications in the fuzzy soft Banach algebra introduced by Bag and Samanta:

$$\text{st-convergence} \Rightarrow \text{s-convergence} \Rightarrow \text{convergence}.$$

2.4 Convergences in Fuzzy Soft Banach algebra and Convergences with Respect to their Corresponding α -norms

In [27], the authors showed that in a fuzzy soft Banach algebra satisfying (N6) and (N7), every sequence is convergent if and only if it is convergent with respect to its corresponding α -norms ($0 < \alpha < 1$). We now show that in a fuzzy soft Banach algebra satisfying (N6), every sequence is convergent if and only if it is convergent with respect to its corresponding α -norms ($0 < \alpha < 1$).

Proposition (2.4.1): Let $\{x_n\}$ be a sequence in a fuzzy soft Banach algebra $(\tilde{X}, I, *)$ satisfying (N6) and let $x \in X$. $\{x_n\}$ converges to x if and only if for all $\alpha \in (0, 1)$ and $\varepsilon \in (0, 1)$ there exists $n_{\alpha, \varepsilon} \in \mathbb{N}$ such that $n \geq n_{\alpha, \varepsilon}$, $\|x_n - x\|_\alpha < \varepsilon$.

Proof. Suppose that $x \in X$. $\{x_n\}$ converges to x . Let $\alpha \in (0, 1)$. Then there exists $\delta \in (0, 1)$ such that $\alpha = 1 - \delta$. Since $\{x_n\}$ converges to x , for $\varepsilon > 0$, there exists $n_{\alpha, \varepsilon} \in \mathbb{N}$, such that if $n \geq n_{\alpha, \varepsilon}$, $N\left(x_n - x, \frac{\varepsilon}{2}\right) > 1 - \delta = \alpha$. This implies that $\|x_n - x\|_\alpha \leq \frac{\varepsilon}{2} < \varepsilon$ for $n \geq n_{\alpha, \varepsilon}$.

Conversely, let $\alpha \in (0, 1)$ and $\varepsilon > 0$. Then there exists $n_{\alpha, \varepsilon} \in \mathbb{N}$, such that $n \geq n_{\alpha, \varepsilon}$, $\|x_n - x\|_\alpha < \varepsilon$. This implies that $N(x_n - x, \varepsilon) \geq \alpha$ for $n \geq n_{\alpha, \varepsilon}$.

since $N(x, \cdot)$ is non-decreasing. Since $\alpha \in (0, 1)$ is arbitrary, x_n converges to x $(\tilde{X}, \Gamma, *)$.

We now consider the representation of st-convergence in fuzzy soft Banach algebra in terms of convergence with respect to its corresponding α -norms.

Theorem (2.4.2): Let $\{x_n\}$ be a sequence in a fuzzy soft Banach algebra $(\tilde{X}, \Gamma, *)$ satisfying (N6) and $x \in X$. $\{x_n\}$ st-converges to x if and only if for all $\alpha \in (0, 1)$ there exists $n_\alpha \in N$ such that $\varepsilon > 0$, $\|x_n - x\|_\alpha < \varepsilon$ for $n \geq n_\alpha$, i.e., $x_n = x$ for $n \geq n_\alpha$.

Proof. Suppose that $\{x_n\}$ st-converges to x . Let $\alpha \in (0, 1)$ Then there exists $\delta \in (0, 1)$ such that $\alpha = 1 - \delta$. Since $\{x_n\}$ st-converges to x , there exists $n_\delta \in N$ such that if $n \geq n_\delta$, $N(x_n - x, t) > 1 - \delta = \alpha$ for all $t > 0$. Let $\varepsilon > 0$. Then $N\left(x_n - x, \frac{\varepsilon}{2}\right) > \alpha$ for $n \geq n_\delta$. This implies that $\|x_n - x\|_\alpha \leq \frac{\varepsilon}{2} < \varepsilon$ for $n \geq n_\delta$.

The converse is clear.

We now investigate the representation of s-convergence in fuzzy soft Banach algebra in terms of convergence with respect to its corresponding α -norms.

Theorem (2.4.3): Let $\{x_n\}$ be a sequence in a fuzzy soft Banach algebra $(\tilde{X}, \Gamma, *)$ satisfying (N6) and let $x \in X$. $\{x_n\}$ s-converges to x if and only if for all $\alpha \in (0, 1)$ there exists $n_\alpha \in N$ such that if $n \geq n_\alpha$, $\|x_n - x\|_\alpha \leq \frac{1}{n}$.

Proof. Suppose that $x \in X$. $\{x_n\}$ s-converges to x . Let $\alpha \in (0, 1)$. Then there exists $n_\alpha \in \mathbb{N}$ such that $n \geq n_\alpha$, $N\left(\left|x_n - x, \frac{1}{n}\right|\right) > \alpha$. This implies that $\|x_n - x\|_\alpha < \frac{1}{n}$ for $n \geq n_{\alpha, \varepsilon}$.

Conversely, let $0 < \varepsilon < 1$. Then $0 < \alpha = 1 - \varepsilon < 1$ and so there exists $n_\varepsilon \in \mathbb{N}$ such that if $n \geq n_\varepsilon$, $\|x_n - x\|_\alpha \leq \frac{1}{n}$. This implies that $N\left(\left|x_n - x, \frac{1}{n}\right|\right) > \alpha = 1 - \varepsilon$, for $n \geq n_\varepsilon$. This means that $\{x_n\}$ s-converges to x .

2.5 Relations of Convergences in Fuzzy Soft Banach algebra Introduced by Felbin

We now introduce the following definitions:

Definition (2.5.1): Let $\{x_n\}$ be a sequence in a fuzzy soft Banach algebra $(\tilde{X}, \Gamma, *)$ introduced by Felbin and $x \in X$. Then

(1) $\{x_n\}$ is said to be convergent to x if, for all $\varepsilon > 0$ and $\alpha \in (0, 1]$, there exists $n_{\alpha, \varepsilon} \in \mathbb{N}$ depending on ε and α , such that $\|x_n - x\|_\alpha^2 < \varepsilon$, for $n \geq n_{\alpha, \varepsilon}$.

(2) $\{x_n\}$ is said to be s-convergent to x if, for all $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that $\|x_n - x\|_{\frac{1}{n}}^2 < \varepsilon$, for $n \geq n_\varepsilon$.

(3) $\{x_n\}$ is said to be st-convergent to x if, for all $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that $\|x_n - x\|_\alpha^2 < \varepsilon$, for $n \geq n_\varepsilon$, and for all $\alpha \in (0, 1]$.

We finally investigate relations of convergence, s-convergence and st-convergence in a fuzzy soft Banach algebra introduced by Felbin.

Proposition (2.5.2): Let $(\tilde{X}, \Gamma, *)$ be a fuzzy soft Banach algebra introduced by Felbin. Then

(1) st-convergence in $(\tilde{X}, \Gamma, *)$ implies s-convergence.

(2) s-convergence in $(\tilde{X}, \Gamma, *)$ implies convergence.

Proof. It is easy to prove (1).

(2) Suppose that $\{x_n\}$ is an s-convergent sequence to x in $(\tilde{X}, \Gamma, *)$. Let $\varepsilon > 0$. Then there exists $n_\varepsilon \in N$ such that if $n \geq n_\varepsilon$, $\|x_n - x\|_{\frac{1}{n}}^2 < \varepsilon$. Let $\alpha \in (0, 1]$. Then there exists $n_\alpha \in N$ such that $\alpha \geq \frac{1}{n_\alpha}$. Let $n_{\varepsilon, \alpha} = \max\{n_\varepsilon, n_\alpha\}$. Then if $n \geq n_{\varepsilon, \alpha}$,

$$\|x_n - x\|_\alpha^2 < \|x_n - x\|_{\frac{1}{n_\alpha}}^2 < \|x_n - x\|_{\frac{1}{n}}^2 < \varepsilon,$$

Since $n \geq n_\varepsilon, \alpha \geq \frac{1}{n_\alpha}$. This means that (x) converges to x in $(\tilde{X}, \Gamma, *)$.

The converses of Proposition 2.5.2 do not hold. The following example is 2

Example (2.5.3): Let $X = R$ and we define a fuzzy soft norm $\|\cdot\|$ on X by

$$\|x\| = \begin{cases} \frac{|x|}{t} & \text{if } |x| \leq t, x \neq 0, \\ 1 & \text{if } t = |x| = 0, \\ 0 & \text{otherwise} \end{cases}$$

It is easy to see that $(\tilde{X}, \Gamma, *)$ is a fuzzy soft Banach algebra and α -level set of is $(\tilde{X}, \Gamma, *)$ given by

$$[\|x\|]_\alpha = \left[|x|, \frac{|x|}{\alpha} \right].$$

Let $x_n = \frac{1}{n}$. Let $\varepsilon > 0$ and $\alpha \in (0, 1]$. Then there exists $n_{\alpha, \varepsilon} \in \cdot$ such that $n_{\alpha, \varepsilon} > \frac{1}{\alpha\varepsilon}$. If $n \geq n_{\alpha, \varepsilon}$, $\|x_n\|_\alpha^2 = \left\| \frac{1}{n} \right\|_\alpha^2 = \frac{1}{n\alpha} \leq \frac{1}{n_{\alpha, \varepsilon}\alpha} < \varepsilon$.

This means that $\{x_n\}$ is convergent in $(\tilde{X}, \Gamma, *)$. We now show that $\{x_n\}$ is not s-convergent in $(\tilde{X}, \Gamma, *)$. Since $\|x_n\|_{\frac{1}{n}}^2 = 1$, $\{x_n\}$ is not s-convergent in $(\tilde{X}, \Gamma, *)$.

Let $y_n = \frac{1}{n^2}$. Let $\varepsilon > 0$. Then there exists $n_\varepsilon \in \cdot$ such that $\frac{1}{n_\varepsilon} < \varepsilon$. If $n \geq n_\varepsilon$, $\|y_n\|_\alpha^2 = \left\| \frac{1}{n} \right\|_\alpha^2 = \frac{1}{n} \leq \frac{1}{n_\varepsilon} < \varepsilon$. This means that (y_n) is s-convergent in $(\tilde{X}, \Gamma, *)$. We now show that (y_n) is not st-convergent in $(\tilde{X}, \Gamma, *)$. Suppose that (y_n) is st-convergent in $(\tilde{X}, \Gamma, *)$.

Let $\varepsilon > 0$. Then there exists $n_\varepsilon \in \cdot$ such that if $n \geq n_\varepsilon$, $\|y_n\|_\alpha^2 = \frac{1}{n^2\alpha} < \varepsilon$ for all $\alpha \in (0, 1]$. This implies that $\alpha > \frac{1}{\varepsilon n_\varepsilon^2}$ for all $\alpha \in (0, 1]$. We get the contradiction. Therefore, $\{y_n\}$ is the s-convergent sequence which is not st-convergent in $(\tilde{X}, \Gamma, *)$. The converses of Proposition 2.5.2 do not hold.

By Proposition 2.5.2, and Example 2.5.3, we get the following strict implication in the fuzzy soft Banach algebra introduced by Felbin:

$$\text{st-convergence} \Rightarrow \text{s-convergence} \Rightarrow \text{convergence}.$$

Chapter Three

**On fuzzy Soft Dual Space
of fuzzy Soft Banach
algebra**

Chapter Three

On fuzzy Soft Dual Space of fuzzy Soft Banach algebra

3.1 Introduction:

In this chapter, the fuzzy soft set was introduced and we presented the definition of the fuzzy soft Banach algebra and the metric space and some of its properties. We introduced the fuzzy soft double space and the fuzzy soft double operator and studied their properties. We stated and proved the theorem about representing a fuzzy soft linear function by the inner product in fuzzy soft Hilbert spaces.

3.2 Properties fuzzy soft Banach algebra

Let X be a vector space over a field F ($F = \mathbb{R}$) and the parameter set E be the real number set $\mathbb{R}$.

Proposition (3.2.1) Every fuzzy soft set can be expressed as a union of all fuzzy soft points belonging to it. Conversely, any set of fuzzy soft points can be considered as a fuzzy soft set.

Let $SSP(\tilde{X})$ be the collection of all fuzzy soft points of $\tilde{X}$ and $\mathbb{R}(A)^*$ denote the set of all non-negative fuzzy soft real numbers.

Definition (3.2.2) A mapping $\tilde{d}: SSP(\tilde{X}) \times SSP(\tilde{X}) \rightarrow \mathbb{R}(A)^*$ is said to be a fuzzy soft metric on the fuzzy soft set $\tilde{X}$ if $\tilde{d}$ satisfies the following conditions:

- (M1) $\tilde{d}(\tilde{x}_{e_1}, \tilde{y}_{e_2}) \geq \tilde{0}$ for all $\tilde{x}_{e_1}, \tilde{y}_{e_2} \in \tilde{X}$,
 (M2) $\tilde{d}(\tilde{x}_{e_1}, \tilde{y}_{e_2}) = \tilde{0}$ if and only if $\tilde{x}_{e_1} = \tilde{y}_{e_2} \in \tilde{X}$,
 (M3) $\tilde{d}(\tilde{x}_{e_1}, \tilde{y}_{e_2}) = \tilde{d}(\tilde{y}_{e_2}, \tilde{x}_{e_1})$ for all $\tilde{x}_{e_1}, \tilde{y}_{e_2} \in \tilde{X}$,
 (M4) For all $\tilde{x}_{e_1}, \tilde{y}_{e_2}, \tilde{z}_{e_3} \in \tilde{X}$, $\tilde{d}(\tilde{x}_{e_1}, \tilde{z}_{e_3}) \leq \tilde{d}(\tilde{x}_{e_1}, \tilde{y}_{e_2}) + \tilde{d}(\tilde{y}_{e_2}, \tilde{z}_{e_3})$.

