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On Geometric Function Theory using Differential Subordinations and Quasi-Subordinations

A Thesis

Submitted to Council of the College of Science, University of Al-Qadisiyah as a partial Fulfillment of the Requirements for the Degree of Master of Science in Mathematics

By

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Supervised by

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2024 A.C .

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بِسْمِ اللَّهِ الرَّحْمَنِ الرَّحِيمِ

﴿ عِلْمُ الْإِنْسَانِ مَا لَمْ يَعْلَمْ ﴾

صدق الله العلي العظيم

سورة العلق / آية : 5



DEDICATION

*I would like to dedicate this thesis to
my dear mother and dear father, to my dear
brothers and sisters, to my dear wife, and to my
supervisor*

(Prof. Dr. Waggas Galib Atshan)

*For his enthusiasm and inspiration.
to my dear friends, to everyone who spared
effort in helping me, and to everyone who
contributed to receiving me, even with a letter,
in my academic life.*

Maitham





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First and foremost, thanks to Allah for everything I would like to express my sincere appreciation to my supervisor, (Prof. Dr. Waggas Galib Atshan), for giving me the major steps to go on exploring the subject, and sharing me the ideas related to my work. I also would like to express my deep thanks to all the members of staff of mathematics at Al-Qadisiyah University for their help.

Maitham



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Supervisor Certification

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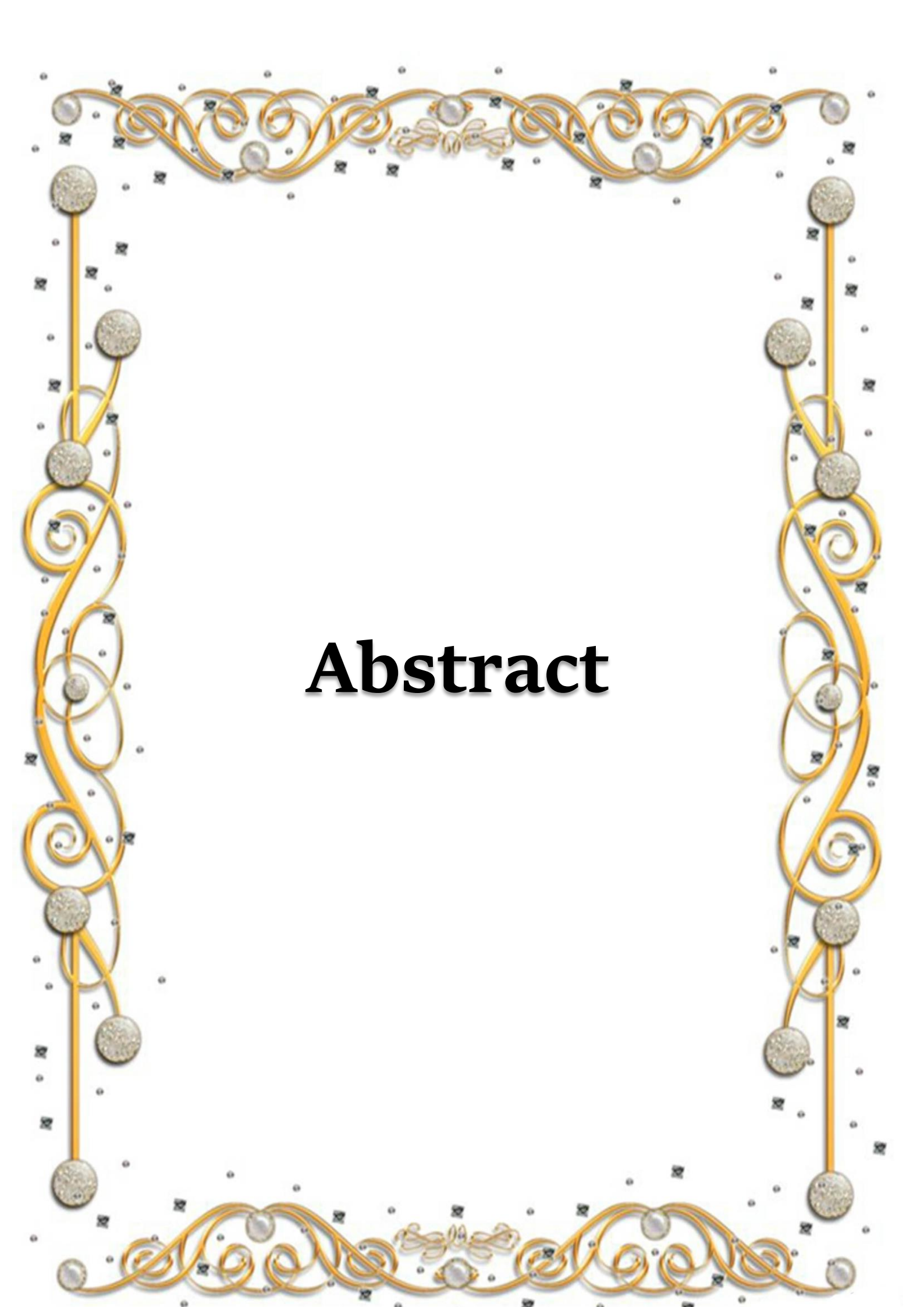
The Council of the college of Science met in its session / /
/ 2024 and decided to grant it a master's degree in Mathematics
sciences / complex analysis.

List of Publications

- 1) Results on New Subclasses for Bi-Univalent Functions Using Quasi-Subordination, Journal of Al-Qadisiyah for Computer Science and Mathematics, 16(2) (2024), PP. Math 70-77.
- 2) Coefficient Bounds and Fekete-Szegő problem of Subclasses of Bi-Univalent Functions, AIP Conference proceedings (Scopus: CiteScore2023: 0.5) (2th The National University of Science and Technology, International Conference for Engineering Sciences (NUSTPAS-2024), (2024), (Accepted for publication).
- 3) New Results on Differential Subordinations and Superordinations for Analytic Univalent Functions Defined by a Differential Operator, Nonlinear Functional Analysis and Applications, (Scopus: CiteScore2023: 1.9) (2024), (Submitted for publication).
- 4) On Differential Sandwich Results for Analytic Meromorphic Multivalent Functions Defined by a Linear Operator, Communications on Applied Nonlinear Analysis, (Scopus: CiteScore2023: 0.3) (2024) (Submitted for publication).

List of Symbols

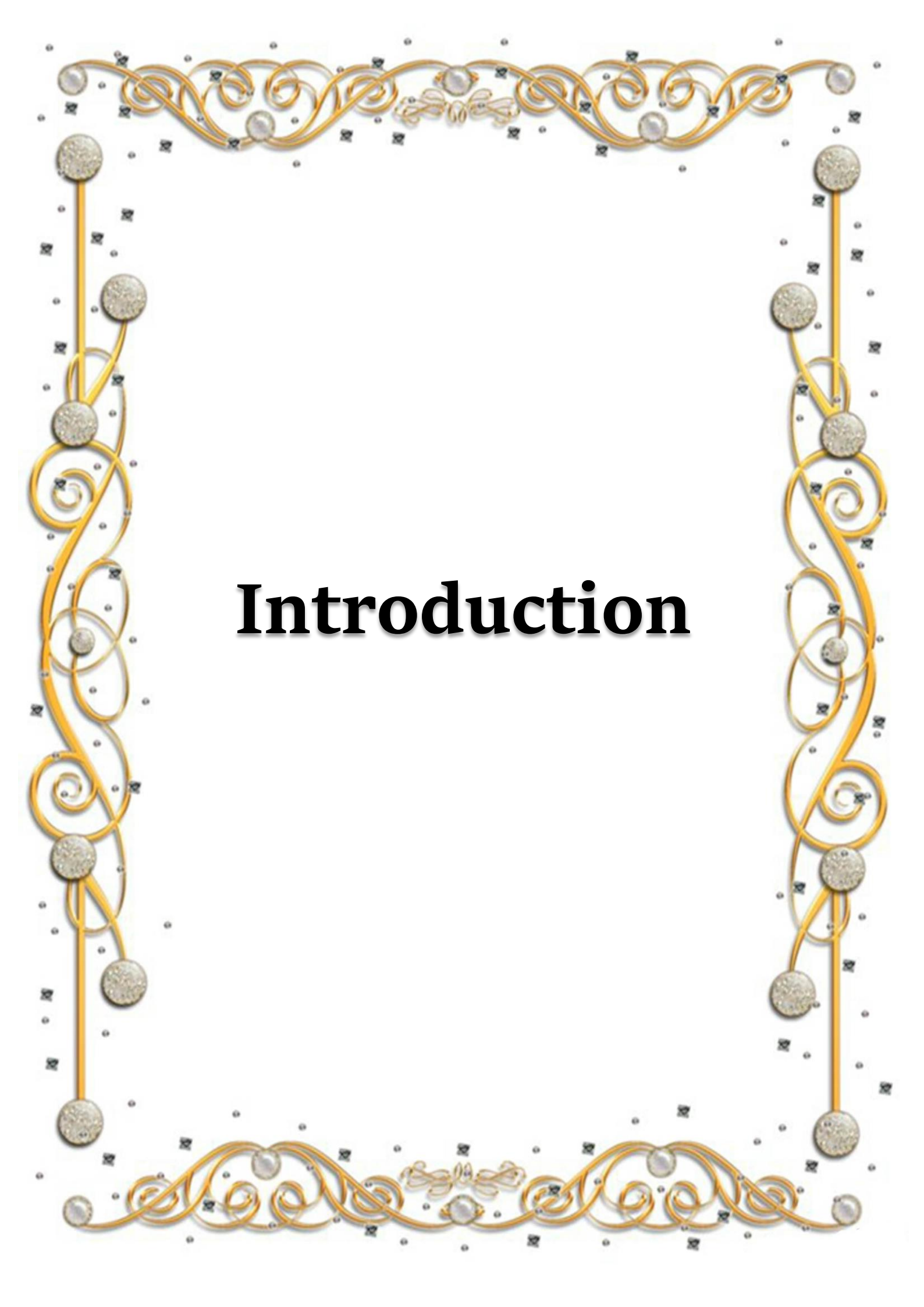
Symbol	Description
$\mathbb{C}$	Complex plane
U	Open unit disk $\{z: z \in \mathbb{C}: z < 1\}$.
U^*	Open punctured unit disk $\{z \in \mathbb{C}: 0 < z < 1\}$
$\bar{U}$	$U \cup \partial U = \{z: z \in \mathbb{C}: z \leq 1\}$.
∂U	The boundary of unit disk U , $\{z \in \mathbb{C}: z = 1\}$
$\mathbb{N}$	Set of natural numbers.
$\mathbb{N}_0$	$\mathbb{N} \cup \{0\}$.
$\mathbb{R}$	Set of all real numbers.
Re	The real part of a complex number.
$f * g$	Hadamard product (or convolution) of functions f and g .
$\mathcal{C}(\beta)$	The class of all convex functions of order β in U .
$\mathcal{S}^*(\beta)$	The class of all starlike functions of order β in U .
$H = H(U)$	Class of analytic functions in U .
$H[a, n]$	Class of analytic functions in U of the form: $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots$
$w(z)$	Schwarz function.
$f \prec g$	f is subordinate to g .
$\mathcal{A}$	Class of all normalized analytic functions in U of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$
$\mathcal{S}$	Class of all normalized univalent functions in U
Σ	Class of bi-univalent functions in U .
$\mathcal{P}$	The family of all functions h which analytic in U and $Re(h(z)) > 0$.
M_p	Class of Meromorphic p -valent functions in U^* of the form: $f(z) = z^{-p} + \sum_{n=1-p}^{\infty} a_n z^n$.
$\ll$	Majorization
$f \prec_q g$	f quasi-subordinate to g
$\Psi_{\Sigma}(\delta, \tau; \alpha)$ $D_{\Sigma}(\delta, \tau; \beta)$ $M_{\Sigma}^q(\beta, \rho, \lambda)$ $W_{\Sigma}^q(\tau, \mu, \eta, \phi)$	New classes of bi-univalent functions in U .

A decorative border surrounds the central text. It features gold-colored swirling lines, silver-colored spheres of various sizes, and small black dots. The border is more ornate at the top and bottom, with larger swirls and clusters of spheres, and simpler on the sides with vertical lines and spaced-out spheres.

Abstract

Abstract

The purpose of this thesis is discussing geometric function theory using differential subordinations and quasi-subordinations. It is study of new results of differential subordinations and superordinations for analytic univalent functions defined by a differential operator. We obtain some sandwich theorems for a subclass of univalent functions defined by the differential operator $\mathcal{N}_{\alpha,\lambda}^n f(z)$. Also, differential sandwich results for meromorphic multivalent functions defined by a linear operator are studied. We introduced sandwich results for meromorphic p -valent functions using differential subordination and superordination results defined by the linear operator $\mathfrak{M}_{\alpha,\beta,p,\eta}^{\gamma,k,\mu} f(z)$. Our thesis discussed coefficient bounds and Fekete-Szegő problem of subclasses of bi-univalent functions and this study focuses on two kinds of bi-univalent functions $\Psi_{\Sigma}(\delta, \tau; \alpha)$ and $D_{\Sigma}(\delta, \tau; \beta)$. We obtain estimates for the Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions in these subclasses, in addition, we provide a solution to the Fekete-Szegő problem for functions in these specific classes. Also, our thesis dealt with results on new subclasses for bi-univalent functions using quasi-subordination and the present study introduced two specific subclasses $M_{\Sigma}^q(\beta, \rho, \lambda)$ and $W_{\Sigma}^q(\tau, \mu, \eta, \phi)$ of bi-univalent functions that are defined by quasi-subordination, we obtain estimates for the Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions within these subclasses. We derived coefficients bounds of subclass of bi-univalent functions associated with Horadam polynomials. Here, we used the Horadam polynomials to introduce a new class of analytic and bi-univalent functions defined in the open unit disk U and establish upper bounds for the second and third coefficients of functions belonging to this new subclass.

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Introduction

Introduction

Geometric Function Theory is a fairly old branch of mathematics, specifically complex analysis catch the attention of several mathematicians and researchers for its geometrical aspects and abundant avenues for research work. The study of univalent and multivalent functions is one of the major branches of Geometric Function Theory and aids in developing complex analysis. One of the essential dilemmas in the study of univalent functions is whether there exists an univalent mapping from a simply connected domain onto a given simply connected domain. However, in view of Riemann Mapping Theorem [31] above simply connected domain such as, starlike, convex and close-to-convex etc.

Let $\mathcal{A}$ refers to the class of all analytic functions on the open unit disk $U = \{z \in \mathbb{C}: |z| < 1\}$ normalized by the conditions $f(0) = 0$ and $f'(0) = 1$, also let S stand for subclass of $\mathcal{A}$ consisting of univalent functions. In 1916, Bieberbach [31] studied the second coefficient a_2 of a function $f \in S$. He demonstrated that $|a_2| \leq 2$, with equality if and only if f is rotation of the Koebe function and he also stated " $|a_n| \leq n$ is generally valid ". This statement is known as the Bieberbach conjecture. This conjecture was a challenge to several mathematicians for many years; it motivated them to develop a variety of new methods in the complex analysis. In 1923, Löwner [46] proved Bieberbach conjecture for $n = 3$, many investigation have been made regards the Bieberbach conjecture for specific values of n . Eventually, the conjecture was established by de Branges [23] in the Summer of 1985 subordination between analytic functions initiated by Littlewood [44,45], where Rogosinski [59,60] introduced the term and established the basic results involving subordination. Lately, investigations to a variety of interesting properties of the generalized hypergeometric function were made by

Srivastava and Owa [69] via applying the concept of subordination. Research scholars and mathematicians, internationally recognized, like, Ruscheweyh, Srivastava, Miller, Mocanu, Duren, Silverman, Owa, Jahangiri et al., have opened new avenues in the field of Complex Analysis, specifically in geometric function theory.

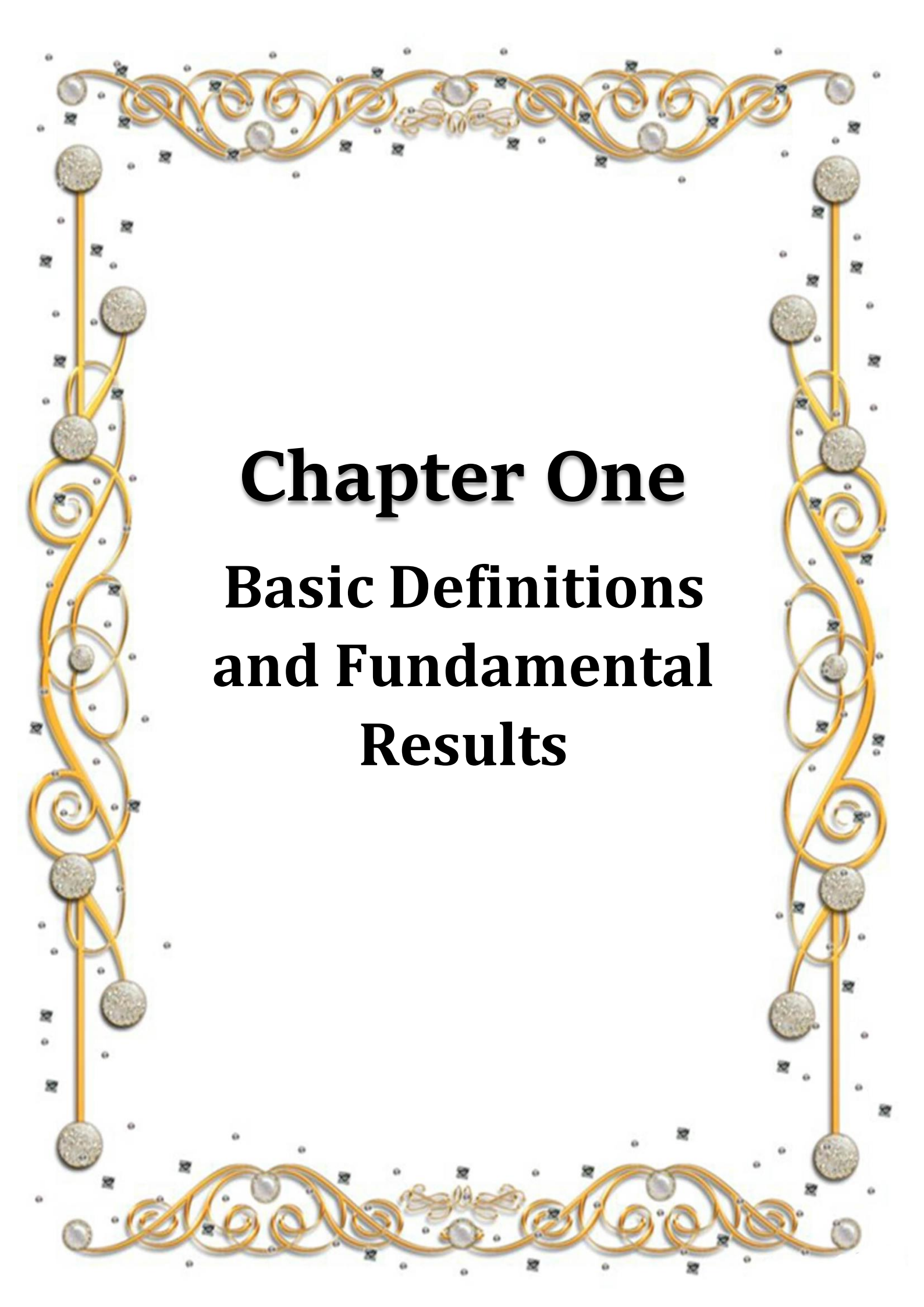
The current work unveils differential subordination and superordination, differential sandwich theorems, new subclasses of bi-univalent functions defined by subordination and quasi-subordination to obtain coefficients estimates, Fekete-Szegő problem.

The first chapter introduces a list of all relevant definitions of analytic, univalent, multivalent (p-valent) and bi-univalent functions, other definitions, some examples, applications of conformal mapping and basic results which are needed during chapters for research.

Chapter two is devoted for the study of new results on sandwich theorems using differential subordination and superordination for subclasses of analytic univalent and meromorphic multivalent functions. This chapter is divided into two sections. The first section deals with new results of differential subordinations and superordinations for analytic univalent functions defined by a differential operator. We obtain some sandwich theorems for a subclass of univalent functions defined by the differential operator $\mathcal{N}_{\alpha,\lambda}^n f(z)$. Section two is concerned with the differential sandwich results for meromorphic multivalent functions defined by a linear operator. We introduced sandwich results for meromorphic p-valent functions using differential subordination and superordination results defined by the linear operator $\mathfrak{M}_{\alpha,\beta,p,\eta}^{\gamma,k,\mu} f(z)$.