The fuzzy soft set $\tilde{X}$ with a fuzzy soft metric $\tilde{d}$ is called a fuzzy soft metric space and denoted by $(\tilde{X}, \tilde{d}, \Gamma_A)$.

Proposition (3.2.3) The set $SSV(\tilde{X})$ is a vector space according to the following operations;

- (1) $\tilde{x}_e + \tilde{y}_{e'} = \widetilde{(x + y)}_{(e + e')}$ for every $\tilde{x}_e, \tilde{y}_{e'} \in SV(\tilde{X})$;
 (2) $\tilde{r} \cdot \tilde{x}_e = (\widetilde{r\tilde{x}})_{(re)}$ for every $\tilde{x}_e \in SV(\tilde{X})$ and for every fuzzy soft real number $\tilde{r}$.

Definition (3.2.4) Let $SSV(\tilde{X})$ be a fuzzy soft Banach algebra and $\tilde{M} \subset SSV(\tilde{X})$ be a subset. If $\tilde{M}$ is a fuzzy soft Banach algebra, then $\tilde{M}$ is said to be a fuzzy soft Banach algebra subspace of $SV(\tilde{X})$ and denoted by $SSV(\tilde{M}) \subset SSV(\tilde{X})$.

Definition (3.2.5) A sequence of fuzzy soft Banach algebra $\{\tilde{x}_{e_n}^n\}$ in $(\tilde{X}, \Gamma, *)$ is said to be convergent to $\tilde{x}_{e_0}^0$, if $\lim_{n \rightarrow \infty} \|\tilde{x}_{e_n}^n - \tilde{x}_{e_0}^0\| = \tilde{0}$ and denoted by $\tilde{x}_{e_n}^n \rightarrow \tilde{x}_{e_0}^0$ as $n \rightarrow \infty$.

Definition (3.2.6) A sequence of fuzzy soft Banach algebra $\{\tilde{x}_{e_n}^n\}$ in $(\tilde{X}, \Gamma, *)$ is said to be a fuzzy soft Cauchy sequence if corresponding to every $\tilde{\varepsilon} \succ \tilde{0}$, $\exists m \in N$ such that

$$\|\tilde{x}_{e_i}^i - \tilde{x}_{e_j}^j\| \prec \tilde{\varepsilon}, \forall i, j \geq m \text{ that is } \|\tilde{x}_{e_i}^i - \tilde{x}_{e_j}^j\| \rightarrow \tilde{0} \text{ as } i, j \rightarrow \infty.$$

Proposition (3.2.7) Every fuzzy soft convergent sequence is a fuzzy soft Cauchy sequence.

Definition (3.2.8) Let $(\tilde{X}, \Gamma, *)$ be a fuzzy soft Banach algebra. Then $(\tilde{X}, \Gamma, *)$ is said to be complete if every fuzzy soft Cauchy sequence in $\tilde{X}$ converges to a soft vector of $\tilde{X}$.

Definition (3.2.9) Every fuzzy soft complete fuzzy soft normed linear space is called a fuzzy soft Banach algebra.

Proposition (3.2.10) Every fuzzy soft normed space is a fuzzy soft metric space.

Definition (3.2.11) Let $T : SSV(\tilde{X}) \rightarrow SSV(\tilde{Y})$ be a fuzzy soft mapping. Then T is said to be fuzzy soft linear operator if

(L1). T is additive, that is $T(\tilde{x}_e, \tilde{y}_{e'}) = T(\tilde{x}_e) + T(\tilde{y}_{e'})$ for every $\tilde{x}_e, \tilde{y}_{e'} \in SSV(\tilde{X})$.

(L2). T is homogeneous, that is, for every fuzzy soft scalar $\tilde{r}$, $T(\tilde{r}\tilde{x}_e) = \tilde{r}.T(\tilde{x}_e)$ for every $\tilde{x}_e \in SSV(\tilde{X})$.

Definition (3.2.12) The fuzzy soft operator $T : SSV(\tilde{X}) \rightarrow SSV(\tilde{Y})$ is said to be fuzzy soft continuous at $\tilde{x}_e \in SSV(\tilde{X})$ if for every fuzzy soft sequence $\{\tilde{x}_{e_n}^n\}$ of fuzzy soft vectors of $\tilde{X}$ with $\tilde{x}_{e_n}^n \rightarrow \tilde{x}_{e_0}^0$ as $n \rightarrow \infty$, we have $T(\tilde{x}_{e_n}^n) \rightarrow T(\tilde{x}_{e_0}^0)$ as $n \rightarrow \infty$. If T is soft continuous at each fuzzy soft Banach algebra of $SSV(\tilde{X})$, then T is said to be fuzzy soft continuous operator.

Definition (3.2.13) The fuzzy soft operator $T : SSV(\tilde{X}) \rightarrow SSV(\tilde{Y})$ is said to be fuzzy soft bounded, if there exists a fuzzy soft real number $\tilde{M}$ such that

$$\|T(\tilde{x}_e)\| \lesssim \tilde{M}\|\tilde{x}_e\|, \text{ for all } \tilde{x}_e \in SSV(\tilde{X}).$$

Theorem (3.2.14) The fuzzy soft operator $T : SSV(\tilde{X}) \rightarrow SSV(\tilde{Y})$ is fuzzy soft continuous if and only if it is fuzzy soft bounded.

Definition (3.2.15) Let $T : SSV(\tilde{X}) \rightarrow SSV(\tilde{Y})$ be a fuzzy soft continuous operator .

$$\|T\| = \inf\{\tilde{M} : \|T(\tilde{x}_e)\| \lesssim \tilde{M}\|\tilde{x}_e\|\}$$

is said to be a norm of T .

It is obvious that $\|T(\tilde{x}_e)\| \lesssim \|T\|\|\tilde{x}_e\|$.

Theorem (3.2.16) Let $T : SSV(\tilde{X}) \rightarrow SSV(\tilde{Y})$ be a fuzzy soft operator. Then,

$$\|T\| = \sup_{\tilde{x}_e \neq \tilde{\theta}_0} \frac{\|T(\tilde{x}_e)\|}{\|\tilde{x}_e\|} = \sup_{\|\tilde{x}_e\| \lesssim 1} \|T(\tilde{x}_e)\|$$

Theorem (3.2.17) Let $T : SSV(\tilde{X}) \rightarrow SSV(\tilde{Y})$ be a fuzzy soft operator then $\|T\|$ is a fuzzy soft norm.

Theorem (3.2.18) Let $T : SSV(\tilde{X}) \rightarrow SSV(\tilde{Y})$ and $S : SSV(\tilde{Y}) \rightarrow SSV(\tilde{Z})$ be two soft operators. Then

a) $\|SoT\| \lesssim \|S\| \|T\|$,

b) If $T : SSV(\tilde{X}) \rightarrow SSV(\tilde{Y})$ is a fuzzy soft operator, then

$$\|T^n\| \lesssim \|T\|^n.$$

is satisfied.

Definition (3.2.19) Let $SSV(\tilde{X})$ be a fuzzy soft Banach algebra. The mapping

$$\langle \cdot \rangle : SSV(\tilde{X}) \rightarrow SSV(\tilde{Y}) \rightarrow \mathbb{R}(A)$$

(is called a fuzzy soft inner product on $SSV(\tilde{X})$ iff it satisfies the following conditions, for every $\tilde{x}_e, \tilde{y}_{e'}, \tilde{z}_{e''} \in SSV(\tilde{X})$ and for every fuzzy soft real number $\tilde{\alpha}$ where $\mathbb{R}(A)$ is the fuzzy soft real number set:

I1. $\langle \tilde{x}_e, \tilde{x}_e \rangle \gtrsim \tilde{0}$ and $\langle \tilde{x}_e, \tilde{x}_e \rangle = \tilde{0} \Leftrightarrow \tilde{x}_e = \tilde{\theta}_0$,

I2. $\langle \tilde{x}_e, \tilde{y}_{e'} \rangle = \langle \tilde{y}_{e'}, \tilde{x}_e \rangle$,

I3. $\langle \tilde{\alpha} \tilde{x}_e, \tilde{y}_{e'} \rangle = \langle \tilde{x}_e, \tilde{\alpha} \tilde{y}_{e'} \rangle = \tilde{\alpha} \langle \tilde{x}_e, \tilde{y}_{e'} \rangle$,

I4. $\langle \tilde{x}_e + \tilde{y}_{e'}, \tilde{z}_{e''} \rangle = \langle \tilde{x}_e, \tilde{z}_{e''} \rangle + \langle \tilde{y}_{e'}, \tilde{z}_{e''} \rangle$.

The triple $(SSV(\tilde{X}), \langle \cdot \rangle, \Gamma_A)$ is called fuzzy soft inner product space.

Proposition (3.2.20) (Parallelogram Law) Let $(SSV(\tilde{X}), < ., >, \Gamma_A)$ be a fuzzy soft inner product space. For every $\tilde{x}_e, \tilde{y}_{e'} \in SSV(\tilde{X})$

$$\|\tilde{x}_e + \tilde{y}_{e'}\|^2 + \|\tilde{x}_e - \tilde{y}_{e'}\|^2 = 2(\|\tilde{x}_e\|^2 + \|\tilde{y}_{e'}\|^2).$$

is satisfied.

Theorem (3.2.21) Let $(SSV(\tilde{X}), < ., >, \Gamma_A)$ be a fuzzy soft inner product space. For every $\tilde{x}_e, \tilde{y}_{e'} \in SSV(\tilde{X})$

$$|< \tilde{x}_e, \tilde{y}_{e'} >| \leq \|\tilde{x}_e\| \|\tilde{y}_{e'}\|$$

is hold.

Proposition (3.2.22) A fuzzy soft inner product function is continuous in a fuzzy soft inner product space. In other words, If $\{\tilde{x}_{e_n}^n\} \rightarrow \tilde{x}_e$ and $\{\tilde{y}_{e'_n}^n\} \rightarrow \tilde{y}_{e'}$, then $< \tilde{x}_{e_n}^n, \tilde{y}_{e'_n}^n > \rightarrow < \tilde{x}_e, \tilde{y}_{e'} >.$

Proposition (3.2.23) Let $(SSV(\tilde{X}), < ., >, \Gamma_A)$ be a fuzzy soft inner product space and $\{\tilde{x}_{e_n}^n\}, \{\tilde{y}_{e'_n}^n\}$ be fuzzy soft Cauchy sequences in this space. In this case, $< \tilde{x}_{e_n}^n, \tilde{y}_{e'_n}^n >$ is also a fuzzy soft Cauchy sequence.

Definition (3.2.24) Let $(SSV(\tilde{X}), < ., >, \Gamma_A)$ be a fuzzy soft inner product space If this space is complete according to the induced norm by the fuzzy soft inner product then $(SSV(\tilde{X}), < ., >, \Gamma_A)$ is said to be a fuzzy soft Hilbert space.

3.3 Fuzzy Soft Dual Spaces

Proposition (3.3.1): Given the fuzzy soft Banach algebra $SSV(\tilde{\mathbb{R}}) = \{\tilde{x}_e : x \in \mathbb{R}, e \in \mathbb{R}\}$ and for every $\tilde{x}_e, \tilde{y}_{e'} \in SSV(\tilde{\mathbb{R}})$, let us define the mapping $\langle . \rangle : SSV(\tilde{\mathbb{R}}) \times SSV(\tilde{\mathbb{R}}) \rightarrow \mathbb{R}(A)$ ($A = \mathbb{R}$) as follows

$$\langle \tilde{x}_e, \tilde{y}_{e'} \rangle = x.y + e.e'$$

In this case the mapping $\langle . \rangle$ is an inner product on $SSV(\tilde{\mathbb{R}})$.

Proof. I1. For every $\tilde{x}_e \in SSV(\tilde{\mathbb{R}})$,

$$\langle \tilde{x}_e, \tilde{x}_e \rangle = x.x + e.e = x^2 + e^2 \succeq \tilde{0}$$

$$\langle \tilde{x}_e, \tilde{x}_e \rangle = 0 \Leftrightarrow \tilde{x}_e = 0$$

$$\Leftrightarrow e = 0 \text{ and } x = \theta$$

$$\Leftrightarrow \tilde{x}_e = \tilde{\theta}_0$$

I2. For every $\tilde{x}_e, \tilde{y}_{e'} \in SSV(\tilde{\mathbb{R}})$,

$$\langle \tilde{x}_e, \tilde{y}_{e'} \rangle = x.y + e.e' = y.x + e.e' = \langle \tilde{y}_{e'}, \tilde{x}_e \rangle.$$

I3. For every $\tilde{\alpha} \in \mathbb{R}(A)$ and $\forall \tilde{x}_e, \tilde{y}_{e'} \in SSV(\tilde{\mathbb{R}})$,

$$\langle \tilde{\alpha}\tilde{x}_e, \tilde{y}_{e'} \rangle = \tilde{\alpha} x.y + \tilde{\alpha} e.e' = \tilde{\alpha} y.x + e.(\tilde{\alpha}e') = \langle \tilde{x}_e, \tilde{\alpha}\tilde{y}_{e'} \rangle.$$

I4. For every $\tilde{x}_e, \tilde{y}_{e'}, \tilde{z}_{e''} \in SSV(\tilde{\mathbb{R}})$,

$$\langle \tilde{x}_e + \tilde{y}_{e'}, \tilde{z}_{e''} \rangle = \langle (\widetilde{x+y})_{(e.e')}, \tilde{z}_{e''} \rangle.$$

$$= (x + y).z + (e + e').e''$$

$$= xz + e'.e'' + y.z + e'.e''$$

$$= \langle \tilde{x}_e, \tilde{z}_{e''} \rangle + \langle \tilde{y}_{e'}, \tilde{z}_{e''} \rangle.$$

Thus, $(SSV(\tilde{\mathbb{R}}), \langle . \rangle, \Gamma_A)$ is a fuzzy soft inner product space.