Chapter three deals with new results and applications on coefficients estimates for subclasses of bi-univalent functions. This chapter is divided

into three sections. The first section discussed coefficient bounds and Fekete-Szegö problem of subclasses of bi-univalent functions. This study focuses on two kinds of bi-univalent functions $\Psi_{\Sigma}(\delta, \tau; \alpha)$ and $D_{\Sigma}(\delta, \tau; \beta)$. We obtain estimates for the Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions in these subclasses. In addition, we provide a solution to the Fekete-Szegö problem for functions in these specific classes. Section two deals with results on new subclasses for bi-univalent functions using quasi-subordination. The present study introduces two specific subclasses $M_{\Sigma}^q(\beta, \rho, \lambda)$ and $W_{\Sigma}^q(\tau, \mu, \eta, \phi)$ of bi-univalent functions that are defined by quasi-subordination. We obtain estimates for the Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions within these subclasses. Section three deals with coefficients bounds of subclass of bi-univalent functions associated with Horadam polynomials. Here, we use the Horadam polynomials to introduce a new class of analytic and bi-univalent functions defined in the open unit disk U and establish upper bounds for the second and third coefficients of functions belonging to this new subclass.

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Chapter One

Basic Definitions and Fundamental Results

Chapter One

Basic Definitions and Fundamental Results

Introduction

This chapter includes two sections, the first section, deals with general notations that we need later, some fundamental definitions of analytic functions, univalent, bi-univalent and meromorphic functions which are needed in subsequent chapters for research. In section two, basic lemmas and theorems, have been mentioned. The detailed proof and complete discussions may be found in standard texts such as Goodman [34] Miller and Mocanu [51,52,53], Duren [31] and other references.

1.1 Basic Definitions

In this section, states some fundamental definitions that are stated in classical studies of complex analysis. Also, some principle notions are introduced in univalent function theory. This section starts by recalling the following definitions.

Definition (1.1.1) [31]: Suppose that $U = \{z: z \in \mathbb{C}: |z| < 1\}$ denotes the open unit disk in the complex plane $\mathbb{C}$. A function f of a complex variable is said to be analytic at a point z_0 if its derivatives exist not only at z_0 but at each point z in some neighborhoods of z_0 . It is analytic in the unit disk U if it is analytic at every point in U . We say that f is entire function if it is analytic at every point in the complex plane $\mathbb{C}$.

Definition (1.1.2) [31]: Let $w = f(z)$ be a mapping defined in a domain $D \subseteq \mathbb{C}$ and let $z_0 \in D$. This mapping is said to be conformal at z_0 if it is one-to-one in a neighborhood of the point z_0 and for every pair of smooth paths γ_1 and γ_2 in D intersecting at z_0 the angle between γ_1 and γ_2 at z_0 is equal to the angle between the image paths $f(\gamma_1)$ and $f(\gamma_2)$ at $f(z_0)$ in magnitude and sense. A function $w = f(z)$ is said to be conformal in the domain $D \subseteq \mathbb{C}$ if it is conformal at each point of the domain.

Definition (1.1.3) [31]: A function f analytic in domain U is said to be univalent there, if it does not take the same value twice, that is $f(z_1) \neq f(z_2)$ for all pairs of distinct points z_1 and z_2 in U . In other words, f is one - to -one (or injective) mapping of U onto another domain.

As examples, the function $f(z) = z$ is univalent in U while $f(z) = z^2$ is not univalent in U .

A univalent function map U conformally onto a domain in the w -plane and known as conformal mapping. But the converse is not always true, for example for the function $f(z) = e^{2\pi z}$, $f'(z) \neq 0$ in $|z| < 1$.

Example (1.1.1) [34]: The function $f(z) = z + \frac{1}{7}z^6$ is univalent function in unit disk U (see Figure. (1.1.1)).

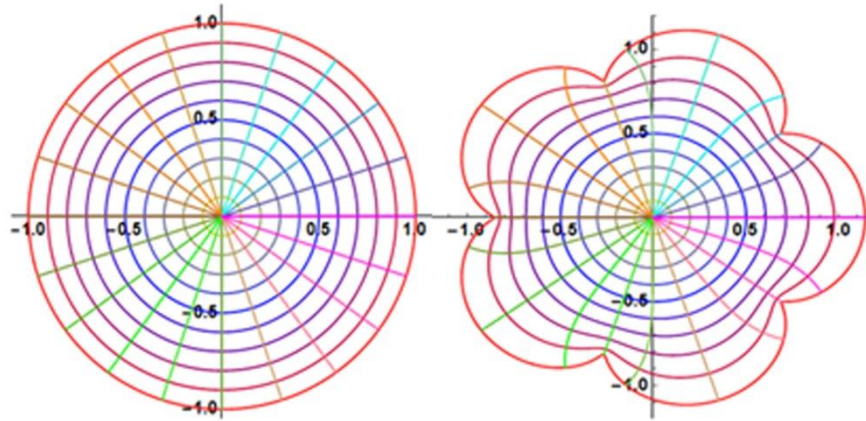


Figure (1.1.1)

Example (1.1.2) [34]: The function $f(z) = z^2 + \frac{2}{7}z^6$ is non-univalent function in unit disk U (see Figure. (1.1.2)).

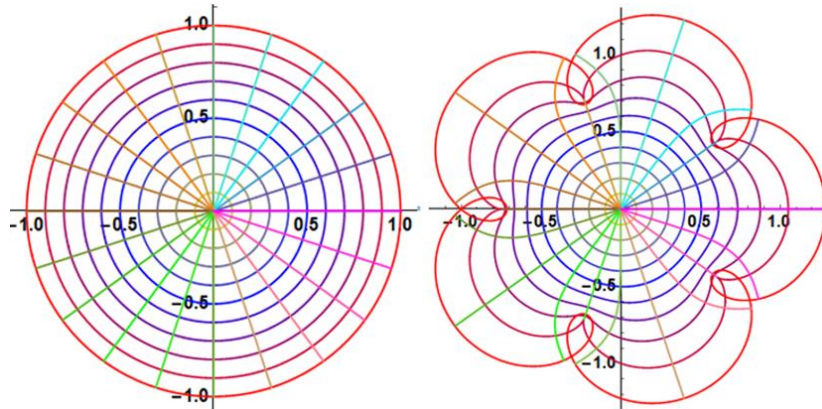


Figure (1.1.2)

We also deal with the functions which are meromorphic multivalent in the punctured unit disk $U^* = \{z \in \mathbb{C}: 0 < |z| < 1\}$. $f(z)$ is said to be meromorphic if it is analytic at every point in U except finite elements in U .

Definition (1.1.4) [31]: A function f is said to be locally univalent at a point $z_0 \in \mathbb{C}$ if it is univalent in some neighborhood of z_0 .

For analytic function f , the condition $f'(z_0) \neq 0$ is equivalent to local univalence at z_0 . f is locally univalent throughout a region D if f is locally univalent at every $z \in D$. In other words, the locally univalent function is also namely as conformal mapping because of its angle and sense preserving property.

Note that $f(z) = z^2$ is not locally univalent at $z = 0$ since $f(z) = f(-z)$ for all z . However, f is locally univalent at all points $z \neq 0$.

Definition (1.1.5) [34]: The functions f of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad n \in \mathbb{N} \quad (1.1)$$

are said to be in the subclass S of class $\mathcal{A}$ if it is analytic and univalent in the open unit disk U , where $\mathcal{A}$ is a class of all analytic functions of the form (1.1).

Definition (1.1.6) [31]: Let M_p denote the class of functions f of the form:

$$f(z) = \frac{1}{z^p} + \sum_{n=1-p}^{\infty} a_n z^n, \quad (z \in U^*), \quad (1.2)$$

which are meromorphic multivalent in the punctured unit disk $U^* = \{z \in \mathbb{C}: 0 < |z| < 1\}$.

Definition (1.1.7) [34]: We say that $f \in \mathcal{A}$ is normalized if f satisfies the conditions $f(0) = 0$ and $f'(0) = 1$.

Geometrically, the normalization $f(0) = 0$ amounts to only a translation of the image domain and $f'(0) = 1$ corresponds to rotation and stretching or shrinking of the image domain.

Definition (1.1.8) [31]: If functions f and g belonging to the class $\mathcal{A}$, given by

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j, \quad g(z) = z + \sum_{j=2}^{\infty} b_j z^j$$

then the Hadamard product or (convolution) of functions f and g denoted by $f * g$ is defined by

$$(f * g)(z) = z + \sum_{j=2}^{\infty} a_j b_j z^j = (g * f)(z), (z \in U, j \in \mathbb{N}).$$

Definition (1.1.9) [34]: A set $E \subseteq \mathbb{C}$ is said to be starlike with respect to $w_0 \in E$ if the linear segment joining w_0 to every other point $w \in E$ lies entirely in E . In a more picturesque language, the requirement is that every point of E be visible from w_0 . The set E is said to be convex if it is starlike with respect to each of its points, that is, if the linear segment joining any two points of E lies entirely in E .

Definition (1.1.10) [31]: A function $f \in \mathcal{A}$ is said to be starlike function of order β if

$$\operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) > \beta, (z \in U; 0 \leq \beta < 1). \quad (1.3)$$

Denotes the class of all starlike functions of order β in U by $S^*(\beta)$ and S^* the class of all starlike functions of order 0, $S^*(0) = S^*$. Geometrically, we can say that a starlike function is conformal mapping of the unit disk onto a starlike domain with respect to the origin. For example, the function

$$f(z) = \frac{z}{(1-z)^{2(1-\beta)}},$$

is starlike function of order β .

Definition (1.1.11) [31]: A function $f \in A$ is said to be convex function of order β if

$$\operatorname{Re} \left(1 + \frac{zf''(z)}{f'(z)} \right) > \beta, (0 \leq \beta < 1, z \in U). \quad (1.4)$$

Denotes the class of all convex functions of order β in U by $C(\beta)$ and C for the convex functions of order 0 by $C(0) = C$.

We note that $C(\beta) \subset S^*(\beta)$.

Definition (1.1.12) [34]: A Möbius transformation, or called a bilinear transformation, is a rational function $f: \mathbb{C} \rightarrow \mathbb{C}$ of the form: $f(z) = \frac{az+b}{cz+d}$, where $a, b, c, d \in \mathbb{C}$ are fixed and $ad - bc \neq 0$. This transformation can be considered as a combination of the transformations of translation, rotation, stretching and inversion.

Example (1.1.3) [34]: Perhaps the most important member of S is the Koebe function which is given by

$$K(z) = \frac{z}{(1-z)^2}.$$

We can compute the Maclaurin series for k by differential the series for $\frac{1}{(1-z)^2}$ and then multiplying by z .

$$K(z) = \frac{z}{(1-z)^2} = \sum_{n=1}^{\infty} nz^2 = z + 2z^2 + 3z^3 + \dots,$$

and maps the unit disk to the complement of the ray $(-\infty, -\frac{1}{4}]$. This can be verified by writing

$$K(z) = \frac{1}{4} \left(\frac{1+z}{1-z} \right)^2 - \frac{1}{4}.$$

Note that for this function $a_n = n$ for all n . We will show that the image of U under k is a slit domain that is a domain consisting of the entire complex plane except that a slit is cut out of it. To determine $k(U)$, consider the next sequence of functions:

$$u_1(z) = \frac{1+z}{1-z}, \quad u_2(z) = \frac{1}{4}u_1^2(z), \quad u_3(z) = u_2(z) - \frac{1}{4},$$

where noting that $\frac{1+z}{1-z}$ maps the unit disk conformally onto the right half-plane $\{Re\{z\} > 0\}$ (see Figure. (1.1.3)).

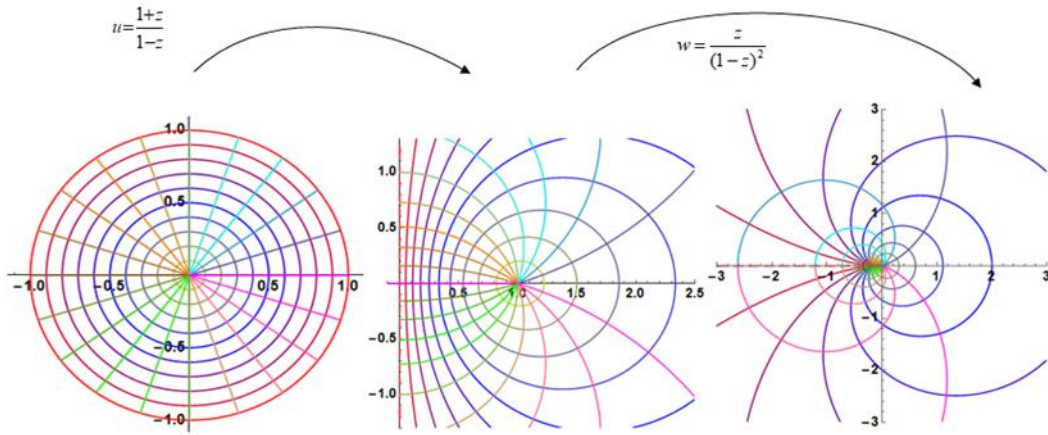


Figure (1.1.3): The Koebe function maps U conformally onto $\mathbb{C} \setminus (-\infty, -\frac{1}{4}]$.

$$\text{Now } u_3 \circ u_2 \circ u_1(z) = \frac{1}{4} \left(\frac{1+z}{1-z} \right)^2 - \frac{1}{4} = \frac{z}{(1-z)^2}.$$

Note that u_1 is the Möbius transformation that functions maps U onto the right half-plane whose boundary is the imaging axis. Also, u_2 is the squaring function, while u_3 translates the image one space to the left and then multiplies it by a factor of $\frac{1}{4}$.

Example (1.1.4) [34]: The special function $L_o(z) = \frac{1+z}{1-z}$ is a convex function because it maps U onto a half-plane. The Koebe function,

$$k(z) = \frac{z}{(1-z)^2} = z + 2z^2 + 3z^3 + \cdots nz^n + \cdots,$$

is a starlike function. In fact the domain $k(U)$ is starlike with respect to each $w_0 > -\frac{1}{4}$.

Definition (1.1.13) [31]: Let $f: \mathbb{R} \rightarrow \mathbb{C}$ be a function. We call f a Schwarz function, if for all $c \in \mathbb{R}$, $n \in \mathbb{N} \cup \{0\}$, then $|f^{(n)}(x)| = O(|z|^c)$, where "capital " O " is defined as follows:

let $\{a_n\}$ and $\{b_n\}$ be any two sequences and $b_n \geq 0$ for all n . If there exists a fixed number $\eta > 0$ such that $a_n \leq \eta b_n$ (for all n), then we write $a_n = O(b_n)$.

Definition (1.1.14) [51]: If f and g are two analytic functions in U , we say that f is subordinate to g or g is said to be superordinate to f , written symbolically as follows :

$$f < g \quad \text{or} \quad f(z) < g(z), \quad (z \in U),$$

if there exists a Schwarz function $w(z)$, with $w(0) = 0$ and $|w(z)| < 1$ ($z \in U$), such that $f(z) = g(w(z))$, ($z \in U$).

If the function g is univalent in U , we have the following equivalence

$$f(z) < g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U).$$

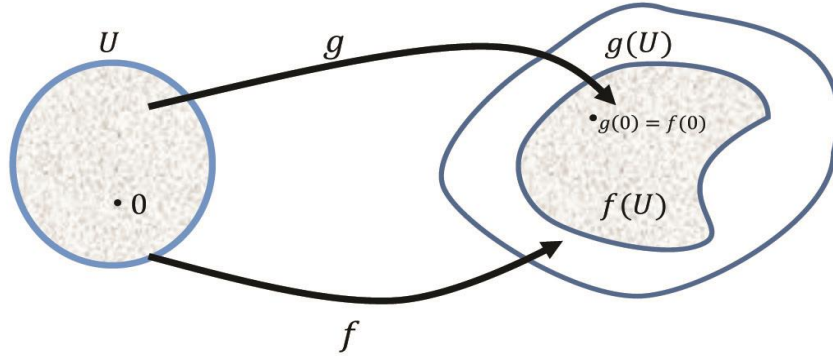


Figure (1.1.4): Subordination

Definition(1.1.15) [51]: Let $\phi: \mathbb{C}^3 \times U \rightarrow \mathbb{C}$ and $h(z)$ be univalent in U . If $p(z)$ is analytic in U and satisfies the second-order differential subordination:

$$\phi(p(z), z p'(z), z^2 p''(z); z) < h(z), \quad (1.5)$$

then $p(z)$ is called a solution of the differential subordination (1.5), and the univalent function $q(z)$ is called a dominant of the solutions of the differential subordination (1.5) or more simply dominant if $p(z) < q(z)$

for all $p(z)$ satisfying (1.5). A univalent dominant $\tilde{q}(z)$ that satisfies $\tilde{q}(z) \prec q(z)$ for all dominant $q(z)$ of (1.5) is said to be the best dominant is unique up to a relation of U .

Definition(1.1.16) [51]: Let $p, h \in \mathcal{A}$ and $\emptyset(r, s, t; z): \mathbb{C}^3 \times U \rightarrow \mathbb{C}$. If p and $\emptyset(p(z), z p'(z), z^2 p''(z); z)$ are univalent functions in U and if p satisfies the second-order differential superordination:

$$h(z) \prec \emptyset(p(z), z p'(z), z^2 p''(z); z), \quad (1.6)$$

then p is called a solution of the differential superordination (1.6). An analytic function $q(z)$ which is called a subordinator of the solutions of the differential superordination (1.6), or more simply a subordinator if $q \prec p$ for all the functions p satisfying (1.6). A univalent subordinator $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all the subordinants q of (1.6) is said to be the best subordinator.

Definition (1.1.17) [51]: Denote by Q the set of all functions f that are analytic and injective on $\bar{U} \setminus E(f)$, where $\bar{U} = U \cup \{z \in \partial U\} = \{z \in \mathbb{C}: |z| \leq 1\}$, and

$$E(f) = \left\{ \zeta \in \partial U: \lim_{z \rightarrow \zeta} f(z) = \infty \right\} \quad (1.7)$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(f)$. Further, let the subclass of Q for which $f(0) = a$ be denoted by $Q(a)$, $Q(0) = Q_0$ and $Q(1) \equiv Q_1 = \{f \in Q: f(0) = 1\}$.

Definition (1.1.18) [52]: An analytic function f is quasi- subordination to analytic function G in the open unit disk U is written

$$f(z) \prec_q G(z),$$

if there exist analytic functions φ and w , with $|\varphi(z)| \leq 1, w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = \varphi(z)G(w(z))$, for all $z \in U$.

Note that, when $\varphi(z) = 1$, then $f(z) = G(w(z))$ so that $f(z) \prec G(z)$ in U .

Furthermore if $w(z) = z$, then $f(z) = \varphi(z)G(z)$ and this case f is majorized by G , written $f(z) \ll G(z)$ in U . Therefore, it is obvious that quasi - subordination is a generalization of subordination as well as of majorization.

Definition (1.1.19) [31]: An analytic function f is said to be bi-univalent in a domain U , if f and its inverse f^{-1} are both univalent in U . The class of all bi-univalent analytic functions f in U is denoted by Σ .

As examples, the functions $f(z) = \frac{z}{1-z}$, $f(z) = \log \frac{1}{1-z}$, $f(z) = \log \sqrt{\frac{1+z}{1-z}}$ are bi-univalent in U while the familiar Koebe function is not bi-univalent in U . Other common examples of functions in U such as

$$f(z) = \frac{2z-z^2}{2} \text{ and } f(z) = \frac{z}{1-z^2}.$$

1.2 Fundamental Results

The following lemmas and theorems are essential and has been used of the our principal results in the next chapters

Lemma (1.2.1) [27]: Let q be a convex univalent function in U and let $\alpha \in \mathbb{C}$, $\beta \in \mathbb{C} \setminus \{0\}$ with $q(0) = 1$ and suppose that

$$\operatorname{Re} \left\{ 1 + \frac{zq''(z)}{q'(z)} \right\} > \max \left\{ 0, -\operatorname{Re} \left(\frac{\alpha}{\beta} \right) \right\}.$$

If p is analytic in U and

$$\alpha p(z) + \beta zp'(z) \prec \alpha q(z) + \beta zq'(z), \quad (1.8)$$

then $p \prec q$ and q is the best dominant of (1.8).

Lemma (1.2.2) [51]: Let q be univalent in the unit disk U and let θ and ϕ be analytic in a domain D containing $q(U)$ with $\phi(w) \neq 0$, when $w \in q(U)$. Set $Q(z) = zq'(z)\phi(q(z))$ and $h(z) = \theta(q(z)) + Q(z)$. Suppose that

i. $Q(z)$ is starlike univalent in U ,

ii. $\operatorname{Re} \left\{ \frac{zh'(z)}{Q(z)} \right\} > 0$ for $z \in U$.

If p is analytic in U , with $p(0) = q(0)$, $p(U) \subseteq D$ and

$$\theta(p(z)) + zp'(z)\phi(p(z)) < \theta(q(z)) + zq'(z)\phi(q(z)), \quad (1.9)$$

then $p < q$ and q is the best dominant of (1.9).

Lemma (1.2.3) [53]: Let q be a convex univalent in U and $q(0) = 1$. Let $\beta \in \mathbb{C}$, that $\operatorname{Re}(\beta) > 0$. If $p \in H[q(0), 1] \cap Q$ and $p(z) + \beta zp'(z)$ is univalent in U , then

$$q(z) + \beta zq'(z) < p(z) + \beta zp'(z), \quad (1.10)$$

which implies that $q < p$ and q is the best subdominant of (1.10).