Remark (3.4.2): The induced norm from the fuzzy soft inner product defined in the preceding proposition is

$$\|\tilde{x}_e\| = \sqrt{x^2 + e^2}$$

where $\|\cdot\| : SSV(\tilde{\mathbb{R}}) \times SSV(\tilde{\mathbb{R}}) \rightarrow \mathbb{R}^+(A)(A = R)$.

Also, the induced fuzzy soft metric from the fuzzy soft norm is

$$\tilde{d}(\tilde{x}_e, \tilde{y}_{e'}) = \sqrt{(x - y)^2 + (e - e')^2}$$

where $\tilde{d} : SSV(\tilde{\mathbb{R}}) \times SSV(\tilde{\mathbb{R}}) \rightarrow \mathbb{R}^+(A)(A = R)$.

Proposition (3.3.3): The fuzzy soft inner product space $(SSV(\tilde{\mathbb{R}}), \langle \cdot, \cdot \rangle, A)(A = R)$ is a fuzzy soft Euclidean space.

Proof. Let us take a fuzzy soft Cauchy sequence $\{\tilde{x}_{e_n}^n\}$ in the fuzzy soft metric space $(SSV(\tilde{\mathbb{R}}), \tilde{d})$. In this case, for every fuzzy soft reel number $\tilde{\varepsilon} \gtrsim \tilde{0}$ there exists a $n_0 \in N$ such that for $\forall m, n \geq n_0$

$$\tilde{d}(\tilde{x}_{e_n}^n, \tilde{x}_{e_m}^m) \lesssim \sqrt{(x^n - x^m)^2 + (e_n - e_m)^2} \lesssim \tilde{\varepsilon}.$$

Thus for $\forall m, n \geq n_0$ we have $|x^n - x^m|^2 + |e_n - e_m|^2 \lesssim \tilde{\varepsilon}^2$. Since $|x^n - x^m| \lesssim \tilde{\varepsilon}$ and $|e_n - e_m| \lesssim \tilde{\varepsilon}$ $\{x^n\}$ and $\{e_n\}$ are Cauchy sequence in $\mathbb{R}$. Since $\mathbb{R}$ is a complete space there exist real numbers x and e such that $\{x_n\} \rightarrow x$ and $\{e_n\} \rightarrow e$. If we take the real number e as a parameter of the real number x then we have the fuzzy soft sequence $\tilde{x}_e$ and we obtain $\{\tilde{x}_{e_n}^n\} \rightarrow \tilde{x}_e$. Finally, since $x \in \mathbb{R}$ and $e \in \mathbb{R}$ we have $\tilde{x}_e \in SSV(\tilde{\mathbb{R}})$.

Since the fuzzy soft Cauchy sequence $\{\tilde{x}_{e_n}^n\}$ is arbitrary the fuzzy soft space $SSV(\tilde{\mathbb{R}})$ is a fuzzy soft Euclidean space.

Definition (3.3.4): Let $(SSV(\tilde{X}), \|\cdot\|, \Gamma_A)$ be a fuzzy soft Banach algebra, $\tilde{x}_e \in SSV(\tilde{X})$, $f : X \rightarrow R$ be a linear functional and $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ be a function. The mapping

$$(f, \varphi) : SSV(\tilde{X}) \rightarrow SSV(\tilde{\mathbb{R}})$$

which is defined as follows

$$(f, \varphi)(\tilde{x}_e) = (f(\tilde{x}))_{\varphi(e)}$$

is called a fuzzy soft functional.

Proposition (3.3.5): The fuzzy soft functional $(f, \varphi) : SSV(\tilde{X}) \rightarrow SSV(\tilde{\mathbb{R}})$ is linear if and only if the function $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ is linear.

Proof. Let $\tilde{x}_e, \tilde{y}_{e'} \in SSV(\tilde{X})$ and $\lambda \in \tilde{\mathbb{R}}$. In this case (f, φ) is linear if and only if

$$\begin{aligned} (f, \varphi)(\tilde{\lambda}(\tilde{x}_e) + \tilde{y}_{e'}) &= (f, \varphi)((\tilde{\lambda}x)_{\lambda e} + \tilde{y}_{e'}) = (f, \varphi)((\lambda x + y)_{\lambda e + e'}) \\ &= f(\lambda x + y)_{\varphi(\lambda e + e')} = (f(\lambda x) + f(y))_{\varphi(\lambda e) + \varphi(e')} \\ &= f(\lambda x)_{\varphi(\lambda e)} + f(y)_{\varphi(e')} = \tilde{\lambda} \cdot (f(x))_{\varphi(e)} + (f(y))_{\varphi(e')} \end{aligned}$$

where $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ is linear.

Proposition (3.3.6): The space of fuzzy soft linear functionals is a fuzzy soft Banach algebra.

Proof. Let $(f, \varphi) : SSV(\tilde{X}) \rightarrow SSV(\tilde{\mathbb{R}})$, $(g, \psi) : SSV(\tilde{X}) \rightarrow SSV(\tilde{\mathbb{R}})$ be fuzzy soft functionals where $f : X \rightarrow \mathbb{R}$, $g : X \rightarrow \mathbb{R}$ and $\tilde{x}_e \in SSV(\tilde{X})$, $\tilde{\lambda} \in \tilde{\mathbb{R}}$. Then we have

$$\begin{aligned} ((f, \varphi) + (g, \psi))(\tilde{x}_e) &= ((f + g), (\varphi + \psi))(\tilde{x}_e) \\ &= (f + g)(x)_{(\varphi + \psi)(e)} = (f(x) + g(x))_{\varphi(e) + \psi(e)} \end{aligned}$$

since X is a Banach algebra for the functionals $f : X \rightarrow \mathbb{R}$, $g : X \rightarrow \mathbb{R}$ we have $f(x) + g(x) \in \mathbb{R}$ and for the function $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ we have $\varphi(e) + \psi(e) \in \mathbb{R}$ which means that $(f(x) + g(x))_{\varphi(e) + \psi(e)} \in SSV(\tilde{\mathbb{R}})$ and consequently we have $(f, \varphi) + (g, \psi) : SSV(\tilde{X}) \rightarrow SSV(\tilde{\mathbb{R}})$.

On the other hand,

$$\tilde{\lambda}(A, \varphi)(\tilde{x}_e) = (\tilde{\lambda}A, \tilde{\lambda}\varphi)(\tilde{x}_e) = (\tilde{\lambda}A)(x)_{\tilde{\lambda}\varphi(e)}.$$

Definition (3.3.7): The fuzzy soft norm of the fuzzy soft linear functional (f, φ) is defined as follows

$$\|(f, \varphi)\| = \inf\{\tilde{M} : \|(f, \varphi)(\tilde{x}_e)\| \preceq \tilde{M}\|\tilde{x}_e\|\}$$

is said to be a norm of (f, φ) where $\tilde{M}$ is a fuzzy soft real number.

It is obvious that $\|(f, \varphi)(\tilde{x}_e)\| \preceq \|(f, \varphi)\|\|\tilde{x}_e\|$.

Remark (3.3.8): Let $(SSV(\tilde{X}), \langle ., . \rangle, A)(A = \mathbb{R})$ be a fuzzy soft inner product space. For the parameter $e = 0$ the soft vector space $SSV(\tilde{X})$ is equal to the vector space X and we have the inner product space $(X, \langle ., . \rangle$

$0) = \text{SSV}(\tilde{X}_0)$ which is a subspace of $(\text{SSV}(\tilde{X}), \langle ., \rangle, A)$. Here, the Banach space X may not be a inner product space.

Proposition (3.3.9): Let $(\text{SSV}(\tilde{X}), \langle ., \rangle, A)$ ($A = \mathbb{R}$) be a fuzzy soft inner product space. For every $\tilde{x}_e \in \text{SSV}(\tilde{X})$ the operator $I : \text{SSV}(\tilde{X}) \rightarrow X$ where $I(\tilde{x}_e) = x \in X$. The restriction of the operator I to $\text{SSV}(\tilde{X}_0)$ denoted by $I/\text{SSV}(\tilde{X}_0) : \rightarrow X$ where $I(\tilde{x}_0) = x$ is a bijection and consequently $\text{SSV}(\tilde{X}_0) \sim X$.

Remark (3.3.10): For the parameter $e = 0$ the fuzzy soft Banach algebra $\tilde{X}_0$ is equal to the vector space X and thus for every $x, y \in X$ and $x_0, y_0 \in \tilde{X}_0$ we can write $\langle x, y \rangle = \langle x_0, y_0 \rangle$.

Theorem (3.3.11): Let $(\text{SSV}(\tilde{X}), \langle ., \rangle, A)$ ($A = \mathbb{R}$) be a fuzzy soft inner product space and $(f, \varphi) : \text{SSV}(\tilde{X}) \rightarrow \text{SSV}(\tilde{\mathbb{R}})$ be a fuzzy soft functional. In this case we can state the term $(f, \varphi)(\tilde{x}_e) = \widetilde{(f(x))}_{\varphi(e)}$ in terms of fuzzy soft inner product as given below

$$(f, \varphi)(\tilde{x}_e) = \langle x, \widetilde{u} \rangle_{\langle e, v \rangle}$$

where $f(x) = \langle x, u \rangle$ and $\varphi(e) = \langle e, v \rangle$. Moreover,

$$\|f\| = \|u\|, \|\varphi\| = \|v\|.$$

Proof. For $(f, \varphi) : \text{SSV}(\tilde{X}) \rightarrow \text{SSV}(\tilde{\mathbb{R}})$ we have the functions $f : X \rightarrow \mathbb{R}$ and $\varphi : \mathbb{R} \rightarrow \mathbb{R}$. If $f : X \rightarrow \mathbb{R}$ then for every $x_0 \in \tilde{X}_0$ there exist a vector $u_0 \in X_0$ such that

$$f(x_0) = \langle x_0, u_0 \rangle$$

Since for the parameter $e = 0$ we have $\langle x, y \rangle = \langle x_0, u_0 \rangle$ then $f(x) = \langle x, u \rangle$

Similarly If $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ then for every parameter $e \in \mathbb{R}$ there exist a real number $v \in \mathbb{R}$ such that

$$\varphi(e) = \langle e, v \rangle.$$

Consequently, we have

$$(f, \varphi)(\tilde{x}_e) = \langle \widetilde{x, u} \rangle_{\langle e, v \rangle}. \quad (3.1)$$

From the equation (3.1) we can write

$$\begin{aligned} \|(f, \varphi)(\tilde{x}_e)\| &= \|\langle x, u \rangle_{\langle e, v \rangle}\| \\ &= \sqrt{\langle x, u \rangle^2 + \langle e, v \rangle^2} \\ &\leq \sqrt{x^2 \cdot u^2 + e^2 \cdot v^2} \\ &\leq \sqrt{u^2 + v^2} \quad (3.2) \end{aligned}$$

On the other hand

$$\begin{aligned} \|(f, \varphi)(\tilde{x}_e)\| &= \left\| (\widetilde{f(x)})_{\varphi(e)} \right\| \\ &= \sqrt{f^2(x) + \varphi^2(e)} \\ &\leq \sqrt{\|f\|^2 \|x\|^2 + \|\varphi\|^2 \|e\|^2} \\ &\leq \sqrt{\|f\|^2 + \|\varphi\|^2} \quad (3.3) \end{aligned}$$

from the inequalities (3.2) and (3.3) we have $\|f\| = \|u\|$ and $\|\varphi\| = \|v\|$

Conclusion (3.3.12): Let $(f, \varphi) : SSV(\tilde{X}) \rightarrow SSV(\tilde{\mathbb{R}})$ be a fuzzy soft functional. for every $\tilde{x}_e \in SSV(\tilde{X})$ there exist $u \in X$ and $v \in \mathbb{R}$ such that $\|f\| = \|u\|$ and $\|\varphi\| = \|v\|$ and consequently $\|(f, \varphi)\| = \|u_v\|$

Definition (3.3.13): The fuzzy soft Banach algebra of fuzzy soft linear functionals is called the fuzzy soft dual space of the fuzzy soft Banach algebra $SSV(\tilde{X})$ and denoted by $SSV(\tilde{X})^*$

Definition (3.3.14): Let $(SSV(\tilde{X}), \|\cdot\|, A)$, $(SSV(\tilde{Y}), \|\cdot\|, A)$ ($A = \mathbb{R}$) be fuzzy soft Banach algebra, $(A, \psi) : SSV(\tilde{X}) \rightarrow SSV(\tilde{Y})$, $(A, \psi)^* : SSV(\tilde{Y})^* \rightarrow SSV(\tilde{X})^*$ be fuzzy soft linear operators where $A : X \rightarrow Y$, $\psi : \mathbb{R} \rightarrow \mathbb{R}$, $A^* : Y^* \rightarrow X^*$, $\psi^* : \mathbb{R}^* \rightarrow \mathbb{R}^*$, $\tilde{x}_e \in SSV(\tilde{X})$ and $(f, \varphi) : SSV(\tilde{Y}) \rightarrow SSV(\tilde{\mathbb{R}})$ be a fuzzy soft linear functional. In this case we have

$$(A, \psi)^*(f, \varphi) = (f, \varphi) \circ (A, \psi)$$

Proposition (3.3.15): Let $(A, \psi) : SSV(\tilde{X}) \rightarrow SSV(\tilde{Y})$ be a fuzzy soft operator. Then $(A, \psi)^* : SSV(\tilde{Y})^* \rightarrow SSV(\tilde{X})^*$ is a fuzzy soft linear operator.

Proof. Let $(f, \varphi), (g, \psi) \in SSV(\tilde{X})^*$ and $\tilde{\lambda} \in \tilde{\mathbb{R}}$. In this case, since $(A, \psi), (f, \varphi)$ and (g, ψ) are linear we have

$$\begin{aligned} (A, \psi)^*(\tilde{\lambda}(f, \varphi) + (g, \psi)) &= (\tilde{\lambda}(f, \varphi) + (g, \psi)) \circ (A, \psi) \\ &= \tilde{\lambda}(f, \varphi) \circ (A, \psi) + (g, \psi) \circ (A, \psi) \\ &= (A, \psi)^*(\tilde{\lambda}(f, \varphi)) + (A, \psi)^*((g, \psi)) \end{aligned}$$

Theorem (3.3.16): If the fuzzy soft linear operator $(A, \psi) : \text{SSV}(\tilde{X}) \rightarrow \text{SSV}(\tilde{Y})$ is continuous then $(A, \psi)^* : \text{SSV}(\tilde{Y})^* \rightarrow \text{SSV}(\tilde{X})^*$ is continuous.

Proof. Let $\tilde{x}_e \in \text{SSV}(\tilde{X})$ be any fuzzy soft Banach algebra.

$$\begin{aligned} \|(A, \psi)^*(f, \varphi)(\tilde{x}_e)\| &= \|(f, \varphi)((A, \psi)(\tilde{x}_e))\| \\ &\leq \|f, \varphi\| \|(A, \psi)(\tilde{x}_e)\| \\ &\leq \|(A, \psi)\| \|(f, \varphi)\| \|\tilde{x}_e\| \end{aligned} \quad (3.4)$$

from the Inequality 3.4 we have

$$\|(A, \psi)^*(f, \varphi)\| \leq \|(A, \psi)\| \|(f, \varphi)\|$$

and consequently,

$$\|(A, \psi)^*\| \leq \|(A, \psi)\|$$

since (A, ψ) is continuous it is bounded according to the Theorem 3.2.16
Since (A, ψ) is bounded then $(A, \psi)^*$ is bounded and as a result it is continuous.

Chapter Four

Applications of fuzzy soft in Banach algebra

Chapter four

Applications of fuzzy soft in Banach algebra

4.1 Introduction:

In this chapter, we presented some applications of fuzzy soft Banach algebra, we studied the fuzzy soft linear space and some definitions and examples of it, and we introduced the balanced fuzzy soft and Absorbing fuzzy soft and fuzzy soft convexity.

4. Further properties of Fuzzy Soft Linear Space

Recall that: Let X be an universe set, $A \subseteq E_1$, where E_1 and E_2 are sets of parameters.

and let $f: E_1 \rightarrow E_2$ be a function

1. Let Γ_A be a fuzzy soft set over X . The image, $f(\Gamma_A)$, of Γ_A under f is defined by

$$f(\Gamma_A)(y) = \begin{cases} \bigcup_{x \in f^{-1}(y)} \Gamma_A(x), & f^{-1}(y) \neq \emptyset \\ \emptyset, & f^{-1}(y) = \emptyset \end{cases}$$

For all $y \in E_2$, Where $f^{-1}(y) = \{x \in E_1 : f(x) = y\}$.

2. Let (Γ_B) be a fuzzy soft set over X . The inverse image, $f^{-1}(\Gamma_B)$, of Γ_B under f is defined by $f^{-1}(\Gamma_B)(x) = \Gamma_B(f(x)) = (\Gamma_B f)(x)$ For all $x \in E_1$.

Obviously $f(\Gamma_A)$ is a fuzzy soft set over x, E_2 and $f^{-1}(\Gamma_B)$ is a fuzzy soft set over (X, E_1) .

Definition (4.2.1): Let X be an initial universe set and E be the set of parameters. $P(X)$ denote the power set of X and $A, B \subseteq E$. Let Γ_A and Γ_B be two fuzzy soft sets over X . The product of Γ_A and (Γ_B) is the fuzzy soft set over X , denoted by $\Gamma_A \times \Gamma_B$ and it is defined by $(\Gamma_A \times \Gamma_B)(e_1, e_2) = \Gamma(A) \cap \Gamma(B)$ for all $(e_1, e_2) \in A \times B$.

Example (4.2.2): Let $X = \{a, b, c, d, e\}, E = \{e_1, e_2, e_3, e_4, e_5\}, A = \{e_1, e_2\}$ and $B = \{e_2, e_3, e_4\}$. Also let Γ_A and Γ_B be two soft sets over X defined by

$$\Gamma(A) = \{a, b, c\}, \Gamma(A) = \{d, e\} \text{ and } \Gamma(B) = \{b, d, e\}, \Gamma(B) = \{a, b\},$$

$$\Gamma(e_4) = \{c, d, e\}. \text{ Then}$$

$$(\Gamma_A \times \Gamma_B)(e_1, e_2) = \Gamma(A) \cap \Gamma(B) = \{a, b, c\} \cap \{b, d, e\} = \{b\}$$

$$(\Gamma_A \times \Gamma_B)(e_1, e_3) = \Gamma(A) \cap \Gamma(B) = \{a, b, c\} \cap \{a, b\} = \{a, b\}$$

$$(\Gamma_A \times \Gamma_B)(e_1, e_4) = \Gamma(A) \cap \Gamma(B) = \{a, b, c\} \cap \{c, d, e\} = \{c\}$$

$$(\Gamma_A \times \Gamma_B)(e_2, e_2) = \Gamma(A) \cap \Gamma(B) = \{d, e\} \cap \{b, d, e\} = \{d, e\}$$

$$(\Gamma_A \times \Gamma_B)(e_2, e_3) = \Gamma(A) \cap \Gamma(B) = \{d, e\} \cap \{a, b\} = \emptyset$$

$$(\Gamma_A \times \Gamma_B)(e_2, e_4) = \Gamma(A) \cap \Gamma(B) = \{d, e\} \cap \{c, d, e\} = \{d, e\}$$

$$\Gamma_A \times \Gamma_B = \{\emptyset, \{b\}, \{c\}, \{a, b\}, \{d, e\}\}$$

Definition (4.2.3): Let X be an universe set , E be a linear space over a field Φ and $\Gamma_A, \Gamma_B \in S(X)$. We define $\Gamma_A + \Gamma_B : \Gamma_A + \Gamma_B = f(\Gamma_A \times \Gamma_B)$, where $(\Gamma_A \times \Gamma_B)(e_1, e_2) = \Gamma(e_1) \cap \Gamma(e_2)$ for all $(e_1, e_2) \in A \times B$ and $f: E \times E \rightarrow E$ is a function defined by $f(x, y) = x + y$ for all $x, y \in E$. i.e. for each $e \in E, (\Gamma_A \times \Gamma_B)(e) =$

$$\bigcup_{(e_1, e_2) \in f^{-1}(e)} (\Gamma_A \times \Gamma_B)(e_1, e_2) = \bigcup_{(e_1, e_2) \in f^{-1}(e)} \Gamma(e_1) \cap \Gamma(e_2)$$

Example (4.2.4): Let $X = \mathbb{R}^2, E = \mathbb{R}, A = \{1, 2, 3\}, B = \{3, 4\}$ and Γ_A, Γ_B be two fuzzy soft sets over X defined by

$$\Gamma(e) = \{(x, y) \in \mathbb{R}^2 : 2x + 3y = e\} \quad \text{for all } e \in A \quad \text{and} \quad \Gamma(e) = \{(x, y) \in \mathbb{R}^2 : 4x + 7y = e\} \quad \text{for all } e \in B.$$

Now $1 + 3 = 4, 1 + 4 = 5, 2 + 3 = 5, 2 + 4 = 6, 3 + 3 = 6, 3 + 4 = 7$. Then

$$(\Gamma_A + \Gamma_B)(4) = \bigcup_{(e_1, e_2) \in f^{-1}(4)} \Gamma(e_1) \cap \Gamma_B(e_2) = \Gamma(1) \cap \Gamma(3) = \{(-1, 1)\}$$

$$\begin{aligned} (\Gamma_A + \Gamma_B)(5) &= \bigcup_{(e_1, e_2) \in f^{-1}(5)} \Gamma(e_1) \cap \Gamma_B(e_2) \\ &= \Gamma(1) \cap \Gamma(4) \cup (\Gamma(2) \cap \Gamma(3)) \end{aligned}$$

$$= \left\{ \left(-\frac{5}{2}, 2 \right), \left(\frac{5}{2}, -1 \right) \right\}$$

$$\begin{aligned} (\Gamma_A + \Gamma_B)(6) &= \bigcup_{(e_1, e_2) \in f^{-1}(6)} \Gamma(e_1) \cap \Gamma_B(e_2) \\ &= (\Gamma(2) \cap \Gamma(4)) \cup (\Gamma(3) \cap \Gamma(3)) = \{(1, 0), (6, -3)\} \end{aligned}$$

$$(\Gamma_A + \Gamma_B)(7) = \bigcup_{(e_1, e_2) \in f^{-1}(7)} \Gamma(e_1) \cap \Gamma_B(e_2) = \Gamma(3) \cap \Gamma(4) = \left\{ \left(\frac{9}{2}, -2 \right) \right\}$$

$$\Gamma_A + \Gamma_B = \left\{ \{(-1, 1)\}, \left\{ \left(\frac{9}{2}, -2 \right) \right\}, \left\{ \left(-\frac{5}{2}, 2 \right), \left(\frac{5}{2}, -1 \right) \right\}, \{(1, 0), (6, -3)\} \right\}$$

Definition (4.2.5): Let X be an universe set , E be a linear space over a field F and $\Gamma_A \in S(X), \lambda \in F$. We define $\lambda\Gamma_A$ as: $\lambda\Gamma_A = g(\Gamma_A)$, where $g: E \rightarrow E$ is a function defined by $g(x) = \lambda x$ for all $x \in E$. i.e. for each $e \in E$,

$$(\lambda\Gamma_A)(e) = g(\Gamma_A)(e) = \bigcup_{x \in g^{-1}(e)} \Gamma(x)$$

Theorem (4.2.6): Let X be an universe set , E be a linear space over a field F and $\Gamma_A \in S(X), \lambda \in F$. Then

$$(\lambda\Gamma_A)(e) = \begin{cases} \Gamma(\lambda^{-1}e), \lambda \neq 0 \\ \Phi, \lambda = 0, & e \neq 0 \text{ for all } e \in E \\ \bigcup_{x \in E} \Gamma(x), \lambda = 0, & e = 0 \end{cases}$$

Proof : Let $e \in E$

1. if $\lambda \neq 0$, then $(\lambda\Gamma_A)(e) =$

$$\bigcup_{x \in g^{-1}(e)} \Gamma(x) = \bigcup_{x \in \lambda^{-1}e} \Gamma(x) = \Gamma(\lambda^{-1}e)$$

2. if $\lambda = 0$ and $e \neq 0$, then there exists no element x such that $e = g(x) = 0$

Hence

$$\bigcup_{x \in g^{-1}(e)} \Gamma(x) = \Phi, \text{ since } (\lambda\Gamma_A)(e) = \bigcup_{x \in g^{-1}(e)} \Gamma(x), \text{ then } (\lambda\Gamma_A)(e) = \Phi$$

3. if $\lambda = 0$ and $e = 0$, then $(\lambda\Gamma_A)(e) =$

$$\bigcup_{x \in g^{-1}(e)} \Gamma(x)$$

Let

$x \in E$. Then $0x = 0$, i. e. $g(x) = 0$ or $x \in g^{-1}(0) = g^{-1}(e)$. Thus $g^{-1}(e) = E$. So $(\lambda\Gamma_A)(e)$

$$= \bigcup_{x \in E} \Gamma(x).$$

Example (4.2.7): Let $X = Z$, $E = R$, $A = R^2$ and Γ_A be a fuzzy soft set over (U, E) defined by $F(e) = \{|e|, |e| + 1, |e| + 2, \dots\}$ for all $e \in A$ and $\lambda \in R$. Then

1. If $\lambda \neq 0$, then $(\lambda\Gamma_A)(e) = F(\lambda^{-1}e) = \{|\lambda^{-1}e|, |\lambda^{-1}e| + 1, |\lambda^{-1}e| + 2, \dots\}$
2. If $\lambda = 0$ and $e \neq 0$, then $(\lambda\Gamma_A)(e) = \emptyset_A$
3. If $\lambda = 0$ and $e = 0$, then $(\lambda\Gamma_A)(e)$

$$= \bigcup_{x \in E} \Gamma(x) = \bigcup_{\chi \in A} \Gamma(\chi) = N$$

Theorem (4.2.8): Suppose E_1 and E_2 are linear spaces over the same field F and $f : E_1 \rightarrow E_2$ is linear function .