Lemma (1.2.4) [27]: Let q be a convex univalent function in the unit disk U and let θ and ϕ be analytic in a domain D containing $q(U)$. Suppose that

$$\text{i. } \operatorname{Re} \left\{ \frac{\theta'(q(z))}{\phi(q(z))} \right\} > 0 \text{ for } z \in U,$$

$$\text{ii. } Q(z) = zq'(z)\phi(q(z)) \text{ is starlike univalent in } U.$$

If $p \in H[q(0), 1] \cap Q$, with $p(U) \subset D$, $\theta(p(z)) + zp'(z)\phi(p(z))$ is univalent in U and

$$\theta(q(z)) + zq'(z)\phi(q(z)) < \theta(p(z)) + zp'(z)\phi(p(z)), \quad (1.11)$$

then $q < p$ and q is the best subdominant of (1.11).

Lemma (1.2.5) [31]: If $\mathcal{P}$ be the class of analytic functions p normalized by $p(z) = 1 + \sum_{k=1}^{\infty} c_k z^k$ and $\operatorname{Re}(p(z)) > 0$, ($z \in U$), then $|c_k| \leq 2, k \geq 1$.

Lemma (1.2.6) [41]: Let w be analytic function in U , with $w(0) = 0, |w(z)| < 1$ and $w(z) = w_1 z + w_2 z^2 + w_3 z^3 + \dots$. Then

$$|w_2 - \mu w_1^2| \leq \max\{1, |\mu|\}, \quad \mu \in \mathbb{C}.$$

The result is sharp for the functions given by $w(z) = z$ or $w(z) = z^2$.

Lemma (1.2.7) [31]: (Schwarz Lemma)

Let f be analytic in U with $f(0) = 0$, and $|f(z)| < 1$ in U . Then $|f'(0)| \leq 1$ and $|f(z)| \leq |z|$ in U . Strict inequality holds in both estimates unless f is a rotation of the disk $f(z) = e^{i\theta} z$, ($0 \leq \theta < 2\pi$).

Theorem (1.2.8) [31]: (Alexander's Theorem)

Let f be an analytic function in U with $f(0) = f'(0) - 1 = 0$. Then $f \in \mathcal{C}$ if and only if $zf' \in S^*$.

Theorem (1.2.9) [31]: (Bieberbach Conjecture)

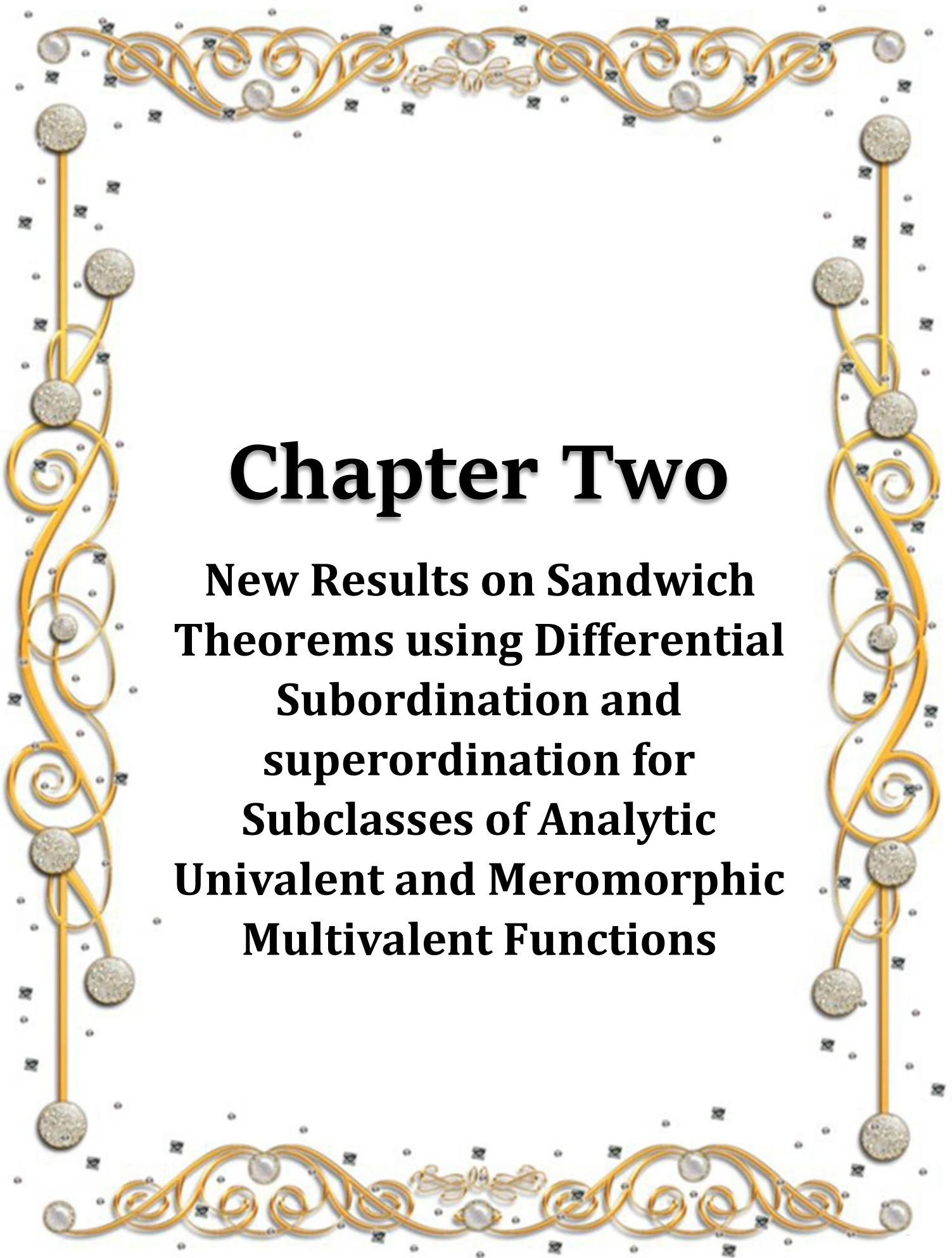
The coefficients of each $f \in \mathcal{A}$ satisfy $|a_n| \leq n$ for $n = 2, 3, \dots$. The strict inequality holds for all n unless f is the Koebe function or one of its rotation.

Theorem (1.2.10)[31]: (Maximum Modulus Theorem)

Suppose that a function f is continuous on boundary of U (U any disk or region). Then, the maximum value of $|f(z)|$, which is always reached, occurs somewhere on the boundary of U and never in the interior.

Theorem(1.2.11)[45]: (Littlewood's Theorem)

For the constant e , the coefficients of each function $f \in \mathcal{A}$ satisfy $|a_n| \leq en$ for $n = 2, 3, \dots$.

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Chapter Two

**New Results on Sandwich
Theorems using Differential
Subordination and
superordination for
Subclasses of Analytic
Univalent and Meromorphic
Multivalent Functions**

Chapter Two

New Results on Sandwich Theorems using Differential Subordination and superordination for Subclasses of Analytic Univalent and Meromorphic Multivalent Functions

Introduction:

In [51] Miller and Mocanu extended the study of differential inequalities of real-valued functions to complex-valued functions defined in the open unit disk U . They developed a new method in geometric function theory known as the method of differential subordination or the method of admissible functions. This method is very effective to obtain new results. Following Miller and Mocanu [52,53], Bulboacă [25] and others [11,55,57] studied different classes of analytic functions, by means of differential subordination and superordination. In this chapter, the focus is primarily on the applications of differential subordination and superordination to univalent and meromorphic multivalent functions with sandwich theorems.

This chapter consists of two sections. The first section is concerned with the new results of differential subordinations and superordinations for analytic univalent functions defined by a differential operator. Here, we derived sandwich theorems for analytic univalent functions using differential subordination and superordination results defined by the differential operator $\mathcal{N}_{\alpha,\lambda}^n f(z)$.

Section two deals with the differential sandwich results for analytic meromorphic multivalent functions defined by a linear operator. We introduced sandwich results for meromorphic multivalent functions using differential subordination and superordination theorems defined by linear operator $\mathfrak{M}_{\alpha,\beta,p,\eta}^{\gamma,k,\mu} f(z)$.

2.1 New Results On Differential Subordinations and Superordinations for Analytic Univalent Functions Defined by a Differential Operator

Let $H = H(U)$ be the class of analytic functions in the open unit disk $U = \{z \in \mathbb{C}: |z| < 1\}$. For $n \in \mathbb{N}$ and $a \in \mathbb{C}$. Let $H[a, n]$ be the subclass of H of the form:

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots \quad (a \in \mathbb{C}).$$

Let $\mathcal{A}$ denote the subclass of H of functions f of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (z \in U), \quad (2.1)$$

which are analytic in the open unit disk $U = \{z \in \mathbb{C}: |z| < 1\}$. We define the Hadamard product (or convolution) of

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad \text{and} \quad g(z) = z + \sum_{k=2}^{\infty} b_k z^k$$

$$\text{as } (f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g * f)(z), \quad (z \in U).$$

Let f and g are analytic functions in H , f is said to be subordinate to g , or g is said to be superordinate to f in U and write $f < g$, if there exists a Schwarz function K in U , such that $K(0) = 0$, and $|K(z)| < 1$, ($z \in U$), where $f(z) = g(K(z))$. In such a case we write $f < g$ or $f(z) < g(z)$ ($z \in U$). If g is univalent in U , then $f < g$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$. ([52,53]).

Several authors [1,2,16,52,65] obtained sufficient conditions on the functions h, p and ϕ for which the following implication holds:

$$h(z) < \phi(p(z), zp'(z), z^2p''(z); z),$$

then

$$q(z) < p(z). \quad (2.2)$$

Using the results (see [3,4,6,12,13, 26,53]) to obtain sufficient conditions for normalized analytic functions to satisfy:

$$q_1(z) < \frac{zf'(z)}{f(z)} < q_2(z),$$

where q_1 and q_2 are given univalent functions in U and $q_1(0) = q_2(0) = 1$. Also, several authors (see[1,3,12,13,15,17]) derived some differential subordination and superordination results with some sandwich theorems.

For $f \in \mathcal{A}$, $\alpha \in \mathbb{N}_0, \lambda \geq 0, n \in \mathbb{N}_0, (\mathbb{N}_0 = \mathbb{N} \cup \{0\})$. Darus and Al-Shaqsi proposed the differential operator in [48], $R_{\alpha, \lambda}^n: \mathcal{A} \rightarrow \mathcal{A}$ defined by

$$R_{\alpha, \lambda}^n f(z) = z + \sum_{k=2}^{\infty} [1 + \lambda(k-1)]^n C(\alpha, k) a_k z^k, (z \in U, f \in \mathcal{A}), \quad (2.3)$$

where

$$C(\alpha, k) = \binom{k+\alpha-1}{\alpha}, \quad (z \in U). \quad (2.4)$$

By generalizing Sălăgean operator, Al-Oboudi [5] introduced the following operator $D_\alpha^n: \mathcal{A} \rightarrow \mathcal{A}$ defined by

$$D_\lambda^n f(z) = z + \sum_{k=2}^{\infty} [1 + \lambda(k-1)]^n a_k z^k. \quad (2.5)$$

The differential operators $R_{\alpha,\lambda}^n$ and D_λ^n were used to generate another convoluted differential operators in [47]. Both of them have a convoluted form as follows :

$$\mathcal{N}_{\alpha,\lambda}^n f(z) : \mathcal{A} \rightarrow \mathcal{A}, \quad (2.6)$$

where

$$\begin{aligned} \mathcal{N}_{\alpha,\lambda}^n f(z) &= R_{\alpha,\lambda}^n f(z) * D_\lambda^n f(z) \\ &= \left(z + \sum_{k=2}^{\infty} [1 + \lambda(k-1)]^n C(\alpha, k) a_k z^k \right) * \left(z + \sum_{k=2}^{\infty} [1 + \lambda(k-1)]^n a_k z^k \right) \\ &= z + \sum_{k=2}^{\infty} [1 + \lambda(k-1)]^{2n} C(\alpha, k) a_k^2 z^k. \end{aligned} \quad (2.7)$$

It is easily seen from (2.7) that

$$z \left(\mathcal{N}_{\alpha,\lambda}^n f(z) \right)' = (\alpha + 1) \mathcal{N}_{\alpha+1,\lambda}^n f(z) - \alpha \mathcal{N}_{\alpha,\lambda}^n f(z). \quad (2.8)$$

The main object of the present investigation is to find sufficient conditions for certain normalized analytic function f to satisfy:

$$q_1(z) < \left[\frac{\mathcal{N}_{\alpha,\lambda}^n f(z)}{z} \right]^\gamma < q_2(z).$$

and

$$q_1(z) < \left[\frac{\mathcal{N}_{\alpha+1,\lambda}^n f(z)}{\mathcal{N}_{\alpha,\lambda}^n f(z)} \right]^\gamma < q_2(z),$$

where q_1 and q_2 are given univalent functions in U with $q_1(0) = q_2(0) = 1$, and $\gamma > 0$.

In this idea, we derive some sandwich theorems, involving the operator $\mathcal{N}_{\alpha,\lambda}^n f(z)$.

Theorem (2.2.1). Let $q(z)$ be convex univalent function in U with $q(0) = 1, 0 \neq \delta \in \mathbb{C}, \gamma > 0$ and suppose that q satisfies:

$$\operatorname{Re} \left\{ 1 + \frac{zq''(z)}{q'(z)} \right\} > \max \left\{ 0, -\operatorname{Re} \left(\frac{\gamma}{\delta} \right) \right\}. \quad (2.9)$$

If $f \in \mathcal{A}$ satisfies the subordination condition :-

$$\begin{aligned} \left[\frac{\mathcal{N}_{\alpha,\lambda}^n f(z)}{z} \right]^\gamma + \delta(\alpha + 1) \left[\frac{\mathcal{N}_{\alpha,\lambda}^n f(z)}{z} \right]^\gamma \left(\frac{\mathcal{N}_{\alpha+1,\lambda}^n f(z)}{\mathcal{N}_{\alpha,\lambda}^n f(z)} - 1 \right) \\ < q(z) + \frac{\delta}{\gamma} zq'(z), \end{aligned} \quad (2.10)$$

then

$$\left[\frac{\mathcal{N}_{\alpha,\lambda}^n f(z)}{z} \right]^\gamma < q(z), \quad (2.11)$$

and $q(z)$ is the best dominant of (2.10).

Proof. Define the function p by

$$p(z) = \left[\frac{\mathcal{N}_{\alpha,\lambda}^n f(z)}{z} \right]^\gamma, \quad (2.12)$$

then the function $p(z)$ is analytic in U and $p(0) = 1$, therefore, differentiating (2.12) with respect to z and using the identity (2.8) in the resulting equation, we obtain

$$\frac{zp'(z)}{p(z)} = \gamma \left[\frac{z \left(\mathcal{N}_{\alpha,\lambda}^n f(z) \right)'}{\mathcal{N}_{\alpha,\lambda}^n f(z)} - 1 \right]. \quad (2.13)$$

Hence

$$\frac{zp'(z)}{p(z)} = \gamma \left[(\alpha + 1) \left(\frac{\mathcal{N}_{\alpha+1,\lambda}^n f(z)}{\mathcal{N}_{\alpha,\lambda}^n f(z)} - 1 \right) \right].$$

Therefore,

$$\frac{zp'(z)}{\gamma} = \left[\frac{\mathcal{N}_{\alpha,\lambda}^n f(z)}{z} \right]^\gamma \left[(\alpha + 1) \left(\frac{\mathcal{N}_{\alpha+1,\lambda}^n f(z)}{\mathcal{N}_{\alpha,\lambda}^n f(z)} - 1 \right) \right].$$

The subordination (2.10) from the hypothesis becomes

$$p(z) + \frac{\delta}{\gamma} zp'(z) < q(z) + \frac{\delta}{\gamma} zq'(z).$$

By Lemma (1.2.1) with $\lambda = \frac{\delta}{\gamma}$ and $\alpha = 1$, we obtain (2.11).

Putting $q(z) = \left(\frac{1+Az}{1+Bz} \right)$, $(-1 \leq B < A \leq 1)$ in Theorem (2.1.1), we obtain the following corollary:

Corollary (2.1.1). Let $0 \neq \delta \in \mathbb{C}, \gamma > 0$ and

$$\operatorname{Re} \left\{ \frac{1 - Bz}{1 + Bz} \right\} > \max \left\{ 0, -\operatorname{Re} \left(\frac{\gamma}{\delta} \right) \right\}.$$

If $f \in \mathcal{A}$ satisfies the subordination condition :

$$\left[\frac{\mathcal{N}_{\alpha, \lambda}^n f(z)}{z} \right]^\gamma + \delta(\alpha + 1) \left[\frac{\mathcal{N}_{\alpha, \lambda}^n f(z)}{z} \right]^\gamma \left(\frac{\mathcal{N}_{\alpha+1, \lambda}^n f(z)}{\mathcal{N}_{\alpha, \lambda}^n f(z)} - 1 \right) < \frac{1 + Az}{1 + Bz} + \frac{\delta}{\gamma} \frac{(A - B)z}{(1 + Bz)^2},$$

then

$$\left[\frac{\mathcal{N}_{\alpha, \lambda}^n f(z)}{z} \right]^\gamma < \left(\frac{1 + Az}{1 + Bz} \right)$$

and $q(z) = \left(\frac{1 + Az}{1 + Bz} \right)$ is the best dominant.

Let $A = 1$ and $B = -1$, in corollary (2.1.1), we obtain following result.

Corollary (2.1.2). If $0 \neq \delta \in \mathbb{C}, \gamma > 0$ and

$$\operatorname{Re} \left\{ 1 + \frac{2z}{1 - z} \right\} > \max \left\{ 0, -\operatorname{Re} \left(\frac{\gamma}{\delta} \right) \right\}.$$

If $f \in \mathcal{A}$ satisfies the subordination

$$\left[\frac{\mathcal{N}_{\alpha, \lambda}^n f(z)}{z} \right]^\gamma + \delta(\alpha + 1) \left[\frac{\mathcal{N}_{\alpha, \lambda}^n f(z)}{z} \right]^\gamma \left(\frac{\mathcal{N}_{\alpha+1, \lambda}^n f(z)}{\mathcal{N}_{\alpha, \lambda}^n f(z)} - 1 \right) < \frac{1 - z^2 + 2 \frac{\delta}{\gamma} z}{(1 - z)^2},$$

then

$$\left[\frac{\mathcal{N}_{\alpha, \lambda}^n f(z)}{z} \right]^\gamma < \left(\frac{1 + z}{1 - z} \right),$$

and $q(z) = \left(\frac{1 + z}{1 - z} \right)$ is the best dominant.

Theorem (2.1.2). Assume that the function $q(z)$ be convex univalent function in U with $q(0) = 1$, $q'(z) \neq 0$ ($z \in U$).

If q satisfy the condition :

$$\operatorname{Re} \left\{ 1 + \frac{a}{\delta} q(z) + \frac{2b}{\delta} (q(z))^2 - z \frac{q'(z)}{q(z)} + z \frac{q''(z)}{q'(z)} \right\} > 0, \quad (2.14)$$

where $a, b \in \mathbb{C}$, $\delta \in \mathbb{C} \setminus \{0\}$ and $z \in U$. Suppose that $z \frac{q'(z)}{q(z)}$ is starlike univalent in U and $f \in \mathcal{A}$ satisfies

$$\psi(m, a, b, \gamma, \alpha, \lambda, n, \delta; z) < m + aq(z) + b(q(z))^2 + \delta z \frac{q'(z)}{q(z)}, \quad (2.15)$$

where,

$$\begin{aligned} \psi(m, a, b, \gamma, \alpha, \lambda, n, \delta; z) = m + \left[\frac{\mathcal{N}_{\alpha+1, \lambda}^n f(z)}{\mathcal{N}_{\alpha, \lambda}^n f(z)} \right]^\gamma & \left(a + b \left[\frac{\mathcal{N}_{\alpha+1, \lambda}^n f(z)}{\mathcal{N}_{\alpha, \lambda}^n f(z)} \right]^\gamma \right) + \\ \delta \gamma \left[(\alpha + 1) \left(\frac{\mathcal{N}_{\alpha+2, \lambda}^n f(z)}{\mathcal{N}_{\alpha+1, \lambda}^n f(z)} - \frac{\mathcal{N}_{\alpha+1, \lambda}^n f(z)}{\mathcal{N}_{\alpha, \lambda}^n f(z)} \right) \right], \end{aligned} \quad (2.16)$$

then

$$\left[\frac{\mathcal{N}_{\alpha+1, \lambda}^n f(z)}{\mathcal{N}_{\alpha, \lambda}^n f(z)} \right]^\gamma < q(z) \quad (2.17)$$

and $q(z)$ is the best dominant of (2.15).

Proof. Define the function $p(z)$ by

$$p(z) = \left[\frac{\mathcal{N}_{\alpha+1, \lambda}^n f(z)}{\mathcal{N}_{\alpha, \lambda}^n f(z)} \right]^\gamma, \quad (2.18)$$

then the function $p(z)$ is analytic in U and $p(0) = 1$. Differentiating (2.18) with respect to z and using the identity (2.8), we get,

$$\frac{zp'(z)}{p(z)} = \gamma \left[(\alpha + 1) \left(\frac{\mathcal{N}_{\alpha+2,\lambda}^n f(z)}{\mathcal{N}_{\alpha+1,\lambda}^n f(z)} - \frac{\mathcal{N}_{\alpha+1,\lambda}^n f(z)}{\mathcal{N}_{\alpha,\lambda}^n f(z)} \right) \right].$$

By setting $\theta(w) = m + aw + bw^2$ and $\phi(w) = \frac{\delta}{w}$, $w \neq 0$.

It is simple to see that $\theta(w)$ is analytic in $\mathbb{C}$, and $\phi(w)$ are analytic in $\mathbb{C} \setminus \{0\}$ and that $\phi(w) \neq 0$, $w \in \mathbb{C} \setminus \{0\}$ Also, we get

$$Q(z) = zq'(z)\phi(q(z)) = \delta z \frac{q'(z)}{q(z)}$$

and

$$h(z) = \theta(q(z)) + Q(z) = m + aq(z) + b(q(z))^2 + \delta z \frac{q'(z)}{q(z)}.$$

It is clear that $Q(z)$ is starlike univalent in U ,

$$Re \left\{ \frac{zh'(z)}{Q(z)} \right\} = Re \left\{ 1 + \frac{a}{\delta} q(z) + \frac{2b}{\delta} (q(z))^2 - z \frac{q'(z)}{q(z)} + z \frac{q''(z)}{q'(z)} \right\} > 0.$$

Through simple calculation, we find that

$$\psi(m, a, b, \gamma, \alpha, \lambda, n, \delta; z) =$$

$$m + \frac{a}{\delta} p(z) + \frac{2b}{\delta} (p(z))^2 - z \frac{p'(z)}{p(z)} + z \frac{p''(z)}{p'(z)}, \quad (2.19)$$

where $\psi(m, a, b, \gamma, \alpha, \lambda, n, \delta; z)$ is given by (2.16).

From (2.15) and (2.19), we have

$$m + ap(z) + b(p(z))^2 + \delta z \frac{p'(z)}{p(z)} <$$

$$m + aq(z) + b(q(z))^2 + \delta z \frac{q'(z)}{q(z)}. \quad (2.20)$$

Therefore, by Lemma (1.2.2), we get $p(z) < q(z)$.

Corollary (2.1.3). Let $-1 \leq B < A \leq 1$ and

$$\operatorname{Re} \left\{ 1 + \frac{a}{\delta} \left(\frac{1 + Az}{1 + Bz} \right) + \frac{2b}{\delta} \left(\frac{1 + Az}{1 + Bz} \right)^2 - \frac{2\beta z}{1 + Bz} + \frac{(A + B)z}{(1 + Bz)(1 + Az)} \right\} > 0,$$

where $m, a, b \in \mathbb{C}$, $\delta \in \mathbb{C} \setminus \{0\}$ and $z \in U$, if $f \in \mathcal{A}$ satisfies

$$\begin{aligned} & \psi(m, a, b, \gamma, \alpha, n; z) \\ & < m + a \frac{1 + Az}{1 + Bz} + b \left(\frac{1 + Az}{1 + Bz} \right)^2 + \delta z \frac{(A - B)z}{(1 + Az)(1 + Bz)}, \end{aligned}$$

where is given $\psi(m, a, b, \gamma, \alpha, \lambda, n, \delta; z)$ by (2.16), then

$$\left[\frac{\mathcal{N}_{\alpha+1, \lambda}^n f(z)}{\mathcal{N}_{\alpha, \lambda}^n f(z)} \right]^\gamma < \left(\frac{1 + Az}{1 + Bz} \right)$$

and $q(z) = \left(\frac{1 + Az}{1 + Bz} \right)$ is the best dominant.

Theorem (2.1.3). Assume that the function $q(z)$ is convex univalent function in U with $q(0) = 1$, $\lambda > 0$ and $\operatorname{Re}\{\delta\} > 0$. Let $f \in \mathcal{A}$ satisfies

$$\left[\frac{\mathcal{N}_{\alpha, \lambda}^n f(z)}{z} \right]^\gamma \in H[q(0), 1] \cap Q$$

and

$$\left[\frac{\mathcal{N}_{\alpha, \lambda}^n f(z)}{z} \right]^\gamma + \delta(\alpha + 1) \left[\frac{\mathcal{N}_{\alpha, \lambda}^n f(z)}{z} \right]^\gamma \left(\frac{\mathcal{N}_{\alpha+1, \lambda}^n f(z)}{\mathcal{N}_{\alpha, \lambda}^n f(z)} - 1 \right)$$

be univalent in U . If

$$q(z) + \frac{\delta}{\gamma} z q'(z) < \left[\frac{\mathcal{N}_{\alpha, \lambda}^n f(z)}{z} \right]^\gamma + \delta(\alpha + 1) \left[\frac{\mathcal{N}_{\alpha, \lambda}^n f(z)}{z} \right]^\gamma \left(\frac{\mathcal{N}_{\alpha+1, \lambda}^n f(z)}{\mathcal{N}_{\alpha, \lambda}^n f(z)} - 1 \right), \quad (2.21)$$

then

$$q(z) < \left[\frac{\mathcal{N}_{\alpha, \lambda}^n f(z)}{z} \right]^\gamma \quad (2.22)$$

and $q(z)$ is the best subordinant of (2.21).

Proof. Define the function $p(z)$ by

$$p(z) = \left[\frac{\mathcal{N}_{\alpha, \lambda}^n f(z)}{z} \right]^\gamma. \quad (2.23)$$

Differentiating (2.23) with respect to z , we get

$$\frac{zp'(z)}{p(z)} = \gamma \left[\frac{z \left(\mathcal{N}_{\alpha, \lambda}^n f(z) \right)'}{\mathcal{N}_{\alpha, \lambda}^n f(z)} - 1 \right]. \quad (2.24)$$

After some computations and using (2.8), from (2.24), we obtain

$$\begin{aligned} \left[\frac{\mathcal{N}_{\alpha, \lambda}^n f(z)}{z} \right]^\gamma + \gamma(\alpha + 1) \left[\frac{\mathcal{N}_{\alpha, \lambda}^n f(z)}{z} \right]^\gamma \left(\frac{\mathcal{N}_{\alpha+1, \lambda}^n f(z)}{\mathcal{N}_{\alpha, \lambda}^n f(z)} - 1 \right) \\ = p(z) + \frac{\delta}{\gamma} zp'(z). \end{aligned}$$

We will use Lemma (1.2.3), to prove our result. Therefore, we get $p < q$ and q is the best subordinant.

The proof is complete.

Putting $q(z) = \left(\frac{1+z}{1-z}\right)$ in Theorem (2.1.3), we obtain the following corollary:

Corollary (2.1.4). Let $\lambda > 0$ and $Re\{\delta\} > 0$. If $f \in \mathcal{A}$ satisfies

$$\left[\frac{\mathcal{N}_{\alpha+1,\lambda}^n f(z)}{\mathcal{N}_{\alpha,\lambda}^n f(z)} \right]^\gamma \in H[q(0), 1] \cap Q$$

and

$$\left[\frac{\mathcal{N}_{\alpha,\lambda}^n f(z)}{z} \right]^\gamma + \delta(\alpha + 1) \left[\frac{\mathcal{N}_{\alpha,\lambda}^n f(z)}{z} \right]^\gamma \left(\frac{\mathcal{N}_{\alpha+1,\lambda}^n f(z)}{\mathcal{N}_{\alpha,\lambda}^n f(z)} - 1 \right)$$

be univalent in U . If

$$\left(\frac{1 - z^2 + \frac{2\delta}{\gamma} z}{(1 - z)^2} \right) < \left[\frac{\mathcal{N}_{\alpha,\lambda}^n f(z)}{z} \right]^\gamma + \gamma(\alpha + 1) \left[\frac{\mathcal{N}_{\alpha,\lambda}^n f(z)}{z} \right]^\gamma \left(\frac{\mathcal{N}_{\alpha+1,\lambda}^n f(z)}{\mathcal{N}_{\alpha,\lambda}^n f(z)} - 1 \right),$$

then

$$\left(\frac{1 + z}{1 - z} \right) < \left[\frac{\mathcal{N}_{\alpha,\lambda}^n f(z)}{z} \right]^\gamma,$$

and $q(z) = \left(\frac{1+z}{1-z}\right)$ is the best subordinant.

Theorem (2.1.4). Assume that the function $q(z)$ be a convex univalent function in U with $q(0) = 1$, $q'(z) \neq 0$ and $\frac{zq'(z)}{q(z)}$ is starlike univalent in U . If $q(z)$ satisfy the next condition:

$$Re \left\{ \frac{q(z)}{\delta} (2bq(z) + a)q'(z) \right\} > 0, \quad (2.25)$$

where $a, b \in \mathbb{C}$, $\delta \in \mathbb{C} \setminus \{0\}$ and $z \in U$.

Let $f \in \mathcal{A}$ satisfies

$$\left[\frac{\mathcal{N}_{\alpha+1,\lambda}^n f(z)}{\mathcal{N}_{\alpha,\lambda}^n f(z)} \right]^\gamma \in H[q(0), 1] \cap Q,$$

and $\psi(m, a, b, \gamma, \alpha, \lambda, n, \delta; z)$ is univalent in U , where is given $\psi(m, a, b, \gamma, \alpha, \lambda, n, \delta; z)$ by (2.16). If

$$m + aq(z) + b(q(z))^2 + \delta z \frac{q'(z)}{q(z)} < \psi(m, a, b, \gamma, \alpha, \lambda, n, \delta; z), \quad (2.26)$$

then

$$q(z) < \left[\frac{\mathcal{N}_{\alpha+1,\lambda}^n f(z)}{\mathcal{N}_{\alpha,\lambda}^n f(z)} \right]^\gamma \quad (2.27)$$

and $q(z)$ is the best subordinant of (2.26).

Proof. Define the function $p(z)$ by

$$q(z) = \left[\frac{\mathcal{N}_{\alpha+1,\lambda}^n f(z)}{\mathcal{N}_{\alpha,\lambda}^n f(z)} \right]^\gamma. \quad (2.28)$$

Differentiating (2.28) with respect to z , we get,

$$\frac{zp'(z)}{p(z)} = \gamma \left[(\alpha + 1) \left(\frac{\mathcal{N}_{\alpha+2,\lambda}^n f(z)}{\mathcal{N}_{\alpha+1,\lambda}^n f(z)} - \frac{\mathcal{N}_{\alpha+1,\lambda}^n f(z)}{\mathcal{N}_{\alpha,\lambda}^n f(z)} \right) \right].$$

By setting $\theta(w) = m + aw + bw^2$ and $\phi(w) = \frac{\delta}{w}$, $w \neq 0$,

we see that $\theta(w)$ is analytic in $\mathbb{C}$, and $\phi(w)$ are analytic in $\mathbb{C} \setminus \{0\}$ and that $\phi(w) \neq 0$, $w \in \mathbb{C} \setminus \{0\}$. Also, as well, if we get,

$$Q(z) = zq'(z)\phi(q(z)) = \delta z \frac{q'(z)}{q(z)}.$$

It is clear that $Q(z)$ is starlike univalent in U . We get,

$$\operatorname{Re} \left\{ \frac{\theta'(q(z))}{\phi(z)} \right\} = \operatorname{Re} \left\{ \frac{q(z)}{\delta} (2bq(z) + a)q'(z) \right\} > 0.$$

Through simple calculation, we find

$$\psi(m, a, b, \gamma, \alpha, \lambda, n, \delta; z) = m + ap(z) + b(p(z))^2 + \delta z \frac{p'(z)}{p(z)}, \quad (2.29)$$

where $\psi(m, a, b, \gamma, \alpha, \lambda, n, \delta; z)$ is given by (2.16).

From (2.26) and (2.29), we have

$$m + aq(z) + b(q(z))^2 + \delta z \frac{q'(z)}{q(z)} < m + ap(z) + b(p(z))^2 + \delta z \frac{p'(z)}{p(z)}.$$

Therefore, by Lemma (1.2.4), we get $q(z) < p(z)$.

Theorem (2.1.5). Assume that the function $q_1(z)$ and $q_2(z)$ be convex univalent function in U with $q_1(0) = q_2(0) = 1$, $\gamma > 0$ and $\operatorname{Re}\{\delta\} > 0, \delta \in \mathbb{C} \setminus \{0\}$ and satisfies (2.9). Let $f \in \mathcal{A}$ satisfies

$$\left[\frac{\mathcal{N}_{\alpha, \lambda}^n f(z)}{z} \right]^\gamma \in H[1, 1] \cap \mathcal{Q}$$

and

$$\left[\frac{\mathcal{N}_{\alpha, \lambda}^n f(z)}{z} \right]^\gamma + \delta(\alpha + 1) \left[\frac{\mathcal{N}_{\alpha, \lambda}^n f(z)}{z} \right]^\gamma \left(\frac{\mathcal{N}_{\alpha+1, \lambda}^n f(z)}{\mathcal{N}_{\alpha, \lambda}^n f(z)} - 1 \right)$$

is univalent in U . If

$$q_1(z) + \frac{\delta}{\gamma} z q_1'(z) < \left[\frac{\mathcal{N}_{\alpha, \lambda}^n f(z)}{z} \right]^\gamma + \gamma(\alpha + 1) \left[\frac{\mathcal{N}_{\alpha, \lambda}^n f(z)}{z} \right]^\gamma \left(\frac{\mathcal{N}_{\alpha+1, \lambda}^n f(z)}{\mathcal{N}_{\alpha, \lambda}^n f(z)} - 1 \right) < \\ q_2(z) + \frac{\delta}{\gamma} z q_2'(z). \text{ Then}$$

$$q_1(z) < \left[\frac{\mathcal{N}_{\alpha,\lambda}^n f(z)}{z} \right]^\gamma < q_2(z),$$

and q_1 and q_2 are respectively the best subdominant and the best dominant.

Theorem (2.1.6). Suppose that $q_1(z)$ and $q_2(z)$ are convex univalent in U with $q_1(0) = q_2(0) = 1$, and let $q_1(z)$ satisfies (2.25) and $q_2(z)$ satisfies (2.14). Let $f \in \mathcal{A}$ satisfies:

$$\left[\frac{\mathcal{N}_{\alpha+1,\lambda}^n f(z)}{\mathcal{N}_{\alpha,\lambda}^n f(z)} \right]^\gamma \in H[1,1] \cap \mathcal{Q}$$

and $\psi(z)$ is univalent in U , where is given $\psi(m, a, b, \gamma, \alpha, \lambda, n, \delta; z)$ by (2.16). If

$$m + a q_1(z) + b (q_1(z))^2 + \delta z \frac{q_1'(z)}{q_1(z)} < \psi(z) < m + a q_2(z) + b (q_2(z))^2 + \delta z \frac{q_2'(z)}{q_2(z)},$$

then

$$q_1(z) < \left[\frac{\mathcal{N}_{\alpha+1,\lambda}^n f(z)}{\mathcal{N}_{\alpha,\lambda}^n f(z)} \right]^\gamma < q_2(z)$$

and q_1 and q_2 are respectively the best subdominant and the best dominant.

2.2 On Differential Sandwich Results for Analytic Meromorphic Multivalent Functions Defined by a Linear Operator

Let M_p denote the class of functions of the form:

$$f(z) = z^{-p} + \sum_{n=1-p}^{\infty} a_n z^n, \quad (2.30)$$

which are meromorphic multivalent in the punctured open unit disk $U^* = \{z: z \in \mathbb{C}, 0 < |z| < 1\} = U \setminus \{0\}$.

Many authors have studied meromorphic functions for another classes and conditions, see [16, 17, 56, 63].

Let H be a class of all analytic functions in U . For a positive integer number n and $a \in \mathbb{C}$, we define a subclass $H[a, n]$ of a class H as follows:

$$H[a, n] = \{f \in H: f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots\}.$$

For two analytic function $f, g \in M_p$, we say that f is subordinate to g in U and write $f(z) \prec g(z)$, if there exists a Schwarz function w , which is analytic in U with $w(0) = 0$ and $|w(z)| < 1, (z \in U)$, such that $f(z) = g(w(z))$, $(z \in U)$.

Furthermore, if the function g is univalent in U , we have the following equivalence relationship (cf. , e.g. [26,27,52,53]):

$$f(z) \prec g(z) \leftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U), (z \in U).$$

Mocanu and Miller [53] and other authors [1,2,3,6,12,13,15,16,17,26,35] and also [39,50,61,62,63,72,73] discovered sufficient conditions of the functions h, p , and θ for which the following implication holds:

$$h(z) \prec \theta(p(z), zp'(z), z^2 p''(z); z) \Rightarrow q(z) \prec p(z) (z \in U). \quad (2.31)$$

If $f \in M_p$ is given by (2.30) and $g \in M_p$ given by

$$g(z) = z^{-p} + \sum_{n=1-p}^{\infty} b_n z^n.$$

The convolution (or Hadamard product) of f and g is defined by

$$(f * g)(z) = z^{-p} + \sum_{n=1-p}^{\infty} a_n b_n z^n = (g * f)(z).$$

By using the results, (see [1,2,6,12,13,17,39,50,61,62,65,66,72,73]) to obtain sufficient conditions for normalized analytic functions to satisfy :

$$q_1(z) < \frac{zf'(z)}{f(z)} < q_2(z),$$

where q_1 and q_2 are given univalent functions in U with $q_1(0) = q_2(0) = 1$.

Also, Shanmugam et al. [65][66], as well as Goyal et al. [35], sandwich results for analytic function classes were recently acquired, (See also [2,3,6,13,27]).

The Mittag-Leffler function [54] $E_{\alpha}(\alpha \in \mathbb{C}, \operatorname{Re}(\alpha) > 0)$, is defined by

$$E_{\alpha}(z) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(n\alpha + 1)} z^n. \quad (2.32)$$

For a more generalization of the function $E_{\alpha}(z)$ is $E_{\alpha,\beta}(z)$ was introduced in [11] and given by:

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha + \beta)}, \quad (z \in \mathbb{C}). \quad (2.33)$$

Srivastava and Tomovski [68] generalized Mittag-Leffler function in the form:

$$E_{\alpha,\beta}^{\gamma,k}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{nk} z^n}{\Gamma(n\alpha + \beta)n!}, \quad (\alpha, \beta, \gamma, k \in \mathbb{C}, \operatorname{Re}(\alpha) > \max\{0, \operatorname{Re}(k) - 1\}, \operatorname{Re}(k) > 0).$$

$$(2.34)$$

Now, by using (2.34) Mostafa and Aouf [56] defined the function $K_{\alpha,\beta}^{\gamma,k}(z)$ by:

$$\begin{aligned} K_{\alpha,\beta,p}^{\gamma,k}(z) &= \Gamma(\beta)z^{-p}E_{\alpha,\beta}^{\gamma,k}(z) \\ &= z^{-p} + \sum_{n=1-p}^{\infty} \frac{\Gamma(k(n+p) + \gamma)\Gamma(\beta)z^n}{\Gamma(\alpha(n+p) + \beta)\Gamma(\gamma)(n+p)!} \quad (z \in \mathbb{C}, p \in \mathbb{N}), \end{aligned} \quad (2.35)$$

and defined the operator

$$\begin{aligned} K_{\alpha,\beta,p}^{\gamma,k}f(z) &= K_{\alpha,\beta,p}^{\gamma,k}(z) * f(z) \\ &= z^{-p} + \sum_{n=1-p}^{\infty} \frac{\Gamma((n+p)k + \gamma)\Gamma(\beta)}{\Gamma[\alpha(n+p) + \beta]\Gamma(\gamma)(n+p)!} a_n z^n. \end{aligned} \quad (2.36)$$

For $(\alpha, \beta, \gamma, k \in \mathbb{C}, \operatorname{Re}(\alpha) > \max\{0, \operatorname{Re}(k) - 1\}, \operatorname{Re}(k) > 0, \eta \geq 0, \text{ and } m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\})$.