Then for any fuzzy soft sets Γ_A and Γ_B over (U, E_1) and for the scalar λ ;

1. $f(\Gamma_A + \Gamma_B) = f(\Gamma_A) + f(\Gamma_B)$;
2. $f(\lambda\Gamma_A) = \lambda f(\Gamma_A)$.

Proof. Let the range of f be M , i.e. $M = f(E_1) = \{f(\chi) : \chi \in E_1\}$

1. Let $y \in Y$. Put $P = f(\Gamma_A + \Gamma_B)(y)$, $Q = (f(\Gamma_A) + f(\Gamma_B))(y)$. We want to prove that $P = Q$

Assume first that $y \notin M \Rightarrow f^{-1}(y) = \{x \in E_1 : f(x) = y\} = \emptyset \Rightarrow P = \emptyset$

Also, if $y_1, y_2 \in Y$ With $y_1 + y_2 = y$, then at least one of the y_1, y_2 is not in M and thus, either

$$f(\Gamma_A)(y_1) = \bigcup_{e \in f^{-1}(y_1)} \Gamma(e) = \emptyset \text{ or } f(\Gamma_A)(y_2) = \bigcup_{e \in f^{-1}(y_2)} \Gamma(e) = \emptyset$$

$$\text{or } f(\Gamma_A)(y_2) = \bigcup_{e \in f^{-1}(y_2)} \Gamma(e) = \emptyset$$

$$\Rightarrow Q = \left(f(\Gamma_A) + f(\Gamma_B)(y) = \bigcup_{y=y_1+y_2} (f(\Gamma_A)(y_1) \cap f(\Gamma_B)(y_2)) \right) = \emptyset$$

Hence, in this case, $P = Q$

Assume next that $y \in M$, and $z \in P$, then $z \in \bigcup_{e \in f^{-1}(y)} (\Gamma_A + \Gamma_B)(e)$, i. e. $z \in (\Gamma_A + \Gamma_B)(e)$ for some $e \in f^{-1}(y)$, i.e. $z \in$

$$\bigcup_{e=e_1+e_2} (\Gamma(e_1) \cap \Gamma(e_2)).$$

Thus there exists $e_1, e_2 \in E$ with $e = e_1 + e_2$ such that $z \in \Gamma(e_1) \cap \Gamma(e_2)$.

On one hand, $Q = (f(\Gamma_A) + f(\Gamma_B)(y)) = \bigcup_{y=y_1+y_2} (f(\Gamma_A)(y_1) \cap f(\Gamma_B)(y_2))$

Since f is linear function, then $y = f(e) = f(e_1 + e_2) = f(e_1) + f(e_2)$, we have

$$f(\Gamma_A(e_1) \cap \Gamma_B(e_2)) \subseteq Q$$

Then, $\{\bigcup_{r_1 \in f^{-1}(e_1)} \Gamma(r_1)\} \cap \{\bigcup_{r_2 \in f^{-1}(e_2)} \Gamma(r_2)\} \subseteq Q$

Thus $\Gamma(e_1)^1\Gamma(e_2) \subseteq Q$, i. e. $z \in Q$. So $P \subseteq Q$.

For the reverse, we take $w \in Q$. Then $w \in \bigcup_{y=y_1+y_2} (f(\Gamma_A)(y_1) \cap f(\Gamma_B)(y_2))$.

Thus, there exist $y_1, y_2 \in M$ with $y = y_1 + y_2$ such that

$$w \in f(\Gamma_A)(y_1) \cap f(\Gamma_B)(y_2) = \left\{ \bigcup_{e_1 \in f^{-1}(y_1)} \Gamma(e_1) \right\} \cap \left\{ \bigcup_{e_2 \in f^{-1}(y_2)} \Gamma(e_2) \right\}$$

So, there exist $e_1 \in f^{-1}(y_1)$ and $e_2 \in f^{-1}(y_2)$ such that $w \in \Gamma(e_1) \cap \Gamma_B(e_2)$, where $y = y_1 + y_2 = f(e_1) + f(e_2) = f(e_1 + e_2)$

Now $P = f(\Gamma_A + \Gamma_B)(y) = \bigcup_{e \in f^{-1}(y)} (\Gamma_A + \Gamma_B)(e)$. Since $y = f(e_1 + e_2)$, $(\Gamma_A + \Gamma_B)(e_1 + e_2) \subseteq P$, i. e.

$\Gamma(e_1) \cap \Gamma(e_2) \subseteq P$. Then $w \in P$. Thus $Q \subseteq P$. So $P = Q$. This completes the proof of (1).

2. Let $y \in Y$. Put $P = f(\lambda\Gamma_A)(y)$, $Q = \lambda f(\Gamma_A)(y)$. We want to prove that $P = Q$

Assume first that $y \notin M \Rightarrow f^{-1}(y) = \{\chi \in E_1 : f(\chi) = y\} = \emptyset$.

$$\begin{aligned} \text{Thus for } \lambda \neq 0, Q &= \lambda f(\Gamma_A)(y) = f(\Gamma_A)(\lambda^{-1}y) = \bigcup_{e \in f^{-1}(\lambda^{-1}y)} \Gamma(e) = \\ &= \bigcup_{\lambda e \in f^{-1}(y)} \Gamma(e) = \emptyset \text{ (as } f^{-1}(y) = \emptyset) \end{aligned}$$

Also $Q = \emptyset$, when $\lambda = 0$. So, if $y \notin M$, then $Q = \emptyset$.

$$\text{Now } P = f(\lambda\Gamma_A)(y) = \bigcup_{e \in f^{-1}(y)} \lambda\Gamma(e) = \emptyset \text{ (as } f^{-1}(y) = \emptyset)$$

Hence $P = Q$, for $y \notin M$.

Assume next that $y \in M$, and $\lambda \neq 0$, then

$$\begin{aligned} Q &= \lambda f(f_A)(y) = f(fA)(\lambda^{-1}y) = \bigcup_{e \in f^{-1}(\lambda^{-1}y)} \Gamma(e) = \bigcup_{f(\lambda e)=y} \Gamma(e) \\ &= \bigcup_{\lambda f(e)=y} \Gamma(e) \end{aligned}$$

And

$$P = f(\lambda \Gamma_A)(y) = \bigcup_{z \in f^{-1}(y)} \lambda \Gamma(z) = \bigcup_{f(z)=y} \Gamma(\lambda^{-1}z) = \bigcup_{f(\lambda e)=y} \Gamma(e),$$

where $\lambda^{-1}z = e$

Thus $P = Q$, for $y \in M$, and $\lambda \neq 0$.

If $y = 0$, $y \in M$, and $\lambda = 0$, then

$$Q = 0f(\Gamma_A)(0) = \bigcup_{e_2 \in E_2} f(\Gamma(e_2)) = \bigcup_{e_2 \in E_2} \bigcup_{e_1 \in f^{-1}(e_2)} \Gamma(e_1) = \bigcup_{e_1 \in E_1} \Gamma(e_1)$$

$$\text{And } P = f(0\Gamma_A)(0) = \bigcup_{e \in f^{-1}(0)} 0\Gamma(e) = 0\Gamma(0)$$

Since if $e \neq 0$ and $e \in f^{-1}(0)$, then $0f(\Gamma_A)(e) = \emptyset$. Hence $P = \bigcup_{e_1 \in E_1} \Gamma(e_1)$

Thus $P = Q$ for $y = 0$, $y \in M$, and $\lambda = 0$. This completes the proof of part (2).

Theorem (4.2.9): If S is an ordinary subset of a linear space E over F , Γ_A is a fuzzy soft set over X and $\chi \in E$, then

1. $(\chi + \Gamma)(e) = \Gamma(e - \chi)$ for all $\chi \in E$ where $\chi + \Gamma$ means $I_\chi + \Gamma$ and

$$I_\chi(e) = \begin{cases} X, & e = \chi \\ \emptyset & e \neq \chi \end{cases}$$

2. $\chi + \Gamma = T_\chi(\Gamma)$ for the translation mapping $T_\chi: E \rightarrow E$ defined by $T_\chi(e) = \chi + e$ for all $e \in E$.

$$3. S + \Gamma = \bigcup_{a \in S} (a + \Gamma)$$

Proof.

$$1. (\chi + \Gamma)(e) = \bigcup_{e=e_1+e_2} \{I_\chi(e_1) \cap \Gamma(e_2)\} =$$

$$\{I_\chi(\chi) \cap \Gamma(e - \chi)\} \bigcup_{e_1 \neq \chi} \{\emptyset \cap \Gamma(e - e_1)\} = \Gamma(e - \chi) \cup \emptyset = \Gamma(e - \chi)$$

2. $T_\chi(\Gamma)(e) = \bigcup_{y \in T_\chi^{-1}(e)} \Gamma(y) = \Gamma(e - \chi)$. Thus, by (1), we get $\chi + \Gamma = T_\chi(\Gamma)$.

$$3. (S + \Gamma)(e) = \bigcup_{e=e_1+e_2} \{I_S(e_1) \cap \Gamma(e_2)\} = \bigcup_{a \in S} \{I_S(a) \cap \Gamma(e - a)\} \bigcup_{a \in S^c} \{\emptyset \cap \Gamma(e - a)\} = \bigcup_{a \in S} \Gamma(e - a) \cup \emptyset = \bigcup_{a \in S} \Gamma(e - a) = \bigcup_{a \in S} (a + \Gamma)(e)$$

Theorem (4.2.10): Let X be an universe set, E be a linear space over a field F , $\alpha, \beta \in K$ and $\Gamma_A, \Gamma_B, \Gamma_C \in S(X)$.

Then the following are equivalent :

1. $\alpha\Gamma_A + \beta\Gamma_B \subseteq \Gamma_C$
2. $\Gamma_C(\alpha\chi + \beta y) \supseteq \Gamma_A \cap \Gamma_B$ for all $\chi, y \in E$

Proof :

$$(1) \Rightarrow (2)$$

$$\begin{aligned}
\Gamma(\alpha\chi + \beta y) &\supseteq (\alpha\Gamma + \beta\Gamma)(\alpha\chi + \beta y) = \bigcup_{\alpha\chi + \beta y} ((\alpha\Gamma(\chi)) \cap (\beta\Gamma(y))) \\
&\supseteq (\alpha\Gamma(\alpha\chi)) \cap (\beta\Gamma(\beta y)) \supseteq \Gamma(\chi) \cap \Gamma(y)
\end{aligned}$$

$$(2) \Rightarrow (1)$$

Let $\Gamma_c(\alpha\chi + \beta y) \supseteq \Gamma(\chi) \cap \Gamma(y)$ for all $\chi, y \in E$

$$\begin{aligned}
(\Gamma\alpha + \beta\Gamma)(z) &= \bigcup_{z=\alpha\chi + \beta y} (\alpha\Gamma(\chi)) \cap (\beta\Gamma(y)) \\
&= \bigcup_{z=\alpha\chi + \beta y} (\Gamma(\alpha^{-1}\chi)) \cap (\Gamma(\beta^{-1}y)) \subseteq \bigcup_{z=\alpha\chi + \beta y} \Gamma(\alpha\alpha^{-1}\chi + \beta\beta^{-1}y)
\end{aligned}$$

Since

$$\bigcup_{z=\alpha\chi + \beta y} \Gamma(\alpha\chi + \beta y) = \Gamma(z)$$

Theorem (4.2.11): Let U be an universe set, E be a linear space over a field F and $\Gamma_A, \Gamma_B \in S(X)$. Then

1. $1\Gamma_A + 0\Gamma_B \subseteq \Gamma_A$
2. $1\Gamma_A + 0\Gamma_B = \Gamma_A$ iff $\bigcup_{e \in E} \Gamma(e) \subseteq \bigcup_{e \in E} \Gamma(e)$ for $\chi, y \in E$

Proof :

1. Let $e_1, e_2 \in E$. Then $\Gamma(1e_1 + 0e_2) = \Gamma(e_1) \supseteq \Gamma(e_1) \cap \Gamma(e_2)$. Thus, by previous theorem, we have

$$1\Gamma_A + 0\Gamma_B \subseteq \Gamma_A.$$

2. Let $\bigcup_{e \in E} \Gamma(e) \subseteq \bigcup_{e \in E} \Gamma(e)$ and $e \in E$. Then

$$\begin{aligned}
(1\Gamma + 0\Gamma)(e) &= \bigcup_{e=e_1+e_2} \{1\Gamma(e_1) \cap 0\Gamma(e_2)\} = 1\Gamma(e) \cap 0\Gamma(0) \\
&= \Gamma(e) \cap \left\{ \bigcup_{\chi \in E} \Gamma(\chi) \right\}
\end{aligned}$$

Since $\Gamma(e) \subseteq \bigcup_{\chi \in E} \Gamma(\chi)$, then $\Gamma(e) \cap \left\{ \bigcup_{\chi \in E} \Gamma(\chi) \right\} = \Gamma(e)$. So $(1\Gamma + 0\Gamma)(e) = \Gamma(e)$

Thus $1\Gamma_A + 0\Gamma_B = \Gamma_A$

Conversely, suppose that $1\Gamma_A + 0\Gamma_B = \Gamma_A$ and $\chi \in \bigcup_{e \in E} \Gamma(e)$

Then $\chi \in \Gamma(e) = (1\Gamma + 0\Gamma)(e)$ for some $e \in E$.

Thus $\chi \in \bigcup_{e=e_1+e_2} \{1\Gamma(e_1) \cap 0\Gamma(e_2)\}$

So $\chi \in 1\Gamma(e) \cap 0\Gamma(e)$, i. e., $\chi \in 1\Gamma(e) \cap 0\Gamma(e) = \bigcup_{e \in E} \Gamma(e)$.