The authors in [56] defined the operator

$\mathfrak{M}_{\alpha,\beta,p,\eta}^{\gamma,k,\mu}f(z): \mathbb{M}_p \rightarrow \mathbb{M}_p$ as follows:

$$\mathfrak{M}_{\alpha,\beta,p,\eta}^{\gamma,k,0}f(z) = K_{\alpha,\beta,p}^{\gamma,k}f(z),$$

$$\mathfrak{M}_{\alpha,\beta,p,\eta}^{\gamma,k,1}f(z) = (1 - \eta)K_{\alpha,\beta,p}^{\gamma,k}f(z) + \eta z^{-p}[z^{p+1}K_{\alpha,\beta,p}^{\gamma,k}f(z)]'$$

$$= z^{-p} + \sum_{n=1-p}^{\infty} \frac{\Gamma(k(n+p) + \gamma)\Gamma(\beta)}{\Gamma(\gamma)\Gamma[\alpha(n+p) + \beta](n+p)!} [1 + \eta(n+p)]a_n z^n$$

$$\begin{aligned}
 \mathfrak{M}_{\alpha,\beta,p,\eta}^{\gamma,k,2} f(z) &= \mathfrak{M}_{\alpha,\beta,p,\eta}^{\gamma,k,1} \left[\mathfrak{M}_{\alpha,\beta,p,\eta}^{\gamma,k,1} f(z) \right] \\
 &= (1 - \eta) \mathfrak{M}_{\alpha,\beta,p,\eta}^{\gamma,k,1} f(z) + \eta z^{-p} \left[z^{p+1} \mathfrak{M}_{\alpha,\beta,p,\eta}^{\gamma,k,1} f(z) \right]' \\
 &= z^{-p} + \sum_{n=1-p}^{\infty} \frac{\Gamma[k(n+p) + \gamma] \Gamma(\beta)}{\Gamma(\gamma) \Gamma[\alpha(n+p) + \beta] (n+p)!} [1 + \eta(n+p)]^2 a_n z^n
 \end{aligned}$$

and in general

$$\mathfrak{M}_{\alpha,\beta,p,\eta}^{\gamma,k,\mu} f(z) = z^{-p} + \sum_{n=1-p}^{\infty} \frac{\Gamma(k(n+p) + \gamma) \Gamma(\beta)}{\Gamma(\gamma) \Gamma[\alpha(n+p) + \beta] (n+p)!} [1 + \eta(n+p)]^{\mu} a_n z^n. \quad (2.37)$$

For $f \in \mathcal{M}_p$, it is easy to see that $\mathfrak{M}_{\alpha,\beta,p,\eta}^{\gamma,k,\mu} f(z)$ achieve the following relation :

$$z \left[\mathfrak{M}_{\alpha,\beta,p,\eta}^{\gamma,k,\mu} f(z) \right]' = \frac{1}{\eta} \mathfrak{M}_{\alpha,\beta,p,\eta}^{\gamma,k,\mu+1} f(z) - \left(p + \frac{1}{\eta} \right) \mathfrak{M}_{\alpha,\beta,p,\eta}^{\gamma,k,\mu} f(z), \quad (2.38)$$

for $(\alpha, \beta, \gamma, k \in \mathbb{C}, \operatorname{Re}(\alpha) > \max\{0, \operatorname{Re}(k) > -1\}, \operatorname{Re}(k) > 0), \eta > 0$, and $\mu \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Remark (2.2.1)[56].

$$\mathfrak{M}_{0,\beta,p,\eta}^{1,1,0} f(z) = f(z), (f \in \mathcal{M}_p);$$

$$\mathfrak{M}_{0,\beta,p,\eta}^{2,1,0} f(z) = (p+1)f(z) + zf'(z), (f \in \mathcal{M}_p);$$

$$\mathfrak{M}_{0,\beta,1,\eta}^{2,1,0} f(z) = 2f(z) + zf'(z), (f \in \mathcal{M}_1);$$

$$\mathfrak{M}_{1,1,p,\eta}^{1,1,0} \left(\frac{1}{z^p(1-z)} \right) = z^{-p} e^z;$$

$$\mathfrak{M}_{1,1,1,\eta}^{1,1,0} \left(\frac{1}{z(1-z)} \right) = z^{-1} e^z;$$

$$\mathfrak{M}_{2,1,p,\eta}^{1,1,0} \left(\frac{1}{z^p(1-z)} \right) = z^{-p} \cosh(\sqrt{z});$$

$$\mathfrak{M}_{2,1,1,\eta}^{1,1,0} \left(\frac{1}{z(1-z)} \right) = z^{-1} \cosh(\sqrt{z});$$

$$\mathfrak{M}_{2,2,p,\eta}^{1,1,0} \left(\frac{1}{z^p(1-z)} \right) = \frac{\sinh(\sqrt{z})}{z^p \sqrt{z}};$$

$$\mathfrak{M}_{2,2,1,\eta}^{1,1,0} \left(\frac{1}{z(1-z)} \right) = \frac{\sinh(\sqrt{z})}{\sqrt{z^3}}.$$

The primary objective of this study is to establish sufficient conditions for a specific normalized analytic function f to obtain the following criteria:

$$q_1(z) \prec \left(\frac{(1-\nu)z\mathfrak{M}_{\alpha,\beta,\eta,p}^{\gamma,k,\mu} f(z) + 2\nu z\mathfrak{M}_{\alpha,\beta,\eta,p}^{\gamma,k,\mu+1} f(z)}{\nu+1} \right)^{\mathcal{L}} \prec q_2(z),$$

and

$$q_1(z) \prec \left[z\mathfrak{M}_{\alpha,\beta,\eta,p}^{\gamma,k,\mu} f(z) \right]^{\mathcal{L}} \prec q_2(z),$$

whenever univalent functions $q_1(z)$ and $q_2(z)$ are given in U with $q_1(0) = q_2(0) = 1$, and $\mathcal{L} \in \mathbb{C} \setminus \{0\}$.

We present certain outcomes related to differential subordination through the application of the operator $\mathfrak{M}_{\alpha,\beta,\eta,p}^{\gamma,k,\mu} f(z)$:

Theorem (2.2.1). Assume that the function $q(z)$ is convex univalent function in open unit disk U with $q(0) = 1$ and $q'(z) \neq 0$, for all $z \in U$. Let $\delta, \mathcal{L} \in \mathbb{C} \setminus \{0\}, v \in \mathbb{R}^+$.

Suppose that q satisfies:

$$Re \left\{ 1 + \frac{zq''(z)}{q'(z)} \right\} > \max \left\{ 0, -Re \left(\frac{\mathfrak{S}}{\delta} \right) \right\}. \quad (2.39)$$

If $f \in \mathcal{M}_p$ satisfies the subordination condition:

$$\Psi(z) < q(z) + \frac{\delta}{\mathcal{L}} zq'(z), \quad (2.40)$$

where

$$\begin{aligned} \Psi(z) = & \left[\frac{(1-v)z\mathfrak{M}_{\alpha,\beta,\eta,p}^{\gamma,k,\mu}(z) + 2vz\mathfrak{M}_{\alpha,\beta,\eta,p}^{\gamma,k,\mu+1}f(z)}{v+1} \right]^{\mathcal{L}} + \\ & \delta \left[\frac{\frac{1}{\eta} \left(\frac{(1-v)z\mathfrak{M}_{\alpha,\beta,\eta,p}^{\gamma,k,\mu}f(z) + 2vz\mathfrak{M}_{\alpha,\beta,\eta,p}^{\gamma,k,\mu+1}f(z)}{v+1} \right)^{\mathcal{L}} \times}{\left(\frac{2v\mathfrak{M}_{\alpha,\beta,\eta,p}^{\gamma,k,\mu+2}f(z) + (1-v)(3+2p\eta-2\eta)\mathfrak{M}_{\alpha,\beta,\eta,p}^{\gamma,k,\mu+1}f(z) - (1-v)(p\eta-\eta+1)\mathfrak{M}_{\alpha,\beta,\eta,p}^{\gamma,k,\mu}f(z)}{(1-v)\mathfrak{M}_{\alpha,\beta,\eta,p}^{\gamma,k,\mu}f(z) + 2v\mathfrak{M}_{\alpha,\beta,\eta,p}^{\gamma,k,\mu+1}f(z)} \right)} \right], \end{aligned} \quad (2.41)$$

then

$$\left[\frac{(1-v)z\mathfrak{M}_{\alpha,\beta,\eta,p}^{\gamma,k,\mu}f(z) + 2vz\mathfrak{M}_{\alpha,\beta,\eta,p}^{\gamma,k,\mu+1}f(z)}{v+1} \right]^{\mathcal{L}} < q(z), \quad (2.42)$$

where q is the best dominant of (2.42).

Proof. Define the function $J(z)$ by

$$J(z) = \left[\frac{(1-v)z\mathfrak{M}_{\alpha,\beta,\eta,p}^{\gamma,k,\mu} f(z) + 2vz\mathfrak{M}_{\alpha,\beta,\eta,p}^{\gamma,k,\mu+1} f(z)}{v+1} \right]^{\mathcal{L}}. \quad (2.43)$$

Then, the function $J(z)$ is analytic in U , and $J(0) = 1$. Thus, by differentiating the expression obtained from (2.43) with respect to z and utilizing identity (2.38), we have:

$$\begin{aligned} \Psi(z) = & \left[\frac{(1-v)z\mathfrak{M}_{\alpha,\beta,\eta,p}^{\gamma,k,\mu} f(z) + 2vz\mathfrak{M}_{\alpha,\beta,\eta,p}^{\gamma,k,\mu+1} f(z)}{v+1} \right]^{\mathcal{L}} + \\ & \delta \left[\frac{\frac{1}{\eta} \left(\frac{(1-v)z\mathfrak{M}_{\alpha,\beta,\eta,p}^{\gamma,k,\mu} f(z) + 2vz\mathfrak{M}_{\alpha,\beta,\eta,p}^{\gamma,k,\mu+1} f(z)}{v+1} \right)^{\mathcal{L}} \times}{\left(\frac{2v\mathfrak{M}_{\alpha,\beta,\eta,p}^{\gamma,k,\mu+2} f(z) + (1-v)(3+2p\eta-2\eta)\mathfrak{M}_{\alpha,\beta,\eta,p}^{\gamma,k,\mu+1} f(z) - (1-v)(p\eta-\eta+1)\mathfrak{M}_{\alpha,\beta,\eta,p}^{\gamma,k,\mu} f(z)}{(1-v)\mathfrak{M}_{\alpha,\beta,\eta,p}^{\gamma,k,\mu} f(z) + 2v\mathfrak{M}_{\alpha,\beta,\eta,p}^{\gamma,k,\mu+1} f(z)} \right)} \right] \end{aligned}$$

Thus the subordination (2.40) is equivalent to

$$J(z) + \frac{\delta}{\mathcal{L}} zJ'(z) < q(z) + \frac{\delta}{\mathcal{L}} zq'(z).$$

By Lemma (1.2.1) with $\lambda = \frac{\delta}{\mathcal{L}}$ and $\alpha = 1$, we obtain (2.42).

By putting $q(z) = \left(\frac{1+Az}{1+Bz} \right)$ in Theorem (2.2.1), we get the following corollary:

Corollary (2.2.1). Let $\mathcal{L}, \delta \in \mathbb{C} \setminus \{0\}$, $v \in \mathbb{R}^+$ and $(-1 \leq B \leq A \leq 1)$

Also

$$Re \left\{ \frac{1+Bz}{1-Bz} \right\} > \max \left\{ 0, -Re \left(\frac{\mathcal{L}}{\delta} \right) \right\}.$$

If $f \in \mathbb{M}_p$ satisfies the subordination

$$\Psi(z) < \frac{1 + Az}{1 + Bz} + \frac{\delta}{\mathcal{L}} \frac{(A - B)z}{1 + Bz}.$$

When $\Psi(z)$ given by (2.41), then

$$\left[\frac{(1 - v)z \mathfrak{M}_{\alpha, \beta, \eta, p}^{\gamma, k, \mu} f(z) + 2vz \mathfrak{M}_{\alpha, \beta, \eta, p}^{\gamma, k, \mu+1} f(z)}{v + 1} \right]^{\mathcal{L}} < \left(\frac{1 + Az}{1 + Bz} \right),$$

and $\frac{1+Az}{1+Bz}$ is the best dominant.

When we put $A = 1, B = -1$, in the previous corollary (2.2.1), we obtain the next result.

Corollary (2.2.2). Let $\mathcal{L}, \delta \in \mathbb{C} \setminus \{0\}, v \in \mathbb{R}^+$ and

$$Re \left\{ \frac{1 + z}{1 - z} \right\} > \max \left\{ 0, -Re \left(\frac{\mathcal{L}}{\delta} \right) \right\}.$$

If $f \in \mathcal{M}_p$ satisfy the subordination condition:

$$\Psi(z) < \frac{1 + z^2 + 2 \frac{\delta}{\mathcal{L}} z}{(1 + z)^2},$$

when $\Psi(z)$ given by (2.41),

$$\text{then } \left[\frac{(1-v)z \mathfrak{M}_{\alpha, \beta, \eta, p}^{\gamma, k, \mu} f(z) + 2vz \mathfrak{M}_{\alpha, \beta, \eta, p}^{\gamma, k, \mu+1} f(z)}{v+1} \right]^{\mathcal{L}} < \left(\frac{1+z}{1-z} \right)$$

and $\frac{1+z}{1-z}$ is the best dominant.

Theorem(2.2.2). In open unit disk U , let $q(z)$ be convex univalent function in U with $q(0) = 1, q'(z) \neq 0$ ($z \in U$) and $z \frac{q'(z)}{q(z)}$ is starlike univalent in U . Let $\delta, \mathcal{L} \in \mathbb{C} \setminus \{0\}, \kappa \in \mathbb{C}$ and suppose that q satisfies :

$$\operatorname{Re} \left\{ 1 + \frac{\kappa}{\delta} (q(z))^\kappa + \frac{\kappa-1}{\delta} (q(z))^{\kappa-1} - z \frac{q'(z)}{q(z)} + z \frac{q''(z)}{q'(z)} \right\} > 0. \quad (2.44)$$

If $f \in \mathcal{M}_p$ satisfies $z\mathfrak{M}_{\alpha,\beta,\eta,p}^{\gamma,k,\mu} f(z) \neq 0$, and

$$\Xi(z) < (1 + q(z)) + (q(z))^\kappa + \delta z \frac{q'(z)}{q(z)}, \quad (2.45)$$

where,

$$\begin{aligned} \Xi(z) = & \left[z\mathfrak{M}_{\alpha,\beta,\eta,p}^{\gamma,k,\mu} f(z) \right]^\mathcal{L} + \left[z\mathfrak{M}_{\alpha,\beta,\eta,p}^{\gamma,k,\mu} f(z) \right]^{(\kappa-1)} + \\ & \frac{\delta \mathcal{L}}{\eta} \left[\frac{z\mathfrak{M}_{\alpha,\beta,\eta,p}^{\gamma,k,\mu+1} f(z)}{z\mathfrak{M}_{\alpha,\beta,\eta,p}^{\gamma,k,\mu} f(z)} - \eta(p-1) - 1 \right], \end{aligned} \quad (2.46)$$

then

$$\left[z\mathfrak{M}_{\alpha,\beta,\eta,p}^{\gamma,k,\mu} f(z) \right]^\mathcal{L} < q(z) \quad (2.47)$$

and q is the best dominant of (2.45).

Proof. Define the function $J(z)$ by

$$J(z) = \left[z\mathfrak{M}_{\alpha,\beta,\eta,p}^{\gamma,k,\mu} f(z) \right]^\mathcal{L}, \quad (2.48)$$

where $J(z)$ is analytic in U and $J(0) = 1$.

Differentiating (2.48) with respect to z , and using the identity (2.38), we get:

$$\frac{zJ'(z)}{J(z)} = \frac{\mathcal{L}}{\eta} \left[\frac{z\mathfrak{M}_{\alpha,\beta,\eta,p}^{\gamma,k,\mu+1} f(z)}{z\mathfrak{M}_{\alpha,\beta,\eta,p}^{\gamma,k,\mu} f(z)} - \eta(p-1) - 1 \right].$$

By setting $\theta(w) = (1+w)^{\kappa-1}w$ and $\phi(w) = \frac{\delta}{w}$, $w \neq 0$,

reveals the $\theta(w)$ is analytic in $\mathbb{C}$, and $\phi(w)$ are analytic in $\mathbb{C} \setminus \{0\}$, $\phi(w) \neq 0$, we get

$$Q(z) = zq'(z)\phi(q(z)) = \delta z \frac{q'(z)}{q(z)}$$

and

$$h(z) = \theta(q(z)) + Q(z) = (1 + q(z))(q(z))^{\kappa-1} + \delta z \frac{q'(z)}{q(z)}.$$

It is clear the $Q(z)$ is starlike univalent in U ,

then, we obtain

$$\operatorname{Re} \left\{ \frac{zh'(z)}{Q(z)} \right\} = \operatorname{Re} \left\{ 1 + \frac{\kappa}{\delta} (q(z))^{\kappa} + \frac{\kappa-1}{\delta} (q(z))^{\kappa-1} - z \frac{q'(z)}{q(z)} + z \frac{q''(z)}{q'(z)} \right\} > 0.$$

By a simple calculation, we obtain

$$E(z) = (1 + J(z))(J(z))^{\kappa-1} + \delta z \frac{J'(z)}{J(z)}, \quad (2.49)$$

where $E(z)$ is given by (2.46).

From (2.45) and (2.49), we have

$$(1 + J(z))(J(z))^{\kappa-1} + \delta z \frac{J'(z)}{J(z)} < (1 + q(z))(q(z))^{\kappa-1} + \delta z \frac{q'(z)}{q(z)}, \quad (2.50)$$

By Lemma(1.2.2), we get $J(z) < q(z)$ and q is the best dominant.

Taking $q(z) = \left(\frac{1+mz}{1+nz} \right)$, $(-1 \leq n < m \leq 1)$ in Theorem (2.2.2), we obtain the next result.

Corollary (2.2.3). Let $0 \neq \delta \in \mathbb{C}$, $\kappa \in \mathbb{C}$ and

$$\operatorname{Re} \left\{ 1 + \frac{\kappa}{\delta} \left(\frac{1+mz}{1+nz} \right)^\kappa + \frac{\kappa-1}{\delta} \left(\frac{1+mz}{1+nz} \right)^{\kappa-1} + \frac{(m-n)z}{(1+nz)(1+mz)} - \frac{2nz}{1+nz} \right\} > 0.$$

If $f \in \mathcal{M}_p$, and

$$\mathcal{E}(z) < \left(1 + \frac{1+mz}{1+nz} \right) \left(\frac{1+mz}{1+nz} \right)^{\kappa-1} + \delta \frac{(m-n)z}{(1+nz)(1+mz)},$$

where $\mathcal{E}(z)$ by (2.46), then

$$\left[{}_Z\mathfrak{M}_{\alpha,\beta,\eta,p}^{\gamma,k,\mu} f(z) \right]^\mathcal{L} < \left(\frac{1+mz}{1+nz} \right)$$

and $q(z) = \left(\frac{1+mz}{1+nz} \right)$ is the best dominant.

Taking $m = 1$ and $n = -1$ in corollary (2.2.3), we get the next result.

Corollary (2.2.4). Let $0 = \delta \in \mathbb{C}$, $\kappa \in \mathbb{C}$ and

$$\operatorname{Re} \left\{ 1 + \frac{\kappa}{\delta} \left(\frac{1+z}{1-z} \right)^\kappa + \frac{\kappa-1}{\delta} \left(\frac{1+z}{1-z} \right)^{\kappa-1} + \frac{2z^2}{1-z^2} \right\} > 0,$$

If $f \in \mathcal{M}_p$, and

$$\mathcal{E}(z) < \left(1 + \frac{1+z}{1-z} \right) \left(\frac{1+z}{1-z} \right)^{\kappa-1} + \delta \frac{2z}{(1-z)(1+z)},$$

where $\mathcal{E}(z)$ by (2.46), then

$$\left[{}_ZM_{\alpha,\beta,\eta,p}^{\gamma,k,\mu} f(z) \right]^\mathcal{L} < \left(\frac{1+z}{1-z} \right)$$

and $q(z) = \left(\frac{1+z}{1-z} \right)$ is the best dominant.

Theorem (2.2.3). Suppose that $q(z)$ is convex univalent function in U with $q(0) = 1$, $\mathcal{L} > 0$, $v \in \mathbb{R}^+$ and $\operatorname{Re}\{\delta\} > 0$. Let $f \in \mathcal{M}_p$ such that

$$\frac{(1-v)z\mathfrak{M}_{\alpha,\beta,\eta,p}^{\gamma,k,\mu}(z) + 2vz\mathfrak{M}_{\alpha,\beta,\eta,p}^{\gamma,k,\mu+1}f(z)}{v+1} \neq 0,$$

$$\left(\frac{(1-v)z\mathfrak{M}_{\alpha,\beta,\eta,p}^{\gamma,k,\mu}f(z) + 2vz\mathfrak{M}_{\alpha,\beta,\eta,p}^{\gamma,k,\mu+1}f(z)}{v+1} \right)^{\mathcal{L}} \in H[q(0), 1] \cap \mathcal{Q}.$$

If $\Psi(z)$ by (2.41) be univalent in U , and holds

$$q(z) + \frac{\delta}{\gamma} z q'(z) < \Psi(z), \quad (2.51)$$

then

$$q(z) < \left(\frac{(1-v)z\mathfrak{M}_{\alpha,\beta,\eta,p}^{\gamma,k,\mu}f(z) + 2vz\mathfrak{M}_{\alpha,\beta,\eta,p}^{\gamma,k,\mu+1}f(z)}{v+1} \right)^{\mathcal{L}} \quad (2.52)$$

and $q(z)$ is the best subordinant of (2.51).

Proof. Express the function $J(z)$ by

$$J(z) = \left(\frac{(1-v)z\mathfrak{M}_{\alpha,\beta,\eta,p}^{\gamma,k,\mu}f(z) + 2vz\mathfrak{M}_{\alpha,\beta,\eta,p}^{\gamma,k,\mu+1}f(z)}{v+1} \right)^{\mathcal{L}}. \quad (2.53)$$

Differentiating (2.53) with respect to z , we get

$$\frac{zJ'(z)}{J(z)} = \mathcal{L} \left[\frac{(1-v)z \left(\mathfrak{M}_{\alpha,\beta,\eta,p}^{\gamma,k,\mu}f(z) \right)' + 2vz \left(\mathfrak{M}_{\alpha,\beta,\eta,p}^{\gamma,k,\mu+1}f(z) \right)'}{(1-v)z\mathfrak{M}_{\alpha,\beta,\eta,p}^{\gamma,k,\mu}f(z) + 2vz\mathfrak{M}_{\alpha,\beta,\eta,p}^{\gamma,k,\mu+1}f(z)} + 1 \right]. \quad (2.54)$$

After some computations and using (2.38), from (2.54), we will get

$$\begin{aligned} \Psi(z) &= \left[\frac{(1-v)z\mathfrak{M}_{\alpha,\beta,\eta,p}^{\gamma,k,\mu} f(z) + 2vz\mathfrak{M}_{\alpha,\beta,\eta,p}^{\gamma,k,\mu+1} f(z)}{v+1} \right]^{\mathcal{L}} + \\ &\quad \frac{\delta}{\eta} \left[\left(\frac{(1-v)z\mathfrak{M}_{\alpha,\beta,\eta,p}^{\gamma,k,\mu} f(z) + 2vz\mathfrak{M}_{\alpha,\beta,\eta,p}^{\gamma,k,\mu+1} f(z)}{v+1} \right)^{\mathcal{L}} \times \right. \\ &\quad \left. \left(\frac{2v\mathfrak{M}_{\alpha,\beta,\eta,p}^{\gamma,k,\mu+2} f(z) + (1-v)(3+2p\eta-2\eta)\mathfrak{M}_{\alpha,\beta,\eta,p}^{\gamma,k,\mu+1} f(z)}{(1-v)\mathfrak{M}_{\alpha,\beta,\eta,p}^{\gamma,k,\mu} f(z) + 2v\mathfrak{M}_{\alpha,\beta,\eta,p}^{\gamma,k,\mu+1} f(z)} - \right. \right. \\ &\quad \left. \left. \frac{(1-v)(p\eta-\eta+1)M_{\alpha,\beta,\eta,p}^{\gamma,k,\mu} f(z)}{(1-v)\mathfrak{M}_{\alpha,\beta,\eta,p}^{\gamma,k,\mu} f(z) + 2v\mathfrak{M}_{\alpha,\beta,\eta,p}^{\gamma,k,\mu+1} f(z)} \right) \right] \\ &= J(z) + \frac{\delta}{\mathcal{L}} zJ'(z). \end{aligned}$$

By using Lemma (1.2.3), we get $p < q$ and q is the best subordinant.

By $q(z) = \left(\frac{1+z}{1-z}\right)$ in Theorem (2.2.3), we obtain the next corollary:

Corollary (2.2.5). Let $Re\{\delta\} > 0$, $0 \neq \mathcal{L} \in \mathbb{C}$, $v \in \mathbb{R}^+$, such that

$$\left(\frac{(1-v)z\mathfrak{M}_{\alpha,\beta,\eta,p}^{\gamma,k,\mu} f(z) + 2vz\mathfrak{M}_{\alpha,\beta,\eta,p}^{\gamma,k,\mu+1} f(z)}{v+1} \right)^{\mathcal{L}} \in H[q(0), 1] \cap Q.$$

If $\Psi(z)$ defined by (2.41) be univalent in U , and $f \in \mathcal{M}_p$ satisfies :

$$\Psi(z) < \frac{1+z}{1-z} + \frac{2z}{(1-z)^2},$$

then

$$\left(\frac{1+z}{1-z}\right) < \left[z\mathfrak{M}_{\alpha,\beta,\eta,p}^{\gamma,k,\mu} f(z)\right]^{\mathcal{L}}$$

and $\left(\frac{1+z}{1-z}\right)$ is the best subordinant.

Theorem (2.2.4). Assume that $q(z)$ be convex univalent function in U^* , with $q(0) = 1$, $q'(z) \neq 0$ and $\frac{zq'(z)}{q(z)}$ is starlike univalent in U . If q satisfies:

$$\operatorname{Re} \left\{ \frac{\kappa}{\delta} (q(z))^\kappa q'(z) + \frac{\kappa - 1}{\delta} (q(z))^{\kappa-1} q'(z) \right\} > 0, \quad (2.55)$$

for $\delta \in \mathbb{C} \setminus \{0\}$, $\kappa \in \mathbb{C}$ and $z \in U^*$, let $f \in M_p$ satisfies the next conditions:

$$\left(z \mathfrak{M}_{\alpha, \beta, \eta, p}^{\gamma, k, \mu} f(z) \right)^\mathcal{L} \in H[q(0), 1] \cap Q, \text{ and } z \mathfrak{M}_{\alpha, \beta, \eta, p}^{\gamma, k, \mu} f(z) \neq 0.$$

If

$$(1 + q(z))(q(z))^{\kappa-1} + \delta z \frac{q'(z)}{q(z)} < \mathcal{E}(z), \quad (2.56)$$

where $\mathcal{E}(z)$ is univalent and given by (2.46),

then

$$q(z) < \left[z \mathfrak{M}_{\alpha, \beta, \eta, p}^{\gamma, k, \mu} f(z) \right]^\mathcal{L} \quad (2.57)$$

and q is the best dominant of (2.57).

Proof. Putting

$$J(z) = \left[z \mathfrak{M}_{\alpha, \beta, \eta, p}^{\gamma, k, \mu} f(z) \right]^\mathcal{L}. \quad (2.58)$$

Differentiating (2.58) with respect to z , we obtain,

$$\frac{zJ'(z)}{J(z)} = \frac{\mathcal{L}}{\eta} \left[\frac{z \mathfrak{M}_{\alpha, \beta, \eta, p}^{\gamma, k, \mu+1} f(z)}{z \mathfrak{M}_{\alpha, \beta, \eta, p}^{\gamma, k, \mu} f(z)} - \eta(p-1) - 1 \right].$$

By setting

$$\theta(w) = (1 + w)^{\kappa-1}w \text{ and } \phi(w) = \frac{\delta}{w}, \quad w \neq 0.$$

We see that $\theta(w)$ is analytic in $\mathbb{C}$, and $\phi(w)$ is analytic in $\mathbb{C} \setminus \{0\}$, $\phi(w) \neq 0$.

Also, we get $Q(z) = zq'(z)\phi(q(z)) = \delta z \frac{q'(z)}{q(z)}$, such that $Q(z)$ is starlike univalent in U , we obtain

$$Re \left\{ \frac{\theta'(q(z))}{\phi(q(z))} \right\} = Re \left\{ \frac{\kappa}{\delta} (q(z))^{\kappa} q'(z) + \frac{\kappa-1}{\delta} (q(z))^{\kappa-1} q'(z) \right\} > 0.$$

Through simple calculation, we have

$$\mathcal{E}(z) = (1 + J(z))(J(z))^{\kappa-1} + \delta z \frac{J'(z)}{J(z)}, \quad (2.59)$$

where $\mathcal{E}(z)$ is given by (2.46).

From (2.56) and (2.59), we get

$$(1 + q(z))(q(z))^{\kappa-1} + \delta z \frac{q'(z)}{q(z)} < (1 + J(z))(J(z))^{\kappa-1} + \delta z \frac{J'(z)}{J(z)}.$$

By using Lemma (1.2.4), we get $q < J$.

and q is the best dominant.

Theorem (2.2.5). Let q_1 and q_2 be convex univalent function in U with $q_1(0) = q_2(0) = 1$, $Re\{\delta\} > 0$, $\mathcal{L} > 0$, $v \in \mathbb{R}^+$ and q_2 satisfies (2.39).

If $f \in \mathcal{M}_p$ such that

$$\left(\frac{(1-v)z \mathfrak{M}_{\alpha, \beta, \eta, p}^{\gamma, k, \mu} f(z) + 2vz \mathfrak{M}_{\alpha, \beta, \eta, p}^{\gamma, k, \mu+1} f(z)}{v+1} \right)^I \in H[1, 1] \cap \mathcal{Q},$$

and $\Psi(z)$ given by (2.41) satisfies :

$$q_1(z) + \frac{\delta}{\mathcal{L}} z q_1'(z) < \Psi(z) < q_2(z) + \frac{\delta}{\mathcal{L}} z q_2'(z),$$

then

$$q_1(z) < \left(\frac{(1-v)z\mathfrak{M}_{\alpha,\beta,\eta,p}^{\gamma,k,\mu} f(z) + 2vz\mathfrak{M}_{\alpha,\beta,\eta,p}^{\gamma,k,\mu+1} f(z)}{v+1} \right)^{\mathcal{L}} < q_2(z),$$

where q_1 and q_2 are respectively the best subdominant and the best dominant.

Theorem (2.2.6). Let q_1 and q_2 are convex univalent in U , with $q_1(0) = q_2(0) = 1$. Assume that q_1 and q_2 satisfy the conditions (2.55) and (2.44) respectively.

If $f \in \mathcal{M}_p$ satisfies

$$\left[z\mathfrak{M}_{\alpha,\beta,\eta,p}^{\gamma,k,\mu} f(z) \right]^{\mathcal{L}} \in H[1,1] \cap \mathcal{Q}$$

and $\left[z\mathfrak{M}_{\alpha,\beta,\eta,p}^{\gamma,k,\mu} f(z) \right]^{\mathcal{L}} \neq 0$, and $\Xi(z)$ is univalent in U , given by (2.46).

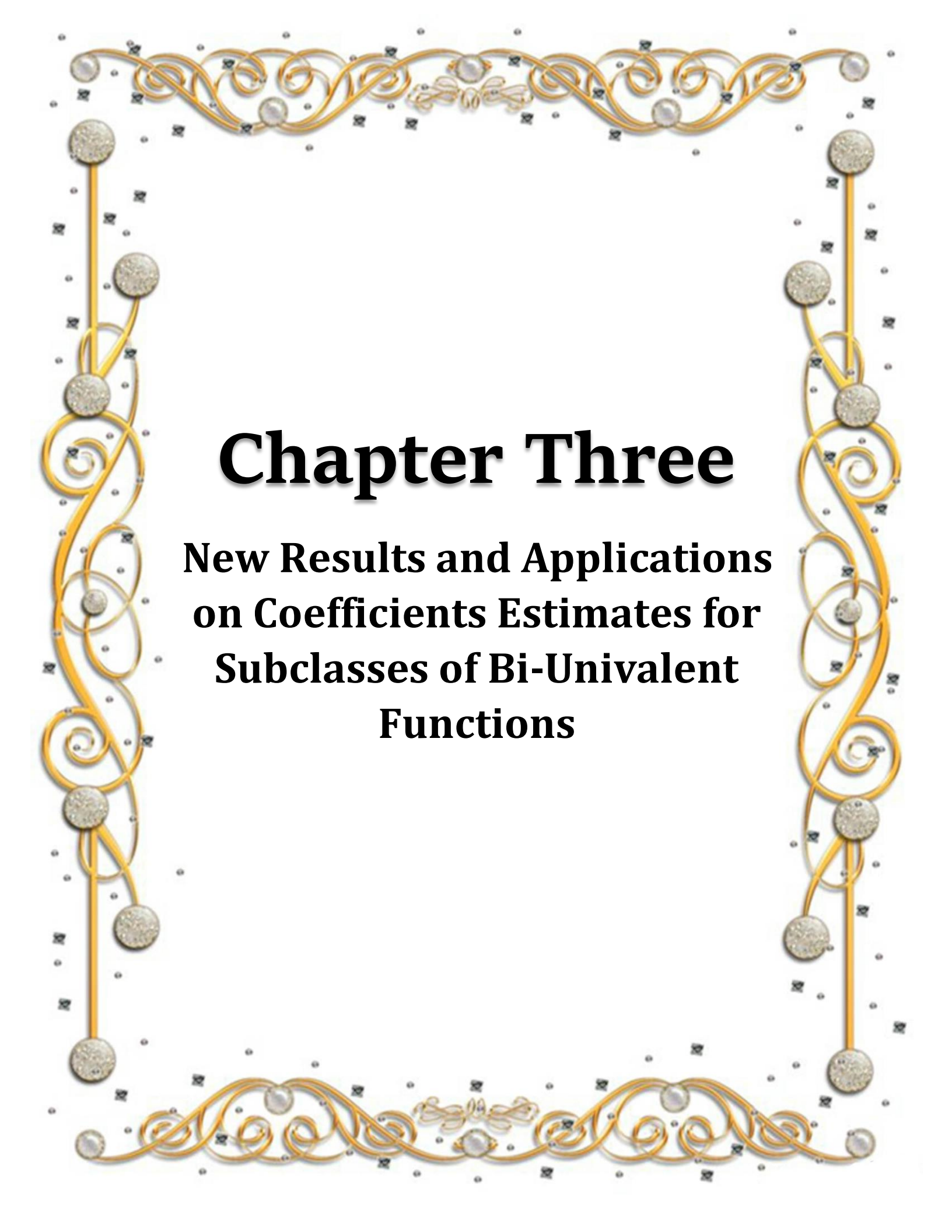
then

$$\begin{aligned} (1 + q_1(z))(q_1(z))^{\kappa-1} + \delta z \frac{q_1'(z)}{q_1(z)} &< \Xi(z) \\ &< (1 + q_2(z))(q_2(z))^{\kappa-1} + \delta z \frac{q_2'(z)}{q_2(z)}, \end{aligned}$$

implies

$$q_1(z) < \left[z\mathfrak{M}_{\alpha,\beta,\eta,p}^{\gamma,k,\mu} f(z) \right]^{\mathcal{L}} < q_2(z),$$

where the best subdominant and the best dominant are q_1 and q_2 respectively.



Chapter Three

**New Results and Applications
on Coefficients Estimates for
Subclasses of Bi-Univalent
Functions**

Chapter Three

New Results and Applications on Coefficients Estimates for Subclasses of Bi-Univalent Functions

Introduction:

Several authors, such as, Lewin [43], introduced the class Σ of bi-univalent analytic functions. He demonstrated that $|a_2| < 1.51$ for every function $f \in \Sigma$. Subsequently, Brannan and Clunie [24] conjecture that $|a_2| \leq \sqrt{2}$. The best known estimate for functions in Σ has been gained in 1984 by Tan [71], that is, $|a_2| < 1.485$. In the recent years, the study of bi-univalent functions has gained momentum mainly due to the work of Srivastava and Owa [70], which has clearly revived the subject.

Motivated by their work, many researchers ([8,58,74]) have lately investigated several interesting subclasses of the bi-univalent function class Σ and found estimates the first two Taylor-Maclaurin coefficients for functions in these subclasses. This chapter is divided into three sections. The first section discussed coefficient bounds and Fekete-Szegő problem of subclasses of bi-univalent functions. This study focuses on two kinds of bi-univalent functions $\Psi_{\Sigma}(\delta, \tau; \alpha)$ and $D_{\Sigma}(\delta, \tau; \beta)$. We obtain estimates for the Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions in these subclasses. In addition, we provide a solution to the Fekete-Szegő problem for functions in these specific classes. Second section deals with results on new subclasses for bi-univalent functions using quasi-subordination. The

present study introduces two specific subclasses $M_{\Sigma}^q(\beta, \rho, \lambda)$ and $W_{\Sigma}^q(\tau, \mu, \eta, \phi)$ of bi-univalent functions that are defined by quasi-subordination. We obtain estimates for the Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions within these subclasses.

Section three deals with coefficient bounds of subclass of bi-univalent functions associated with Horadam polynomials. Here, we use the Horadam polynomials to introduce a new class of analytic and bi-univalent functions defined in the open unit disk U and establish upper bounds for the second and third coefficients of functions belonging to this new subclass.

3.1 Coefficient Bounds and Fekete-Szegő Problem of Subclasses of Bi-Univalent Functions

Consider $\mathcal{A}$ denote the class of all normalized analytic functions f within an open unit disk U , defined as $U = \{z: z \in \mathbb{C}, |z| < 1\}$, and of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (z \in U). \quad (3.1)$$

A function f holds an inverse f^{-1} if it fulfills the conditions $f^{-1}(f(z)) = z$ for all z in U , with $f(f^{-1}(w)) = w$, $(|w| < r_0(f), r_0(f) \geq \frac{1}{4})$,

where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \dots, (w \in U). \quad (3.2)$$

If both f , as well as f^{-1} are functions that map a domain U to itself in a one-to-one manner, then f is referred to as bi-univalent in U . The set of all bi-univalent functions defined in U is denoted by the symbol Σ . (refer to references [22,33,36,42]).

If f and g are two analytic functions on the unit disc U , we define f as being subordinate to g or g as being superordinate to f , denoted symbolically as follows (see references [30,31,51]):

$$f < g \quad \text{or} \quad f(z) < g(z), \quad (z \in U).$$

If there is a Schwarz function $w(z)$ that satisfies the conditions $w(0) = 0$ such that $|w(z)| < 1$ for z in U , and if there exists a function $f(z)$ with $f(z) = g(w(z))$, ($z \in U$).

If g is univalent in U , we may establish the following equivalence:

$$f(z) < g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U).$$

Ma and Minda [49] introduced a category of starlike and convex functions by employing the technique of subordination. They examined the classes $S^*(\phi)$ and $G^*(\phi)$ defined by this way

$$S^*(\phi) = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} < \phi(z), z \in U \right\},$$

and

$$G^*(\phi) = \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} < \phi(z), z \in U \right\}.$$

By $S_{\Sigma}^*(\phi)$ with $G_{\Sigma}^*(\phi)$, we refer to functions that are both bi-starlike as well as bi-convex of Ma-Minda type [49].

In the sequel, it assumed that ϕ of the form:

$$\phi(z) = 1 + c_1 z + c_2 z^2 + \dots, \quad (3.3)$$

where $\phi(0) = 1$ and $\phi'(0) > 0$.

Nevertheless, the literature has a restricted number of articles that define the precise boundaries of the overall coefficients $|a_2|$ and $|a_3|$ ([8,7,9,20,21,14,18,29,32,64]) for analytic bi-univalent functions.

Lemma (3.1.1) [31]: If $p \in P$, then $|p_i| \leq 2$ for each i , where P is the class of all functions that are analytic in U and have a real part greater than 0 at every point in U , where

$$p(z) = 1 + p_1 z + p_2 z^2 + \dots, (z \in U). \quad (3.4)$$

Definition (3.1.1): A function f , which is a member of the class Σ and defined by equation (3.1), is said to belong to the class $\Psi_{\Sigma}(\delta, \tau; \alpha)$ if it satisfies the following criteria:

$$\left| \arg \left[(1 - \delta) \left(1 + \frac{zf''(z)}{f'(z)} \right) + \delta \left(\frac{zf'(z)}{f(z)} \right) - \frac{zf'(z) + \tau z^2 f''(z)}{f(z)} + 1 \right] \right| < \frac{\alpha\pi}{2} \quad (3.5)$$

and

$$\left| \arg \left[(1 - \delta) \left(1 + \frac{wg''(w)}{g'(w)} \right) + \delta \left(\frac{wg'(w)}{g(w)} \right) - \frac{wg(w) + \tau w^2 g''(w)}{g(w)} + 1 \right] \right| < \frac{\alpha\pi}{2}, \quad (3.6)$$

where $(0 < \alpha \leq 1, 0 \leq \tau \leq 1, 0 \leq \delta < 1), z, w \in U$ with $g = f^{-1}$ is given by (3.2).

Remark (3.1.1): A function f belonging to the class Σ is believed to be in the class $\Psi_{\Sigma}(0, \tau; \alpha)$ if

$$\left| \arg \left[2 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z) + \tau z^2 f''(z)}{f(z)} \right] \right| < \frac{\alpha\pi}{2} \quad (3.7)$$

and

$$\left| \arg \left[2 + \frac{wg''(w)}{g'(w)} - \frac{wg(w) + \tau w^2 g''(w)}{g(w)} \right] \right| < \frac{\alpha\pi}{2}, \quad (3.8)$$

where $(0 \leq \tau \leq 1, 0 < \alpha \leq 1), z, w \in U$ and $g = f^{-1}$ is given by (3.2).