Hence $\bigcup_{e \in E} \Gamma(e) \subseteq \bigcup_{e \in E} \Gamma(e)$.

4.3 Balanced fuzzy soft sets

Balanced soft sets are a mathematical framework used for dealing with uncertainty and imprecision in decision-making processes. They extend classical set theory by incorporating the notion of degree of membership, allowing elements to belong to a set with varying degrees of certainty. The balance concept in balanced soft sets provides a mechanism to handle conflicting information or preferences, making them useful in various fields such as decision analysis, pattern recognition, and artificial intelligence.

Definition (4.3.1): Let X be an universe set , E be a linear space over a field F . A soft set Γ_A over X is said to be balanced fuzzy soft set if $\lambda\Gamma_A \subseteq \Gamma_A$ for all $\lambda \in \Gamma$ with $|\lambda| \leq 1$.

Theorem (4.3.2): Let X be an universe set , E be a linear space over a field F . If Γ_A is a soft set over X , then $\bigcup_{|\lambda| \leq 1} \lambda\Gamma_A$ is a balanced fuzzy soft set.

Proof : Let $|\alpha| \leq 1$ and $e \in E$.

Case 1 : $0 < |\alpha| \leq 1$.

$$\alpha \left(\bigcup_{|\lambda| \leq 1} \lambda\Gamma_A \right) (e) = \left(\bigcup_{|\lambda| \leq 1} \lambda\Gamma_A \right) \left(\frac{1}{\alpha} e \right) = \bigcup_{|\lambda| \leq 1} \lambda\Gamma_A \left(\frac{1}{\alpha} e \right) = \bigcup_{|\lambda| \leq 1} \alpha\lambda\Gamma_A(e) \subseteq \bigcup_{|\lambda| \leq 1} \lambda\Gamma_A(e) \text{ since } |\alpha| \leq 1 \text{ and } |\lambda| \leq 1, |\alpha\lambda| \leq 1$$

Case 2 : $\alpha = 0$.

Subcase 1. If $e = 0$, then

$$\begin{aligned} & 0 \left(\bigcup_{|\lambda| \leq 1} \lambda\Gamma_A \right) (0) \\ &= \bigcup_{x \in E} \left\{ \left(\bigcup_{|\lambda| \leq 1} \lambda\Gamma_A \right) (x) \right\} = \bigcup_{x \in E} \left\{ \left(\bigcup_{|\lambda| \leq 1} \lambda\Gamma_A \right) (x) \right\} = \{0\Gamma_A(0)\} \\ & \quad \cup \left\{ \bigcup_{x(\neq 0) \in E} 0\Gamma_A(x) \right\} \end{aligned}$$

$$\cup \left\{ \bigcup_{\chi \in E} \left\{ \bigcup_{0 < |\lambda| \leq 1} \lambda \Gamma_A(\chi) \right\} \right\}$$

$$= \{0\Gamma_A(0)\} \cup \left\{ \bigcup_{\chi \in E} \left\{ \bigcup_{0 < |\lambda| \leq 1} \Gamma_A\left(\frac{1}{\lambda}\chi\right) \right\} \right\}, \text{ as } \bigcup_{\chi(\neq 0) \in E} 0\Gamma_A(\chi) = \Phi$$

$$0\left(\bigcup_{|\lambda| \leq 1} \lambda \Gamma_A\right)(0) = 0\Gamma_A(0)$$

$$\text{Again, } \left(\bigcup_{|\lambda| \leq 1} \lambda \Gamma_A\right)(0) = \bigcup_{|\lambda| \leq 1} \lambda \Gamma_A(0) = \left\{ \bigcup_{0 < |\lambda| \leq 1} \lambda \Gamma_A(0) \right\} \cup \{0\Gamma_A(0)\} =$$

$$\left\{ \bigcup_{0 < |\lambda| \leq 1} \Gamma_A(0) \right\} \cup \{0\Gamma_A(0)\} = 0\Gamma_A(0)$$

Thus, $0\left(\bigcup_{|\lambda| \leq 1} \lambda \Gamma_A\right)(0) = \left(\bigcup_{|\lambda| \leq 1} \lambda \Gamma_A\right)(0)$. Hence, $\bigcup_{|\lambda| \leq 1} \lambda \Gamma_A$ is a balanced fuzzy soft set.

Theorem (4.3.3): Let X be an universe set , E be a linear space over a field F . Let Γ_A be a balanced fuzzy soft set over X .

1. If $\alpha, \beta \in \Gamma$ and $|\alpha| \leq |\beta|$, then $\alpha\Gamma_A \subseteq \beta\Gamma_A$
2. $\alpha \in \Gamma$ and $|\alpha| = 1$, then $\alpha\Gamma_A = \Gamma_A$

Proof :

1. Case I . $\alpha = 0$.

Subcase 1. If $\beta = 0$, then clearly, $\alpha\Gamma_A = \beta\Gamma_A$.

Subcase 2. If $\beta \neq 0$, then for any non-zero $e \in E$, $\alpha\Gamma(e) = \Phi \subseteq \beta\Gamma(e)$ and for zero element of E , we have $\alpha\Gamma_A(0) = 0\Gamma_A(0) \subseteq \Gamma_A(0)$ as Γ_A is a balanced fuzzy soft set. That is, $\alpha\Gamma_A(0) \subseteq \Gamma_A\left(\frac{1}{\beta}0\right) = \beta\Gamma_A(0)$. Therefore in this case $\alpha\Gamma_A \subseteq \beta\Gamma_A$.

Case II . $\alpha \neq 0$. Since $\left|\frac{\alpha}{\beta}\right| \leq 1$, we have $\frac{\alpha}{\beta}\Gamma_A \subseteq \Gamma_A$

Let $e \in E$ and $\frac{1}{\beta}e = e_1$. Now since $\frac{\alpha}{\beta}\Gamma_A \subseteq \Gamma_A$, then $\frac{\alpha}{\beta}\Gamma(e_1) \subseteq \Gamma(e_1)$ or, $\Gamma\left(\frac{\beta}{\alpha}e_1\right) \subseteq \Gamma(e_1)$ or, $\Gamma\left(\frac{\beta}{\alpha}\frac{1}{\beta}e_1\right) \subseteq \Gamma\left(\frac{1}{\beta}e_1\right)$ or, $\Gamma\left(\frac{1}{\alpha}e_1\right) \subseteq \Gamma\left(\frac{1}{\beta}e_1\right)$ or, $\alpha\Gamma(e) \subseteq \beta\Gamma(e)$ that is, $\alpha\Gamma_A \subseteq \beta\Gamma_A$.

2. Let $|\alpha| \leq 1$. Then $\alpha\Gamma_A \subseteq \Gamma_A$. Again let $|\alpha| \geq 1$. Then $\left|\frac{1}{\alpha}\right| \leq 1$ which implies that $\frac{1}{\alpha}\Gamma_A \subseteq \Gamma_A$.

So as the procedure of case II for proof (1), we get $\Gamma_A \subseteq \alpha\Gamma_A$. Thus, $\alpha\Gamma_A = \Gamma_A$ if $|\alpha| = 1$.

Theorem (4.3.4): Let X be an universe set , E be a linear space over a field F . If $\{\Gamma_{A\alpha} : \alpha \in \Lambda\}$ is a collection of balanced fuzzy soft sets over X , then $\bigcap_{\alpha \in \Lambda} \Gamma_{A\alpha}$ is balanced.

Proof :

Let $\{\Gamma_{A\alpha} : \alpha \in \Lambda\}$ be a collection of balanced fuzzy soft sets over X and $\Gamma_A = \bigcap_{\alpha \in \Lambda} \Gamma_{A\alpha}$.

Also let $|t| \leq 1$ and $e \in E$.

Case 1. $t = 0$.

Subcase 1. If $e \neq 0$, then $t\Gamma(e) = \emptyset$. So $t\Gamma_A \subseteq \Gamma_A$.

Also let $|t| \leq 1$ and $e \in E$.

Case 1. $t = 0$.

Subcase 1. If $e \neq 0$, then $t\Gamma(e) = \emptyset$. So $t\Gamma_A \subseteq \Gamma_A$.

Subcase 2. If $e = 0$, then

$$\begin{aligned}
t\Gamma(e) &= 0\Gamma(0) = \bigcup_{p \in E} \Gamma(p) (\cap_{\alpha \in \Lambda} \Gamma_{A\alpha}(p)) \\
&\subseteq \cap_{\alpha \in \Lambda} \left(\bigcup_{p \in E} \Gamma_{A\alpha}(p) \right) \cap_{\alpha \in \Lambda} 0\Gamma_{A\alpha}(0) \subseteq \cap_{\alpha \in \Lambda} \Gamma_{A\alpha}(0) = \Gamma(0).
\end{aligned}$$

Case 2. $t \neq 0$.

$$\begin{aligned}
t\Gamma(e) &= \Gamma(t^{-1}e) = (\cap_{\alpha \in \Lambda} \Gamma_{A\alpha})(t^{-1}e) = \cap_{\alpha \in \Lambda} \Gamma_{A\alpha}(t^{-1}e) = \\
&\cap_{\alpha \in \Lambda} t\Gamma_{A\alpha}(e) \subseteq \cap_{\alpha \in \Lambda} \Gamma_{A\alpha}(e) = \Gamma(e)
\end{aligned}$$

Hence Γ_A is also a balanced fuzzy soft set.

Theorem(4.3.5): Let X be an universe set , E be a linear space over a field F . If $\{\Gamma_{A\alpha} : \alpha \in \Lambda\}$ is a collection of balanced fuzzy soft sets over X , then $\bigcup_{\alpha \in \Lambda} \Gamma_{A\alpha}$ is balanced.

Proof :

Let $\Gamma_{A\alpha} : \alpha \in \Lambda$ be a collection of balanced fuzzy soft sets over X and $\Gamma_A = \bigcup_{\alpha \in \Lambda} \Gamma_{A\alpha}$

Also let $|t| \leq 1$ and $e \in E$.

Case 1. $t = 0$.

Subcase 1. If $e \neq 0$, then $t\Gamma(e) = \emptyset$. So $t\Gamma_A \subseteq \Gamma_A$.

Subcase 2. If $e = 0$, then

$$\begin{aligned}
t\Gamma(e) &= 0\Gamma(0) = \bigcup_{p \in E} \Gamma(p) = \bigcup_{p \in E} \left(\bigcup_{\alpha \in \Lambda} \Gamma_{A\alpha} \right) (p) = \bigcup_{p \in E} \left(\bigcup_{\alpha \in \Lambda} \Gamma_{A\alpha}(p) \right) \\
&= \bigcup_{\alpha \in \Lambda} \left(\bigcup_{p \in E} \Gamma_{A\alpha}(p) \right) (P) \\
&= \bigcup_{\alpha \in \Lambda} 0\Gamma_{A\alpha}(0) \subseteq \bigcup_{\alpha \in \Lambda} \Gamma_{A\alpha}(0) = \Gamma(0)
\end{aligned}$$

Case 2. $t \neq 0$.

$$t\Gamma(e) = \Gamma(t^{-1}e) = (\cup_{\alpha \in \Lambda} \Gamma_{A\alpha})(t^{-1}e) = \cup_{\alpha \in \Lambda} \Gamma_{A\alpha}(t^{-1}e) = \\ \cup_{\alpha \in \Lambda} t\Gamma_{A\alpha}(e) \subseteq \cup_{\alpha \in \Lambda} \Gamma_{A\alpha}(e) = \Gamma(e) \text{ (as each } \Gamma_{A\alpha} \text{ is balanced)}$$

Hence Γ_A is also a balanced fuzzy soft set.

Theorem(4.3.6): Let X be an universe set , E be a linear space over a field F . If $\lambda \in F$ and Γ_A is balanced fuzzy soft set over X , then $\lambda\Gamma_A$ is balanced.

Proof : Let $|t| \leq 1$; if $\lambda = 0$, there is nothing to prove. So we take, $\lambda \neq 0$.

Case 1. $t = 0$.

Subcase 1. If $e \neq 0$, then $t\lambda\Gamma(e) = 0\Gamma(e) = \phi \subseteq \lambda\Gamma(e)$.

Subcase 2. If $e = 0$, then $0\lambda\Gamma(0) = 0\Gamma(0) \subseteq \Gamma(0) = \Gamma(\lambda^{-1}0) = \lambda\Gamma(0)$.

Case 2. $t \neq 0$.

$$t\lambda\Gamma(e) = \Gamma((t\lambda)^{-1}e) = \Gamma(\lambda^{-1}t^{-1}e) = \Gamma(t^{-1}\lambda^{-1}e) = t\Gamma(\lambda^{-1}e) \subseteq \Gamma(\lambda^{-1}e) \\ = \lambda\Gamma(e)$$

Hence, $\lambda\Gamma_A$ is balanced fuzzy soft set.

Theorem (4.3.7): Let X be an universe set , E be a linear space over a field F . If Γ_A and Γ_B are balanced fuzzy soft sets over X and $\alpha, \beta \in \Gamma$, then $\alpha\Gamma_A + \beta\Gamma_B$ is balanced fuzzy soft set.

Proof : Since Γ_A and Γ_B are balanced fuzzy soft sets, $\alpha\Gamma_A$ and $\beta\Gamma_B$ are balanced fuzzy soft sets.

Let $|t| \leq 1$.