Theorem (3.1.1): Consider the function $f \in \Psi_{\Sigma}(\delta, \tau; \alpha)$ $(0 < \alpha \leq 1, 0 \leq \tau \leq 1, 0 \leq \delta < 1)$ is defined by equation (3.1). Subsequently

$$|a_2| \leq \frac{2\alpha}{\sqrt{|2\alpha(1 - 4\tau - \delta) - (\alpha - 1)(1 - 2\tau - \delta)^2|}}, \quad (3.9)$$

and

$$|a_3| \leq \frac{8\alpha^2}{(1 - \delta - 2\tau)^2} + \frac{\alpha}{(2 - 2\delta - 3\tau)}. \quad (3.10)$$

Proof: It follows from the conditions (3.5) and (3.6) that

$$(1 - \delta) \left(1 + \frac{zf''(z)}{f'(z)} \right) + \delta \left(\frac{zf'(z)}{f'(z)} \right) - \frac{zf'(z) + \tau z^2 f''(z)}{f(z)} + 1 = [p(z)]^{\alpha} \quad (3.11)$$

and

$$(1 - \delta) \left(1 + \frac{wg''(w)}{g'(w)} \right) + \delta \left(\frac{wg'(w)}{g'(w)} \right) - \frac{wg(w) + \tau w^2 g''(w)}{g(w)} + 1 = [q(w)]^{\alpha},$$

$$(3.12)$$

where $g = f^{-1}$ and p, q in $\mathcal{P}$ that have the next series representations:

$$p(z) = 1 + p_1z + p_2z^2 + p_3z^3 + \cdots \quad (3.13)$$

and

$$q(w) = 1 + q_1w + q_2w^2 + q_3w^3 + \cdots. \quad (3.14)$$

Comparing the corresponding coefficients of (3.11) and (3.12), we have

$$(1 - \delta - 2\tau)a_2 = \alpha p_1, \quad (3.15)$$

$$2(2 - 2\delta - 3\tau)a_2 - (3 - 3\delta - 2\tau)a_2^2 = \alpha p_2 + \frac{\alpha(\alpha - 1)}{2}p_1^2, \quad (3.16)$$

$$-(1 - \delta - 2\tau)a_2 = \alpha q_1, \quad (3.17)$$

and

$$5(1 - \delta - 2\tau)a_2^2 - 2(2 - 2\delta - 3\tau)a_3 = \alpha q_2 + \frac{\alpha(\alpha - 1)}{2}q_1^2. \quad (3.18)$$

By utilizing equations (3.15) and (3.17), we get

$$p_1 = -q_1 \quad (3.19)$$

and

$$2(1 - \delta - 2\tau)^2 a_2^2 = \alpha^2(p_1^2 + q_1^2). \quad (3.20)$$

Also, from (3.16), (3.17) and (3.20), we find that

$$\begin{aligned} 2(1 - \delta - 4\tau)a_2^2 &= \alpha(p_2 + q_2) + \frac{\alpha(\alpha - 1)}{2}(p_1^2 + q_1^2) \\ &= \frac{(\alpha - 1)(1 - \delta - 2\tau)^2}{\alpha} a_2^2. \end{aligned}$$

Therefore, we have

$$a_2^2 = \frac{\alpha^2(p_2 + q_2)}{2\alpha(1 - \delta - 4\tau) - (\alpha - 1)(1 - \delta - 2\tau)^2}. \quad (3.21)$$

By computing the magnitude of (3.21) and applying Lemma (3.1.1) to the coefficients p_2 and q_2 , we have

$$|a_2| \leq \frac{2\alpha}{\sqrt{|2\alpha(1 - \delta - 4\tau) - (\alpha - 1)(1 - \delta - 2\tau)^2|}}$$

In order to find the bound on $|a_3|$, by subtracting (3.18) from (3.16), we get

$$4(2 - 2\delta - 3\tau)(a_3 - a_2^2) = \alpha(p_2 - q_2) + \frac{\alpha(\alpha - 1)}{2}(p_1^2 - q_1^2). \quad (3.22)$$

It follows from (3.19), (3.20) and (3.22) that

$$a_3 = \frac{\alpha^2(p_1^2 + q_1^2)}{2(1 - \delta - 2\tau)^2} + \frac{\alpha(p_2 - q_2)}{4(2 - 2\delta - 3\tau)}. \quad (3.23)$$

Taking the absolute value of (3.23) and applying Lemma (3.1.1) once again for the coefficients p_1, q_1, p_2 and q_2 , we obtain

$$|a_3| \leq \frac{4\alpha^2}{(1 - \delta - 2\tau)^2} + \frac{\alpha}{(2 - 2\delta - 3\tau)},$$

which completes the proof Theorem (3.1.1).

If we put $\delta = 0$, in the Theorem (3.1.1), we have the following corollary:

Corollary (3.1.1): Let $f \in \Psi_{\Sigma}(0, \tau; \alpha)$ ($0 \leq \tau \leq 1, 0 < \alpha \leq 1$) be given by (3.1). Then

$$|a_2| \leq \frac{2\alpha}{\sqrt{|2\alpha(1 - 4\tau) - (\alpha - 1)(1 - 2\tau)^2|}} \quad (3.24)$$

and

$$|a_3| \leq \frac{8\alpha^2}{(1-2\tau)^2} + \frac{\alpha}{(2-3\tau)}. \quad (3.25)$$

Definition (3.1.2): A function f belonging to the class Σ , defined by equation (3.1), is considered to be in the class $D_\Sigma(\delta, \tau; \beta)$ if it satisfies the following conditions:

$$Re \left\{ (1-\delta) \left(1 + \frac{zf''(z)}{f'(z)} \right) + \delta \left(\frac{zf'(z)}{f'(z)} \right) - \frac{zf'(z) + \tau z^2 f''(z)}{f(z)} + 1 \right\} > \beta, \quad (3.26)$$

and

$$Re \left\{ (1-\delta) \left(1 + \frac{wg''(w)}{g'(w)} \right) + \delta \left(\frac{wg'(w)}{g'(w)} \right) - \frac{wg(w) + \tau w^2 g''(w)}{g(w)} + 1 \right\} > \beta, \quad (3.27)$$

where $(0 \leq \beta < 1, 0 \leq \delta < 1, 0 \leq \tau \leq 1)$, $z, w \in U$ and $g = f^{-1}$ is given by (3.2).

Remark (3.1.2): A function f belonging to the class Σ is believed to be in the class $D_\Sigma(0, \tau; \beta)$ if:

$$Re \left\{ 2 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z) + \tau z^2 f''(z)}{f(z)} \right\} > \beta \quad (3.28)$$

and

$$Re \left\{ 2 + \frac{wg''(w)}{g'(w)} - \frac{wg(w) + \tau w^2 g''(w)}{g(w)} \right\} > \beta, \quad (3.29)$$

are satisfied, where $(0 \leq \beta < 1, 0 \leq \tau \leq 1)$, $z, w \in U$ and $g = f^{-1}$ is given by (3.2).

Theorem (3.1.2): Let $f \in D_{\Sigma}(\delta, \tau; \beta)$, $(0 \leq \beta < 1, 0 \leq \tau \leq 1, 0 \leq \delta < 1)$ by given by (3.1). Then

$$|a_2| \leq \sqrt{\left| \frac{2(1-\beta)}{1-2\tau-\delta} \right|} \quad (3.30)$$

and

$$|a_3| \leq \frac{4(1-\beta)^2}{(1-2\tau-\delta)^2} + \frac{1-\beta}{(2-2\delta-3\tau)}. \quad (3.31)$$

Proof: From the requirements (3.26) and (3.27), it can be shown that there exist p and q , which are elements of P , such that

$$(1-\delta) \left(1 + \frac{zf''(z)}{f'(z)} \right) + \delta \left(\frac{zf'(z)}{f(z)} \right) - \frac{zf'(z) + \tau z^2 f''(z)}{f(z)} + 1 = \beta + (1-\beta)p(z) \quad (3.32)$$

and

$$(1-\delta) \left(1 + \frac{wg''(w)}{g'(w)} \right) + \delta \left(\frac{wg'(w)}{g(w)} \right) - \frac{wg(w) + \tau w^2 g''(w)}{g(w)} + 1 = \beta + (1-\beta)q(w), \quad (3.33)$$

where $p(z)$ and $q(w)$ have the forms (3.13) and (3.14), respectively.

Equating coefficients (3.32) and (3.33) yields

$$(1-\delta-2\tau)a_2 = (1-\beta)p_1, \quad (3.34)$$

$$2(2-2\delta-3\tau)a_3 - (3-3\delta-2\tau)a_2^2 = (1-\beta)p_2, \quad (3.35)$$

$$-(1-\delta-2\tau)a_2 = (1-\beta)q_1, \quad (3.36)$$

and

$$5(1 - \delta - 2\tau)a_2^2 - 2(2 - 2\delta - 3\tau)a_3 = (1 - \beta)q_2. \quad (3.37)$$

From (3.34) and (3.36), we get

$$p_1 = -q_1 \quad (3.38)$$

and

$$2(1 - \delta - 2\tau)^2 a_2^2 = (1 - \beta)^2 (p_1^2 + q_1^2). \quad (3.39)$$

Adding (3.35) and (3.37), we obtain

$$2(1 - \delta - 4\tau)a_2^2 = (1 - \beta)(p_2 + q_2). \quad (3.40)$$

Therefore, we have

$$a_2^2 = \frac{(1 - \beta)(p_2 + q_2)}{4(1 - \delta - 4\tau)}.$$

Applying Lemma (3.1.1) for the coefficients p_2 and q_2 , we obtain

$$|a_2| \leq \sqrt{\left| \frac{2(1 - \beta)}{1 - \delta - 2\tau} \right|}.$$

In order to find the bound on $|a_3|$, by subtracting (3.37) from (3.35), we get

$$4(2 - 2\delta - 3\tau)(a_3 - a_2^2) = (1 - \beta)(p_2 - q_2),$$

or equivalently

$$a_3 = a_2^2 + \frac{(1 - \beta)(p_2 - q_2)}{4(2 - 2\delta - 3\tau)}. \quad (3.41)$$

Upon substituting the value of a_2^2 from (3.39), it follows that

$$a_3 = \frac{(1 - \beta)^2 (p_1^2 + q_1^2)}{2(1 - \delta - 2\tau)^2} + \frac{(1 - \beta)(p_2 - q_2)}{4(2 - 2\delta - 3\tau)}.$$

Applying Lemma (3.1.1) once again for the coefficients p_1, q_1, p_2 and q_2 , we obtain

$$|a_3| \leq \frac{4(1-\beta)^2}{(1-\delta-2\tau)^2} + \frac{(1-\beta)}{(2-2\delta-3\tau)},$$

which completes the proof Theorem (3.1.2).

If we put $\delta = 0$, in the Theorem (3.1.2), we have the following corollary:

Corollary (3.1.2): Let $f \in D_{\Sigma}(0, \tau; \beta)$, $(0 \leq \beta < 1, 0 \leq \tau \leq 1)$ be given by (3.1). Then

$$|a_2| \leq \sqrt{\left| \frac{2(1-\beta)}{1-2\tau} \right|} \quad (3.42)$$

and

$$|a_3| \leq \frac{4(1-\beta)^2}{(1-2\tau)^2} + \frac{1-\beta}{(2-3\tau)}. \quad (3.43)$$

In the next theorems, we present the “Fekete-Szegő inequality” for $f \in \Psi_{\Sigma}(\delta, \tau; \alpha)$ and $f \in D_{\Sigma}(\delta, \tau; \beta)$ respectively.

Theorem (3.1.3): For $(0 \leq \delta < 1, 0 \leq \tau \leq 1, 0 < \alpha \leq 1)$ and $\mu \in \mathbb{R}$, let $f \in \mathcal{A}$ be in the class $\Psi_{\Sigma}(\delta, \tau; \alpha)$. Then

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{|\alpha|}{|2-2\delta-3\tau|} ; \\ \left(|\mu - 1| \leq \left| \frac{2\alpha(1-\delta-4\tau) - (\alpha-1)(1-\delta-2\tau)^2}{4\alpha(2-2\delta-3\tau)} \right| \right), \\ \frac{4\alpha^2|\mu-1|}{|2\alpha(1-\delta-4\tau) - (\alpha-1)(1-\delta-2\tau)^2|} ; \\ \left(|\mu - 1| \geq \left| \frac{2\alpha(1-\delta-4\tau) - (\alpha-1)(1-\delta-2\tau)^2}{4\alpha(2-2\delta-3\tau)} \right| \right). \end{cases}$$

Proof: From (3.21) and (3.22), we get

$$\begin{aligned} a_3 - \mu a_2^2 &= \frac{\alpha(p_2 - q_2)}{4(2 - 2\delta - 3\tau)} + \frac{\alpha^2(p_2 + q_2)(1 - \mu)}{2\alpha(1 - \delta - 4\tau) - (\alpha - 1)(1 - \delta - 2\tau)^2} \\ &= \alpha \left[\left(\varphi(\alpha, \delta, \tau, \mu) + \frac{1}{4(2-2\delta-3\tau)} \right) p_2 + \left(\varphi(\alpha, \delta, \tau, \mu) - \frac{1}{4(2-2\delta-3\tau)} \right) q_2 \right], \end{aligned}$$

where

$$\varphi(\alpha, \delta, \tau, \mu) = \frac{\alpha(1 - \mu)}{2\alpha(1 - \delta - 4\tau) - (\alpha - 1)(1 - \delta - 2\tau)^2}.$$

Applying Lemma (3.1.1), we find that

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{|\alpha|}{|2 - 2\delta - 3\tau|} ; & \left(0 \leq |\varphi(\alpha, \delta, \tau, \mu)| \leq \frac{1}{4(2 - 2\delta - 3\tau)} \right), \\ 2|\alpha||\varphi(\alpha, \delta, \tau, \mu)| ; & \left(|\varphi(\alpha, \delta, \tau, \mu)| \geq \frac{1}{4(2 - 2\delta - 3\tau)} \right), \end{cases}$$

hence, after some calculations, gives

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{|\alpha|}{|2 - 2\delta - 3\tau|} ; \\ \left(|\mu - 1| \leq \left| \frac{2\alpha(1 - \delta - 4\tau) - (\alpha - 1)(1 - \delta - 2\tau)^2}{4\alpha(2 - 2\delta - 3\tau)} \right| \right), \\ \frac{4\alpha^2|\mu - 1|}{|2\alpha(1 - \delta - 4\tau) - (\alpha - 1)(1 - \delta - 2\tau)^2|} ; \\ \left(|\mu - 1| \geq \left| \frac{2\alpha(1 - \delta - 4\tau) - (\alpha - 1)(1 - \delta - 2\tau)^2}{4\alpha(2 - 2\delta - 3\tau)} \right| \right). \end{cases}$$

By substituting $\delta = 0$ into Theorem (3.1.3), we derive the following outcome:

Corollary (3.1.3): If $f \in \mathcal{A}$ defined by (3.1) be in the class $\Psi_{\Sigma}(\delta, \tau; \alpha)$, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{|\alpha|}{|2 - 3\tau|} ; \\ \left(|\mu - 1| \leq \left| \frac{2\alpha(1 - 4\tau) - (\alpha - 1)(1 - 2\tau)^2}{4\alpha(2 - 3\tau)} \right| \right), \\ \frac{4\alpha^2|\mu - 1|}{|2\alpha(1 - 4\tau) - (\alpha - 1)(1 - 2\tau)^2|} ; \\ \left(|\mu - 1| \geq \left| \frac{2\alpha(1 - 4\tau) - (\alpha - 1)(1 - 2\tau)^2}{4\alpha(2 - 3\tau)} \right| \right). \end{cases}$$

Theorem (3.1.4): For $(0 \leq \delta < 1, 0 \leq \tau \leq 1, 0 \leq \beta < 1)$ and $\mu \in \mathbb{R}$, let $f \in \mathcal{A}$ be in the class $D_{\Sigma}(\delta, \tau; \beta)$. Then

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{|(1-\beta)|}{|2-2\delta-3\tau|} ; & \left(|\mu - 1| \leq \left| \frac{(1-\delta-4\tau)}{2(2-2\delta-3\tau)} \right| \right), \\ \frac{2(1-\beta)|\mu - 1|}{(1-\delta-4\tau)} ; & \left(|\mu - 1| \geq \left| \frac{(1-\delta-4\tau)}{2(2-2\delta-3\tau)} \right| \right). \end{cases}$$

Proof: From (3.40) and (3.41), we get

$$\begin{aligned} a_3 - \mu a_2^2 &= \frac{(1-\beta)(p_2 - q_2)}{4(2-2\delta-3\tau)} + \frac{(1-\beta)(p_2 + q_2)(1-\mu)}{2(1-\delta-4\tau)} \\ &= (1-\beta) \left[\left(\theta(\delta, \tau, \mu) + \frac{1}{4(2-2\delta-3\tau)} \right) p_2 + \left(\theta(\delta, \tau, \mu) - \frac{1}{4(2-2\delta-3\tau)} \right) q_2 \right], \end{aligned}$$

where

$$\theta(\delta, \tau, \mu) = \frac{(1-\mu)}{2(1-\delta-4\tau)}.$$

Applying Lemma (3.1.1), we find that

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{|(1-\beta)|}{|2-2\delta-3\tau|} ; & \left(0 \leq |\theta(\delta, \tau, \mu)| \leq \frac{1}{4(2-2\delta-3\tau)} \right), \\ 2|(1-\beta)||\theta(\delta, \tau, \mu)| ; & \left(|\theta(\delta, \tau, \mu)| \geq \frac{1}{4(2-2\delta-3\tau)} \right), \end{cases}$$

hence, after some calculations, gives

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{|(1 - \beta)|}{|2 - 2\delta - 3\tau|}; & \left(|\mu - 1| \leq \left| \frac{(1 - 4\tau - \delta)}{2(2 - 2\delta - 3\tau)} \right| \right), \\ \frac{2(1 - \beta)|\mu - 1|}{(1 - 4\tau - \delta)}; & \left(|\mu - 1| \geq \left| \frac{(1 - 4\tau - \delta)}{2(2 - 2\delta - 3\tau)} \right| \right). \end{cases}$$

By substituting $\delta = 0$ into Theorem (3.1.4), we derive the following outcome:

Corollary (3.1.4): If $f \in \mathcal{A}$ defined by (3.1) be in the class $\Psi_{\Sigma}(\delta, \tau; \alpha)$, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{|(1 - \beta)|}{|2 - 3\tau|}; & \left(|\mu - 1| \leq \left| \frac{(1 - 4\tau)}{2(2 - 3\tau)} \right| \right), \\ \frac{2(1 - \beta)|\mu - 1|}{(1 - 4\tau)}; & \left(|\mu - 1| \geq \left| \frac{(1 - 4\tau)}{2(2 - 3\tau)} \right| \right). \end{cases}$$

3.2 Results on New Subclasses for Bi-Univalent Functions Using Quasi-Subordination

Letting $\mathcal{A}$ denote the class of all normalized analytic functions f in an open unit disk $U = \{z: z \in \mathbb{C}, |z| < 1\}$ of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (z \in U). \quad (3.44)$$

A function f has an inverse f^{-1} is satisfying $f^{-1}(f(z)) = z, (z \in U)$, and $f(f^{-1}(w)) = w, \left(|w| < r_0(f), r_0(f) \geq \frac{1}{4} \right)$,

where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \dots, (w \in U). \quad (3.45)$$

If f and f^{-1} are both univalent functions in the open unit disk U , then f is said to be bi-univalent in U and the set of bi-univalent functions defined in U is denoted by Σ . Refer to reference [42].

Let f and g are analytic functions in $\mathcal{A}$. Then f is said to be quasi-subordinate to g in U and written as follows:

$$f(z) \prec_q g(z), \quad (z \in U),$$

if there exists $\theta(z)$ and $w(z)$ be two analytic functions in U , with $w(0) = 0$ such that $|\theta(z)| < 1, |w(z)| < 1$ and $f(z) = \theta(z)g(w(z))$. If $\theta(z) = 1$, then $f(z) = g(w(z))$, so that $f(z) \prec g(z)$ in U . If $w(z) = z$, then $f(z) = \theta(z)g(z)$, and it is said that f is majorized by g and written $f(z) \ll g(z)$ in U . (see [10], [58]).

Ma and Manda [49] introduced a category of starlike and convex functions through the use of the subordination approach. They examined the classes $S^*(\phi)$ and $G^*(\phi)$ defined by this method.

$$S^*(\phi) = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \phi(z), z \in U \right\},$$

and

$$G^*(\phi) = \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \phi(z), z \in U \right\}.$$

By $S_{\Sigma}^*(\phi)$ and $G_{\Sigma}^*(\phi)$, we denote to bi-starlike and bi-convex functions f is bi-starlike and bi-convex of Ma-Minda type respectively [49].

In the sequel, it assumed that ϕ of the form:

$$\phi(z) = 1 + E_1 z + E_2 z^2 + \dots, \quad (3.46)$$

where $\phi(0) = 1$ and $\phi'(0) > 0$, also

$$\theta(z) = F_0 + \sum_{i=1}^{\infty} F_i z^i, \quad (3.47)$$

which are both analytical and bounded within the domain U . Nevertheless, the literature has only a limited number of studies that establish the overall coefficient bounds $|a_2|$ and $|a_3|$ for analytic bi-univalent functions. These studies are referenced in [7,8,9,21,25,32,36] and [40], while further references may be found in [21,25,29,30].

Definition (3.2.1). A function f , which is a member of the class Σ and defined by equation (3.44), is considered to be in the class $M_{\Sigma}^q(\beta, \rho, \lambda)$ if it satisfies the following quasi-subordination conditions:

$$\begin{aligned} \frac{1}{\beta} \left[\left(\frac{zf'(z)}{f(z)} \right) \left(1 + \frac{zf''(z)}{f'(z)} \right) + \left(\frac{\rho z^2 f''(z) + zf'(z)}{\rho z f'(z) + (1-\rho)f(z)} \right) \right. \\ \left. - \left(\frac{zf'(z) + \lambda z^2 f''(z)}{f(z)} \right) - 1 \right] \prec_q (\phi(z) - 1) \end{aligned} \quad (3.48)$$

and

$$\frac{1}{\beta} \left[\left(\frac{wg'(w)}{g(w)} \right) \left(1 + \frac{wg''(w)}{g'(w)} \right) + \left(\frac{\rho w^2 g''(w) + wg'(w)}{\rho wg'(w) + (1-\rho)g(w)} \right) - \left(\frac{wg'(w) + \lambda w^2 g'(w)}{g(w)} \right) - 1 \right] <_q (\emptyset(w) - 1), \quad (3.49)$$

where $(0 \leq \lambda \leq 1, 0 \leq \rho < 1)$, and $\beta \in \mathbb{C} \setminus \{0\}$, $z, w \in U$, and $g = f^{-1}$, and the functions $g, \emptyset$ are given by (3.45) and (3.46) respectively.