Case 1. $t = 0$.

Subcase 1. If $e \neq 0$, then $t(\alpha\Gamma_A + \beta\Gamma_B)(e) = \emptyset \subseteq (\alpha\Gamma_A + \beta\Gamma_B)(e)$.

Subcase 2. If $e = 0$, then

$$\begin{aligned}
 0(\alpha\Gamma_A + \beta\Gamma_B)(0) &= \bigcup_{e \in E} (\alpha\Gamma_A + \beta\Gamma_B)(e) = \bigcup_{e \in E} \{ \bigcup_{e=e_1, e_2} (\alpha\Gamma_A(e_1) \cap \beta\Gamma_B(e_2)) \} \\
 &= \bigcup_{e_1, e_2 \in E} (\alpha\Gamma_A(e_1) \cap \beta\Gamma_B(e_2)) \subseteq \{ \bigcup_{e_1 \in E} (\alpha\Gamma_A(e_1)) \} \cap \{ \bigcup_{e_2 \in E} \beta\Gamma_B(e_2) \} \\
 &= 0\alpha\Gamma_A(0) \cap 0\beta\Gamma_B(0) \subseteq \alpha\Gamma_A(0) \subseteq \bigcup_{e \in E} (\alpha\Gamma_A(e) \cap \beta\Gamma_B(e)) \\
 &= \bigcup_{0=e+(-e)} (\alpha\Gamma_A(e) \cap \beta\Gamma_B(-e)) = (\alpha\Gamma_A \cap \beta\Gamma_B)(0)
 \end{aligned}$$

Case 2. $t \neq 0$.

$$\begin{aligned}
 t(\alpha\Gamma_A + \beta\Gamma_B)(e) &= (\alpha\Gamma_A + \beta\Gamma_B)(t^{-1}e) = \bigcup_{t^{-1}e=e_1+e_2} (\alpha\Gamma_A(e_1) \cap \beta\Gamma_B(e_2)) \\
 &= \bigcup_{e=te_1+te_2} (\alpha\Gamma_A(e_1) \cap \beta\Gamma_B(e_2)) = \bigcup_{e=x_1+x_2, x_1=te_1, x_2=te_2} (\alpha\Gamma_A(t^{-1}X_1) \cap \beta\Gamma_B(t^{-1}X_2)) \\
 &= \bigcup_{e=x_1+x_2} (t\alpha\Gamma_A(X_1) \cap t\beta\Gamma_B(X_2)) \subseteq \bigcup_{e=x_1+x_2} (\alpha\Gamma_A(X_1) \cap \beta\Gamma_B(X_2)) = (\alpha\Gamma_A + \beta\Gamma_B)(e)
 \end{aligned}$$

This completes the proof of the theorem.

Definition (4.3.8): Let X be an universe set, E be a linear space over a field F . Let Γ_A be a fuzzy soft set over X .

1. The intersection of all balanced fuzzy soft sets over X , each containing Γ_A , is called the balanced hull of Γ_A .
2. The union of all balanced fuzzy soft sets over X , each contained in Γ_A , is called the balanced core of Γ_A .

Theorem (4.3.9): Let X be an universe set , E be a linear space over a field F . Let H and C respectively denotes the balanced hull and the balanced core of a fuzzy soft set Γ_A over X . Then

1. $H = \cup \{\lambda \Gamma_A : |\lambda| \leq 1\}$
2. $C = \cap \{\lambda \Gamma_A : |\lambda| \geq 1\}$ if $\Gamma_A(0) = 0\Gamma_A(0)$.

Proof :

1. Let $B = \cup \{\lambda \Gamma_A : |\lambda| \leq 1\}$. Then B is a balanced fuzzy soft set by theorem (4.3.2).

Also clearly, $\Gamma_A \subseteq B$. Then by definition of H , $H \subseteq B$. Again $\Gamma_A \subseteq H$. Then $\lambda \Gamma_A \subseteq \lambda H \subseteq H$ for

$|\lambda| \leq 1$, as H is balanced. So $B = \cup \{\lambda \Gamma_A : |\lambda| \leq 1\} \subseteq H$. Hence $H = B$.

2. Let G_D be a fuzzy soft set such that $G_D \subseteq C$.

Then we have $\alpha G_D \subseteq \alpha C \subseteq C$ for all $|\alpha| \leq 1$, as C is balanced

$$\Rightarrow \alpha G_D \subseteq \Gamma_A \text{ for all } |\alpha| \leq 1, \Rightarrow G_D \subseteq \frac{1}{\alpha} \Gamma_A \text{ for all } 0 < |\alpha| \leq 1$$

$$\Rightarrow G_D \subseteq \lambda \Gamma_A \text{ for all } |\lambda| \geq 1 \Rightarrow G_D \subseteq \bigcap_{|\lambda| \geq 1} \lambda \Gamma_A$$

Therefore for any fuzzy soft subset G_D of C we have $G_D \subseteq \bigcap_{|\lambda| \geq 1} \lambda \Gamma_A$. Since C is also a fuzzy soft subset of C ,

$$G_D \subseteq \bigcap_{|\lambda| \geq 1} \lambda \Gamma_A. \text{ Again let, } G_D \subseteq \lambda \Gamma_A \text{ for all } |\lambda| \geq 1 \Rightarrow \frac{1}{\lambda} G_D \subseteq \Gamma_A \text{ for all } |\lambda| \geq 1$$

$$\Rightarrow \alpha G_D \subseteq \Gamma_A \text{ for all } 0 < |\alpha| \leq 1 \Rightarrow \bigcup_{0 < |\alpha| \leq 1} \alpha G_D \subseteq \Gamma_A \Rightarrow$$

$$\bigcup_{0 \leq |\alpha| \leq 1} \alpha G_D \subseteq \Gamma_A, \text{ as } 0G_D(0) \subseteq 0\Gamma_A(0) = \Gamma_A(0).$$

Again $\bigcup_{0 \leq |\alpha| \leq 1} \alpha G_D$ is balanced by last theorem, So $\bigcup_{0 \leq |\alpha| \leq 1} \alpha G_D \subseteq C$.

Then $G_D \subseteq C$.

Therefore for any fuzzy soft subset $\mathbf{G}_D$ of $\cap_{|\lambda| \geq 1} \lambda \Gamma_A$ we have $\mathbf{G}_D \subseteq \mathbf{C}$. So,
 $\cap_{|\lambda| \geq 1} \lambda \Gamma_A \subseteq \mathbf{C}$.

Hence $\mathbf{C} = \cap \{\lambda \Gamma_A : |\lambda| \geq 1\}$.

Example (4.3.10): Let $X = \mathbb{N}$, $E = \mathbb{R}$ and $A = E$. Let Γ_A be a fuzzy soft set defined by $\Gamma_A(\chi) = \{\lfloor |\chi| \rfloor, \lfloor |\chi| \rfloor + 1, \lfloor |\chi| \rfloor + 2, \dots\}$ for $\chi \in A$ and $\chi \neq 0$, then $\Gamma_A(\chi) = \{2, 3, 4, \dots\}$ for $\chi = 0$, where $\lfloor |\chi| \rfloor$ is the greatest integer less than equal to $|\chi|$.

Let $D = [2, \infty) \cup (-\infty, -2]$ and Γ_A be a fuzzy soft set defined by $\Gamma_A(\chi) = \Gamma_A(\chi)$ for all $\chi \in D$.

$\Gamma_A(\chi) = \{2, 3, 4, \dots\}$ for $\chi \in A \mid D = (-2, 2)$

It is easy to see that $\Gamma_A(y) \subseteq \Gamma_A(x)$ if $x, y \in A$ with $|x| \leq |y|$

We now show that Γ_A is a balanced fuzzy soft set. Let $\chi \in A$.

If $0 < |\lambda| \leq 1$, then $\lambda \Gamma_A(\chi) = \Gamma_A\left(\frac{1}{\lambda} \chi\right) \subseteq \Gamma_A(\chi)$, as $|\chi| \leq \left|\frac{1}{\lambda} \chi\right|$.

If $\lambda = 0$, then

Case 1. for $x \neq 0$, we have $0 \Gamma_A(\chi) = \emptyset \subseteq \Gamma_A(\chi)$.

Case 2. for $x = 0$, we have

$$\begin{aligned} 0 \Gamma_A(0) &= \bigcup_{e \in E} \Gamma_A(e) = \left\{ \bigcup_{e \in D} \Gamma_A(e) \right\} \cup \left\{ \bigcup_{e \in A \setminus D} \Gamma_A(e) \right\} \\ &= \left\{ \bigcup_{e \in D} \Gamma_A(e) \right\} \cup \{2, 3, 4, \dots\} = \{2, 3, 4, \dots\} \\ &= \Gamma_A(0) \end{aligned}$$

Thus, Γ_A is a balanced fuzzy soft set.

We now show that Γ_A is the greatest balanced fuzzy soft set contained in Γ_A .

Let Γ_P be any balanced fuzzy soft set contained in Γ_A . Then obviously, $P \subseteq A$.

Let $x \in P$. Then either $x \in D$ or $x \in A|D$. If $x \in D$, then $\Gamma_P(x) \subseteq \Gamma_A(x) = \Gamma_A(x)$.

Again $\cup_{e \in P} \Gamma_P(e) = 0\Gamma_P(0) \subseteq \Gamma_P(0) \subseteq \Gamma_A(0) = \{2, 3, 4, \dots\}$.

So, if $x \in A|D$, then $\Gamma_P(x) \subseteq \{2, 3, 4, \dots\} = \Gamma_A(x)$. Thus, $\Gamma_P(x) \subseteq \Gamma_A(x)$ for all $x \in P$ and so, Γ_A

is the greatest balanced fuzzy soft set contained in Γ_A . Therefore the balanced core of Γ_A is Γ_A .

Remark (4.3.11): In the above example the balanced core $\Gamma_A \neq \cap \{\lambda\Gamma_A : |\lambda| \geq 1\}$. In fact, if $x = 1$, then

$$\Gamma_A(1) = \{2, 3, 4, \dots\} \text{ and } (\cap_{|\lambda| \geq 1} \lambda\Gamma_A)(1) = (\cap_{|\lambda| \geq 1} \Gamma_A)\left(\frac{1}{\lambda}\right) = \{0, 1, 2, 3, 4, \dots\}$$

Again

$0\Gamma_A(0) = \cup_{e \in E} \Gamma_A(e) = \{0, 1, 2, 3, 4, \dots\} \neq \Gamma_A(0)$. So, if $0\Gamma_A(0) \neq \Gamma_A(0)$, then the balanced core of Γ_A may not be equal to $\cap \{\lambda\Gamma_A : |\lambda| \geq 1\}$.

Theorem (4.3.13): Let X be an universe set, E_1 and E_2 be linear spaces over a field F , $A \subseteq E_1$, $B \subseteq E_2$ and $f: X \rightarrow Y$ be a linear function.

1. If Γ_A is a balanced fuzzy soft set over (X, E_1) , then $f(\Gamma_A)$ is a balanced fuzzy soft set over (X, E_2) .

2. If Γ_B is a balanced fuzzy soft set over (X, E_2) , then $f^{-1}(\Gamma_B)$ is also a balanced fuzzy soft set over (X, E_1) .

Proof :

1. Let $\lambda \in \Gamma$ such that $|\lambda| \leq 1$. We now verify that $\lambda f(\Gamma_A) \subseteq f(\Gamma_A)$.

2. Let $y \in E_2$.

Case 1. $\lambda = 0$.

Subcase 1. If $y = 0$, then

$$\begin{aligned} \mathbf{0}f(\Gamma_A)(\mathbf{0}) &= \bigcup_{z \in E_2} f(\Gamma_A)(z) = z \in E_2 \bigcup_{\chi \in f^{-1}(z)} \Gamma_A(\chi) = \bigcup_{\chi \in X} \Gamma_A(\chi) \\ &= \mathbf{0}\Gamma_A(\mathbf{0}) \subseteq \Gamma_A(\mathbf{0}) \subseteq \bigcup_{e \in f^{-1}(\mathbf{0})} \Gamma_A(e) = f(\Gamma_A)(\mathbf{0}) \end{aligned}$$

Subcase 2. If $y \neq \mathbf{0}$, then of $(\Gamma_A)(y\mathbf{0} = \phi \subseteq f(\Gamma_A)(y))$.

Case 2. $\lambda \neq \mathbf{0}$.

$$\begin{aligned} \lambda f(\Gamma_A)(y) &= f(\Gamma_A)(\lambda^{-1}y) = \bigcup_{\chi \in f^{-1}(\lambda^{-1}y)} \Gamma_A(\chi) = \bigcup_{w \in f^{-1}(y)} \Gamma_A(\lambda^{-1}w) \\ &= \bigcup_{w \in f^{-1}(y)} \lambda \Gamma_A(w) \subseteq \bigcup_{w \in f^{-1}(y)} \Gamma_A(w) = f(\Gamma_A)(y), \end{aligned}$$

Where $\lambda\chi = w$, i. e. $\chi = \lambda^{-1}w$

Thus, from case 1 and case 2, we have $\lambda f(\Gamma_A)(e) \subseteq f(\Gamma_A)(e)$ for every $|\lambda| \leq \mathbf{1}$ and $y \in E_2$.

Hence, $\lambda f(\Gamma_A) \subseteq f(\Gamma_A)$ for all $|\lambda| \leq \mathbf{1}$. This completes the proof.

2. Let $\lambda \in \Gamma$ such that $|\lambda| \leq \mathbf{1}$. We now verify that $\lambda f^{-1}(\Gamma_B) \in f^{-1}(\Gamma_B)$.

Let $\chi \in E_1$.

Case 1 $\lambda = \mathbf{0}$.

Subcase 1. If $x = \mathbf{0}$, then

$$\begin{aligned} \mathbf{0}f^{-1}(\Gamma_B)(\mathbf{0}) &= z \in E_1 f^{-1}(\Gamma_B)(z) = \bigcup_{z \in E_1} \Gamma_B(f(z)) \subseteq \bigcup_{y \in E_2} \Gamma_B(y) \\ &= \mathbf{0}\Gamma_B(\mathbf{0}) \subseteq \Gamma_B(\mathbf{0}) \\ &= \Gamma_B(f(\mathbf{0})) = f^{-1}(\Gamma_B(\mathbf{0})) \end{aligned}$$

Subcase 2. If $x \neq \mathbf{0}$, then $\mathbf{0}f^{-1}(\Gamma_B)(\chi) = \phi \subseteq f^{-1}(\Gamma_B)(\chi)$.

Case 2. $\lambda \neq \mathbf{0}$, $\mathbf{6} = \mathbf{0}$.

$$\begin{aligned}
\lambda f^{-1}(\Gamma_B)(\chi) &= f^{-1}(\Gamma_B)(\lambda^{-1}\chi) = \Gamma_B(f(\lambda^{-1}\chi)) = \Gamma_B(\lambda^{-1}f(\chi)) \\
&= \lambda \Gamma_B(f(\chi)) \subseteq \Gamma_B(f(\chi)) \\
&= f^{-1}(\Gamma_B)(\chi)
\end{aligned}$$

Thus, from case 1 and case 2, we get $\lambda f^{-1}(\Gamma_B)(\chi) \subseteq f^{-1}(\Gamma_B)(\chi)$ for every $|\lambda| \leq 1$ and $\chi \in X$.

Hence, $\lambda f^{-1}(\Gamma_B) \subseteq f^{-1}(\Gamma_B)$ for all $|\lambda| \leq 1$. This completes the proof.

4.4 Absorbing fuzzy soft sets

Absorbing fuzzy soft sets are a further extension of soft set theory, where the focus is on the absorptive nature of elements within sets. In this framework, elements can absorb other elements, leading to a hierarchical structure within the set. This absorption mechanism allows for the representation of complex relationships and dependencies between elements, making absorbing fuzzy soft sets particularly useful in modeling systems where elements interact and influence each other's behavior.

Definition (4.4.1): Let X be an universe set, E be a linear space over a field F . A fuzzy soft set Γ_A over X is called an absorbing fuzzy soft set if $\bigcup_{\lambda > 0} \lambda \Gamma_A = A$, where $\Gamma(e) = X$ for all $e \in A$.

Example (4.4.2): Let $X = \mathbf{R}, E = \mathbf{R}$ and $0 \in A \subseteq E$ such that $A \cap (-1, 1) \setminus \{0\} \neq \emptyset$ and for $e \in A$, soft set defined by

$\Gamma(e) = \{\lfloor |e| \rfloor + 1, \lfloor |e| \rfloor + 2, \lfloor |e| \rfloor + 3, \dots\}$, where $\lfloor |e| \rfloor$ is the greatest integer less than equal to $|e|$.

Then obviously $\bigcup_{\lambda > 0} \lambda \Gamma_A = \tilde{A}$ and so Γ_A is an absorbing fuzzy soft set.

Example (4.4.3): Let $Y = \mathbf{N}, E = \mathbf{R}$. Let B be an unbounded (both below and above) subset of E containing 0 and Γ_B be a fuzzy soft set defined by $\Gamma(e) = \{0, 1, 2, \dots, \lfloor |e| \rfloor\}$ for $e (\neq 0) \in B$ and $\Gamma(0) = Y$, where $\lfloor |e| \rfloor$ is

the greatest integer less than equal to $|e|$. Then obviously $\bigcup_{\lambda > 0} \lambda \Gamma_B = \tilde{B}$ and so Γ_B is an absorbing fuzzy soft set.

Remark (4.4.4): Consider the absorbing fuzzy soft set F_A of the example 2.27 and the absorbing fuzzy soft set Γ_B of the example 2.28. If $A \cap B = \{0\}$, then obviously $\Gamma_A \cap \Gamma_B = X$. Again if $\{0\} \subset A \cap B$, let $e (\neq 0) \in A \cap B$.

Then $(\Gamma \cap \Gamma)(e) = \Gamma(e) \cap \Gamma(e) = \phi$. Hence, $\Gamma_A \cap \Gamma_B$ is not an absorbing fuzzy soft set. Thus, intersection of two absorbing fuzzy soft sets may or may not be an absorbing fuzzy soft set.

Theorem (4.4.5): Union of two absorbing fuzzy soft sets is an absorbing fuzzy soft set.

Example (4.4.6): Consider the fuzzy soft set Γ_A as defined in example (8.25) and we take $A = E$. Here it is obvious that $0\Gamma_A(0) = \Gamma_A(0)$. Therefore the balanced core of Γ_A is $\bigcap_{|\lambda| \geq 1} \lambda \Gamma_A$. Let $e \in E$. Now

$$\begin{aligned} |\lambda| \geq 1 \lambda \Gamma(e) &= \bigcap_{|\lambda| \geq 1} (\lambda \Gamma)(e) = \bigcap_{|\lambda| \geq 1} \Gamma(\lambda^{-1}e) = |\lambda| \\ &\geq 1\{\llbracket |\lambda^{-1}e| \rrbracket + 1, \llbracket |\lambda^{-1}e| \rrbracket + 2, \llbracket |\lambda^{-1}e| \rrbracket + 3, \dots\} \\ &= \{\llbracket |e| \rrbracket + 1, \llbracket |e| \rrbracket + 2, \llbracket |e| \rrbracket + 3, \dots\} = \Gamma(e) \end{aligned}$$

Since Γ_A is absorbing fuzzy soft set, $\bigcap_{|\lambda| \geq 1} \lambda \Gamma_A$ is absorbing fuzzy soft set.

Example(4.4.7): Consider the fuzzy soft set Γ_B as defined in example (8.26). Here it is obvious that $0\Gamma_A(0) = \Gamma_A(0)$. Therefore the balanced core of Γ_B is $\bigcap_{|\lambda| \geq 1} \lambda \Gamma_B$. Let $e \in E$. If $e \neq 0$, then

$$\begin{aligned} \bigcap_{|\lambda| \geq 1} \lambda \Gamma(e) &= \bigcap_{|\lambda| \geq 1} (\lambda \Gamma)(e) = \bigcap_{|\lambda| \geq 1} \Gamma(\lambda^{-1}e) = \bigcap_{|\lambda| \geq 1} \{0, 1, 2, \dots, \llbracket |\lambda^{-1}e| \rrbracket\} \\ &= \{0\} \end{aligned}$$

If $e = 0$, then

$$\bigcap_{|\lambda| \geq 1} \lambda \Gamma(\mathbf{0}) = \bigcap_{|\lambda| \geq 1} (\lambda \Gamma)(\mathbf{0}) = \bigcap_{|\lambda| \geq 1} \Gamma(\mathbf{0}) = \mathbf{Y}$$

So, it is clear that balanced core of $\Gamma_{\mathbf{B}}$ is not an absorbing fuzzy soft set.

Remark

Balanced core of an absorbing fuzzy soft set may or may not be an absorbing fuzzy soft set.

Theorem (4.4.8): Let $\mathbf{X}$ be an universe set , $\mathbf{E}_1$ and $\mathbf{E}_2$ be linear spaces over a field F , $\mathbf{A} \subseteq \mathbf{E}_1$, $\mathbf{B} \subseteq \mathbf{E}_2$ and $\mathbf{f}: \mathbf{X} \rightarrow \mathbf{Y}$ be a linear function.

1.If $\Gamma_{\mathbf{A}}$ is an absorbing fuzzy soft set over $(\mathbf{X}, \mathbf{E}_1)$, then $\mathbf{f}(\Gamma_{\mathbf{A}})$ is an absorbing fuzzy soft set over $(\mathbf{X}, \mathbf{E}_2)$.

2.If $\Gamma_{\mathbf{B}}$ is an absorbing fuzzy soft set over $(\mathbf{X}, \mathbf{E}_2)$, then $\mathbf{f}^{-1}(\Gamma_{\mathbf{B}})$ is also an absorbing fuzzy soft set over $(\mathbf{X}, \mathbf{E}_1)$.

Proof :

1. Let $\mathbf{y} \in \mathbf{E}_2$. Now

$$\begin{aligned} \left(\bigcup_{\lambda > 0} \lambda \mathbf{f}(\Gamma_{\mathbf{A}}) \right)(\mathbf{y}) &= \bigcup_{\lambda > 0} (\lambda \mathbf{f}(\Gamma_{\mathbf{A}}))(\mathbf{y}) = \bigcup_{\lambda > 0} \mathbf{f}(\Gamma_{\mathbf{A}})(\lambda^{-1} \mathbf{y}) \\ &= \bigcup_{\lambda > 0} \bigcup_{\chi \in \mathbf{f}^{-1}(\lambda^{-1} \mathbf{y})} \Gamma_{\mathbf{A}}(\chi) = \bigcup_{\lambda > 0} \bigcup_{\lambda \chi \in \mathbf{f}^{-1}(\mathbf{y})} \Gamma_{\mathbf{A}}(\chi) = \bigcup_{\lambda > 0} \bigcup_{\mathbf{z} \in \mathbf{f}^{-1}(\mathbf{y})} \Gamma_{\mathbf{A}}(\lambda^{-1} \mathbf{z}), \end{aligned}$$

where $\lambda \mathbf{x} = \mathbf{z}$

$(\bigcup_{\lambda > 0} \lambda \mathbf{f}(\Gamma_{\mathbf{A}}))(\mathbf{y}) \supseteq \bigcup_{\lambda > 0} \Gamma_{\mathbf{A}}(\lambda^{-1} \mathbf{z}) = \bigcup_{\lambda > 0} \lambda \Gamma_{\mathbf{A}}(\mathbf{z}) = \mathbf{X}$, for some $\mathbf{z} \in \mathbf{f}^{-1}(\mathbf{y})$, as $\Gamma_{\mathbf{A}}$ is an absorbing fuzzy soft set.

Therefore, $\bigcup_{\lambda > 0} \lambda \mathbf{f}(\Gamma_{\mathbf{A}}) = \tilde{\mathbf{X}}$. Hence, $\mathbf{f}(\Gamma_{\mathbf{A}})$ is an absorbing fuzzy soft set.

2. Let $\chi \in \mathbf{E}_1$. Now

$$\left(\bigcup_{\lambda > 0} \lambda \mathbf{f}^{-1}(\Gamma_{\mathbf{B}}) \right)(\chi) = \bigcup_{\lambda > 0} (\lambda \mathbf{f}^{-1}(\Gamma_{\mathbf{B}}))(\chi) = \bigcup_{\lambda > 0} \mathbf{f}^{-1}(\Gamma_{\mathbf{B}})(\lambda^{-1} \chi)$$

$$= \bigcup_{\lambda > 0} \Gamma_B(\lambda^{-1}f(\chi)) = \bigcup_{\lambda > 0} \lambda \Gamma_B(f(\chi)) = X,$$

as Γ_B is an absorbing fuzzy soft set.

Thus, $\bigcup_{\lambda > 0} \lambda f^{-1}(\Gamma_B) = \tilde{\mathbf{1}}$. Hence, $f^{-1}(\Gamma_B)$ is an absorbing fuzzy soft set.

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المستخلص

الهدف الرئيسي من هذه الأطروحة هو دراسة التقاربات وأنواع التقاربات في جبر باناخ الناعم الضبابي على نطاق واسع ونظرية الثنائية في جبر باناخ الناعم الضبابي، ومساحة جبر باناخ الناعم الضبابي ، ومساحة الوحدة الناعمة الضبابية و بعض تطبيقات المجموعات الناعمة الضبابية في جبر باناخ. هذه النتائج يتم عرضها في أربعة محاور: أولاً: تعريف المجموعات الضبابية، والأعداد الضبابية، و المجموعات الناعمة الضبابية وتعريف جبر باناخ الناعم الضبابي. ثانياً: تم دراسة التقارب في جبر باناخ الناعم الضبابي وأنواع التقاربات في جبر باناخ.

ثالثاً: من خلال جبر باناخ الناعم الضبابي تم تقديم مساحة في جبر باناخ الناعم الضبابي، باستخدام هذه المساحة القياسية، حددنا دالة خطية ناعمة ضبابية ودراسة بعض خواصها.

وتم تقديم المساحة الثنائية الناعمة الضبابية والمؤثر الثنائي الناعم الضبابي ودراسة خصائصها. تم إثبات النظرية حول تمثيل دالة خطية ناعمة ضبابية بواسطة الضرب الداخلي في حالة ناعمة ضبابية في مساحات فضاء هيلبرت.

رابعاً: تم تقديم بعض تطبيقات جبر باناخ الضبابي الناعم ، مثل الفضاء الخطي الناعم الغامض وبعض التعاريف و الأمثلة، المجموعات المتوازنة الناعمة الضبابية وبعض خصائصها، و المجموعة المحدبة الناعمة الضبابية.



جمهورية العراق

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رسالة

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الماجستير في علوم الرياضيات

من قبل

قاسم علي هاتف الاعرجي

بأشراف

أ.د. نوري فرحان المياحي