By substituting $\lambda = 0$ into Definition (3.2.1), we may deduce the following Remark: $M_{\Sigma}^q(\beta, \rho, 0) = M_{\Sigma}^q(\beta, \rho, \lambda)$.

Remark (3.2.1). A function $f \in \Sigma$, as defined by (3.44), is believed to be in the class $M_{\Sigma}^q(\beta, \rho)$ if it fulfills the following conditions:

$$\frac{1}{\beta} \left[\left(\frac{zf'(z)}{f(z)} \right) \left(1 + \frac{zf''(z)}{f'(z)} \right) + \left(\frac{\rho z^2 f''(z) + zf'(z)}{\rho zf'(z) + (1-\rho)f(z)} \right) - \left(\frac{zf'(z)}{f(z)} \right) - 1 \right] <_q (\emptyset(z) - 1) \quad (3.50)$$

and

$$\frac{1}{\beta} \left[\left(\frac{wg'(w)}{g(w)} \right) \left(1 + \frac{wg''(w)}{g'(w)} \right) + \left(\frac{\rho w^2 g''(w) + wg'(w)}{\rho wg'(w) + (1-\rho)g(w)} \right) - \left(\frac{wg'(w)}{g(w)} \right) - 1 \right] <_q (\emptyset(w) - 1). \quad (3.51)$$

Theorem (3.2.1). Consider the function f defined by equation (3.44) to belong to the class $M_{\Sigma}^q(\beta, \rho, \lambda)$. Thus

$$|a_2| \leq \min \left\{ \frac{|\beta F_0 E_1|}{|3 + \rho - 2\lambda|}, \sqrt{\frac{\beta F_0 E_2}{5 + \rho(2 - \rho) - 6\lambda}} \right\} \quad (3.52)$$

and

$$|a_3| \leq \min \left\{ \frac{\beta^2 F_0^2 E_1^2}{(3 + \rho - 2\lambda)^2} + \frac{\beta F_1 E_1}{8 + 4\rho - 6\lambda}, \frac{\beta F_0 E_2}{5 + \rho(2 - \rho) - 6\lambda} + \frac{\beta F_1 E_1}{8 + 4\rho - 6\lambda} \right\}. \quad (3.53)$$

Proof. Let $f \in \mathcal{L}_{\Sigma}^q(\beta, \lambda, \phi)$ and $g = f^{-1}$. Then, there are two analytic functions $u, v: U \rightarrow U$ with $v(0) = 0$ and $u(0) = 0, |u(z)| < 1, |v(w)| < 1$, satisfying

$$\frac{1}{\beta} \left[\left(\frac{zf'(z)}{f(z)} \right) \left(1 + \frac{zf''(z)}{f'(z)} \right) + \left(\frac{\rho z^2 f''(z) + z f'(z)}{\rho z f'(z) + (1-\rho)f(z)} \right) - \left(\frac{zf'(z) + \lambda z^2 f''(z)}{f(z)} \right) - 1 \right] = (\theta(z)[\phi(u(z)) - 1]) \quad (3.54)$$

and

$$\frac{1}{\beta} \left[\left(\frac{wg'(w)}{g(w)} \right) \left(1 + \frac{wg'(w)}{g'(w)} \right) + \left(\frac{\rho w^2 g''(w) + wg'(w)}{\rho wg'(w) + (1-\rho)g(w)} \right) - \left(\frac{wg'(w) + \lambda w^2 g''(w)}{g(w)} \right) - 1 \right] = (\theta(w)[\phi(v(w)) - 1]). \quad (3.55)$$

We define the function $p(z)$ and $q(w)$ by:

$$p(z) = \frac{1 + u(z)}{1 - u(z)} = 1 + d_1 z + d_2 z^2 + \dots, \quad (3.56)$$

and

$$q(w) = \frac{1 + v(w)}{1 - v(w)} = 1 + e_1 w + e_2 w^2 + \dots. \quad (3.57)$$

Or equivalent,

$$u(z) = \frac{p(z) - 1}{p(z) + 1} = \frac{1}{2} \left[d_1 z + \left(d_2 - \frac{d_1^2}{2} \right) z^2 + \dots \right], \quad (3.58)$$

and

$$v(w) = \frac{q(w) - 1}{q(w) + 1} = \frac{1}{2} \left[e_1 w + \left(e_2 - \frac{e_1^2}{2} \right) w^2 + \dots \right], \quad (3.59)$$

then $p(z)$ and $q(w)$ are analytic functions in U , with $p(0) = q(0) = 1$. Since, $v, u: U \rightarrow U$, have a positive real part in U , and $|d_i| \leq 2$ and $|e_i| \leq 2$, for $i = 1, 2$. Using (3.58) and (3.57) in (3.54) and (3.55), respectively, we have

$$\frac{1}{\beta} \left[\left(\frac{zf'(z)}{f(z)} \right) \left(1 + \frac{zf''(z)}{f'(z)} \right) + \left(\frac{\rho z^2 f''(z) + zf'(z)}{\rho z f'(z) + (1-\rho)f(z)} \right) - \left(\frac{zf'(z) + \lambda z^2 f''(z)}{f(z)} \right) - 1 \right] = \theta(z) \left(\phi \left[\frac{p(z)-1}{p(z)+1} \right] - 1 \right) \quad (3.60)$$

and

$$\frac{1}{\beta} \left[\left(\frac{wg'(w)}{g(w)} \right) \left(1 + \frac{wg'(w)}{g'(w)} \right) + \left(\frac{\rho w^2 g''(w) + wg'(w)}{\rho w g'(w) + (1-\rho)g(w)} \right) - \left(\frac{wg'(w) + \lambda w^2 g''(w)}{g(w)} \right) - 1 \right] = \theta(w) \left(\phi \left[\frac{q(w)-1}{q(w)+1} \right] - 1 \right). \quad (3.61)$$

Given that $f \in \Sigma$ possesses the Maclurian series specified by equation (3.44), a calculation reveals that its inverse $g = f^{-1}$ can be expressed using the expansion described in equation (3.45). Therefore, we can conclude that

$$\frac{1}{\beta} \left[\left(\frac{zf'(z)}{f(z)} \right) \left(1 + \frac{zf''(z)}{f'(z)} \right) + \left(\frac{\rho z^2 f''(z) + zf'(z)}{\rho z f'(z) + (1-\rho)f(z)} \right) - \left(\frac{zf'(z) + \lambda z^2 f''(z)}{f(z)} \right) - 1 \right] = \frac{1}{\beta} [(3 + \rho - 2\lambda)a_2 z + [2(4 + 2\rho - 3\lambda)a_3 - (3 + \rho(2 + \rho) - 2\lambda)a_2^2]z^2 + \dots] \quad (3.62)$$

and

$$\frac{1}{\beta} \left[\left(\frac{wg'(w)}{g(w)} \right) \left(1 + \frac{wg'(w)}{g'(w)} \right) + \left(\frac{\rho w^2 g''(w) + wg'(w)}{\rho w g'(w) + (1-\rho)g(w)} \right) - \left(\frac{wg'(w) + \lambda w^2 g''(w)}{g(w)} \right) - 1 \right] = \frac{1}{\beta} [-(3 + \rho - 2\lambda)a_2 w + [(13 + \rho(6 - \rho) - 10\lambda)a_2^2 - 2(4 + 2\rho - 3\lambda)a_3]w^2 + \dots] \quad (3.63)$$

By combining equations (3.58) and (3.59) with equations (3.46) and (3.47), it becomes clear that

$$\theta(z) \left(\phi \left[\frac{p(z)-1}{p(z)+1} \right] - 1 \right) = \frac{1}{2} F_0 E_1 d_1 z + \left(\frac{1}{2} F_1 E_1 d_1 + \frac{1}{2} F_0 E_1 \left(d_2 - \frac{d_1^2}{2} \right) + \frac{1}{4} F_0 E_2 d_1^2 \right) z^2 + \dots, \quad (3.64)$$

and

$$\theta(w) \left(\phi \left[\frac{q(w)-1}{q(w)+1} \right] - 1 \right) = \frac{1}{2} F_0 E_1 e_1 w + \left(\frac{1}{2} F_1 E_1 e_1 + \frac{1}{2} F_0 E_1 \left(e_2 - \frac{e_1^2}{2} \right) + \frac{1}{4} F_0 E_2 e_1^2 \right) w^2 + \dots. \quad (3.65)$$

By utilising equations (3.64) and (3.62) and comparing the coefficients of z and z^2 , we have

$$\frac{1}{\beta} (3 + \rho - 2\lambda) a_2 = \frac{1}{2} F_0 E_1 d_1, \quad (3.66)$$

and

$$\begin{aligned} \frac{1}{\beta} [2(4 + 2\rho - 3\lambda) a_3 - (3 + \rho(2 + \rho) - 2\lambda) a_2^2] &= \frac{1}{2} F_1 E_1 d_1 + \\ \frac{1}{2} F_0 E_1 \left(d_2 - \frac{d_1^2}{2} \right) + \frac{1}{4} F_0 E_2 d_1^2. \end{aligned} \quad (3.67)$$

By utilising equations (3.65) and (3.63) and comparing the coefficients of w and w^2 , we obtain

$$-\frac{1}{\beta} (3 + \rho - 2\lambda) a_2 = \frac{1}{2} F_0 E_1 e_1, \quad (3.68)$$

and

$$\begin{aligned} \frac{1}{\beta} [(13 + \rho(6 - \rho) - 10\lambda) a_2^2 - 2(4 + 2\rho - 3\lambda) a_3] &= \frac{1}{2} F_1 E_1 e_1 + \\ \frac{1}{2} F_0 E_1 \left(e_2 - \frac{e_1^2}{2} \right) + \frac{1}{4} F_0 E_2 e_1^2. \end{aligned} \quad (3.69)$$

Making use of (3.66) and (3.68), we obtain

$$d_1 = -e_1 \quad (3.70)$$

and

$$8(3 - 2\lambda + \rho)^2 a_2^2 = \beta^2 F_0^2 E_1^2 (d_1^2 + e_1^2). \quad (3.71)$$

Also, from (3.67), (3.69) and (3.71), we find that

$$a_2^2 = \frac{\beta[2F_0E_1(d_2 + e_2) + F_0(E_2 - E_1)(d_1^2 + e_1^2)]}{8(5 - 6\lambda + \rho(2 - \rho))}. \quad (3.72)$$

Applying $|d_i| \leq 2$ and $|e_i| \leq 2$ for the coefficients d_1 , e_1 , d_2 and e_2 , we immediately have

$$|a_2| \leq \frac{|\beta F_0 E_1|}{|3 + \rho - 2\lambda|}$$

and

$$|a_2| \leq \sqrt{\frac{\beta F_0 E_2}{5 - 6\lambda + \rho(2 - \rho)}}.$$

Furthermore, to determine the upper limit of $|a_3|$, we can subtract equations (3.67) and (3.69), resulting in

$$\frac{2}{\beta}(16 + 8\rho - 12\lambda)(a_3 - a_2) = 2F_1E_1d_1 + F_0E_1(d_2 - e_2), \quad (3.73)$$

$$\frac{2}{\beta}(16 + 8\rho - 12\lambda)a_3 = \frac{2}{\beta}(16 + 8\rho - 12\lambda)a_2^2 + 2F_1E_1d_1 + F_0E_1(d_2 - e_2),$$

therefore

$$a_3 = \frac{\beta[2F_1E_1d_1 + F_0E_1((d_2 - e_2))]}{2(16 - 12\lambda + 8\rho)} + a_2^2. \quad (3.74)$$

In light of (3.71), (3.72) and putting (3.74), we have

$$|a_3| \leq \frac{\beta[F_1 E_1]}{(8 - 6\lambda + 4\rho)} + \frac{\beta[F_0 E_2]}{(5 - 6\lambda + \rho(2 - \rho))}$$

and

$$|a_3| \leq \frac{\beta[F_1 E_1]}{(8 - 6\lambda + 4\rho)} + \frac{\beta^2 F_0^2 E_1^2}{(3 - 2\lambda_- + \rho)^2}.$$

This completes the proof Theorem (3.2.1).

By substituting $\lambda = 0$ into Theorem (3.2.1), we obtain the following corollary:

Corollary (3.2.1). Consider the function f defined by equation (3.44) to belong to the class $M_\Sigma^q(\beta, \rho, 0)$. Subsequently

$$|a_2| \leq \min \left\{ \frac{|\beta F_0 E_1|}{|3 + \rho|}, \sqrt{\frac{\beta F_0 E_2}{5 + \rho(2 - \rho)}} \right\}$$

and

$$|a_3| \leq \min \left\{ \frac{\beta^2 F_0^2 E_1^2}{(3 + \rho)^2} + \frac{\beta F_1 E_1}{8 + 4\rho}, \frac{\beta F_0 E_2}{5 + \rho(2 - \rho)} + \frac{\beta F_1 E_1}{8 + 4\rho} \right\}.$$

Definition (3.2.2). A function $f \in \Sigma$ is believed to be in the class $W_\Sigma^q(\tau, \mu, \eta, \phi)$ if it fulfills the quasi-subordination conditions described by equation (3.44).

$$\frac{1}{\tau} \left[(1 - \mu + 2\eta) \frac{f(z)}{z} + (\mu - 2\eta) f'(z) + \eta z f''(z) - 1 \right] \prec_q (\phi(w) - 1) \quad (3.75)$$

and

$$\frac{1}{\tau} \left[(1 - \mu + 2\eta) \frac{g(w)}{w} + (\mu - 2\eta) g'(w) + \eta w g''(w) - 1 \right] \prec_q (\phi(w) - 1), \quad (3.76)$$

where $(\tau \in \mathbb{C} \setminus \{0\}, \mu \geq 0, \eta \geq 0, z, w \in U)$ and $g = f^{-1}$ and the function g, ϕ are given by (3.45) and (3.46) respectively.

If we put $\mu = 1 + 2\eta$ in Definition (3.2.2), we obtain the following Remark such that $W_{\Sigma}^q(\tau, \mu, \eta, \phi) = W_{\Sigma}^q(\tau, 1 + 2\eta, \eta, \phi)$.

Remark (3.2.2). A function $f \in \Sigma$ is considered to be in the class $W_{\Sigma}^q(\tau, 1 + 2\eta, \eta, \phi)$ if it fulfills the given quasi-subordination requirements defined by equation (3.44).

$$\frac{1}{\tau} [f'(z) + \eta z f''(z) - 1] \prec_q (\phi(z) - 1) \quad (3.77)$$

and

$$\frac{1}{\tau} [g'(w) + \eta w g''(w) - 1] \prec_q (\phi(w) - 1), \quad (3.78)$$

where $(\tau \in \mathbb{C} \setminus \{0\}, \eta \geq 0, z, w \in U)$ and $g = f^{-1}$ and the function g, ϕ are given by (3.45) and (3.46) respectively.

Theorem (3.2.2). Let $f \in W_{\Sigma}^q(\tau, \mu, \eta, \phi)$ ($\tau \in \mathbb{C} \setminus \{0\}, \eta \geq 0, z, w \in U$) be given by (3.44). Then

$$|a_2| \leq \min \left\{ \frac{2\tau F_0 E_1}{1 + \mu}, 2 \sqrt{\frac{|\tau F_0 E_2|}{1 + 2\mu + 2\eta}} \right\} \quad (3.79)$$

and

$$|a_3| \leq \min \left\{ \frac{4F_0^2 E_1^2 \tau^2}{(1 + \mu)^2} + \frac{\tau E_1 (F_1 + F_0)}{1 + 2\mu + 2\eta}, \frac{4\tau F_0 E_2}{1 + 2\mu + 2\eta} + \frac{\tau E_1 (F_1 + F_0)}{1 + 2\mu + 2\eta} \right\}. \quad (3.80)$$

Proof. Let $f \in W_{\Sigma}^q(\tau, \mu, \eta, \phi)$ and $g = f^{-1}$. Then, there are two analytic functions $u, v: U \rightarrow U$ with $v(0) = 0$ and $u(0) = 0, |u(z)| < 1, |v(w)| < 1$, satisfying:

$$\frac{1}{\tau} \left[(1 - \mu + 2\eta) \frac{f(z)}{z} + (\mu - 2\eta) f'(z) + \eta z f''(z) - 1 \right] = \theta(z) (\phi(u(z)) - 1) \quad (3.81)$$

and

$$\frac{1}{\tau} \left[(1 - \mu + 2\eta) \frac{g(w)}{w} + (\mu - 2\eta) g'(w) + \eta w g''(w) - 1 \right] = \theta(w) (\phi(v(w)) - 1). \quad (3.82)$$

Since $f \in \Sigma$ has the Maclurian series defined by (3.44), a computation shows that its inverse $g = f^{-1}$ has the expansion by (3.45), we get

$$\frac{1}{\tau} \left[(1 - \mu + 2\eta) \frac{f(z)}{z} + (\mu - 2\eta) f'(z) + \eta z f''(z) - 1 \right] = \frac{1}{\tau} (1 + \mu) a_2 z + \frac{1}{\tau} (1 + 2\mu + 2\eta) a_3 z^2 + \dots, \quad (3.83)$$

and

$$\begin{aligned} \frac{1}{\tau} \left[(1 - \mu + 2\eta) \frac{g(w)}{w} + (\mu - 2\eta) g'(w) + \eta w g''(w) - 1 \right] = \\ - \frac{1}{\tau} (1 + \mu) a_2 w + \frac{1}{\tau} (1 + 2\mu + 2\eta) (2a_2^2 - a_3) w^2 - \dots. \end{aligned} \quad (3.84)$$

By utilizing equations (3.64) and (3.83) and comparing the coefficients z and z^2 , we obtain

$$\frac{1}{\tau}(1 + \mu)a_2 = \frac{F_0E_1}{2}d_1, \quad (3.85)$$

and

$$\frac{1}{\tau}(1 + 2\mu + 2\eta)a_3 = \frac{F_1E_1}{2}d_1 + \frac{F_0E_1}{2}\left(d_2 - \frac{d_1^2}{2}\right) + \frac{F_0E_2}{4}d_1^2. \quad (3.86)$$

Furthermore, by employing equations (3.65) and (3.84) and comparing the coefficients of w and w^2 , we obtain

$$-\frac{1}{\tau}(1 + \mu)a_2 = \frac{F_0E_1}{2}e_1, \quad (3.87)$$

and

$$\frac{1}{\tau}(1 + 2\mu + 2\eta)(2a_2^2 - a_3) = \frac{F_1E_1}{2}e_1 + \frac{F_0E_1}{2}\left(e_2 - \frac{e_1^2}{2}\right) + \frac{F_0E_2}{4}e_1^2. \quad (3.88)$$

From (3.85) and (3.87), we obtain

$$d_1 = -e_1 \quad (3.89)$$

and

$$\frac{2}{\tau^2}(1 + \mu)^2a_2^2 = F_0^2E_1^2(d_1^2 + e_1^2). \quad (3.90)$$

Now, adding (3.86), (3.88), we obtain

$$\begin{aligned} \frac{2}{\tau}(1 + 2\mu + 2\eta)a_2^2 \\ = 2F_0E_1(d_2 + e_2) + F_0(E_2 - E_1)(d_1^2 + e_1^2). \end{aligned} \quad (3.91)$$

Applying $|d_i| \leq 2$ and $|e_i| \leq 2$ for the coefficients d_1 , e_1 , d_2 and e_2 , we immediately have

$$|a_2| \leq \frac{2\tau F_0E_1}{1 + \mu},$$

and

$$|a_2| \leq 2 \sqrt{\frac{|\tau F_0 E_2|}{1 + 2\mu + 2\eta}}.$$

Furthermore, to derive the upper limit of $|a_3|$, we can remove equation (3.86) from equation (3.88), resulting in

$$a_3 = a_2^2 + \frac{2\tau F_1 E_1 d_1 + \tau F_0 E_1 (d_2 - e_2)}{4(1 + 2\mu + 2\eta)}. \quad (3.92)$$

In light of (3.90), (3.91) and putting (3.92), we have

$$|a_3| \leq \frac{4F_0^2 E_1^2 \tau^2}{(1 + \mu)^2} + \frac{\tau E_1 (F_1 + F_0)}{1 + 2\mu + 2\eta}$$

and

$$|a_3| \leq \frac{4\tau F_0 E_2}{1 + 2\mu + 2\eta} + \frac{\tau E_1 (F_1 + F_0)}{1 + 2\mu + 2\eta}.$$

This completes the proof Theorem (3.2.2).

By putting $\mu = 1 + 2\eta$ in Theorem (3.2.2), we get the next Corollary:

Corollary (3.2.2). Let f defined by (3.44) belongs to the class $W_\Sigma^q(\tau, \eta, \phi)$.

Then

$$|a_2| \leq \min \left\{ \frac{\tau F_0 E_1}{1 + \eta}, 2 \sqrt{\frac{|\tau F_0 E_2|}{3(1 + 2\eta)}} \right\}$$

and

$$|a_3| \leq \min \left\{ \frac{F_0^2 E_1^2 \tau^2}{(1 + \eta)^2} + \frac{\tau E_1 (F_1 + F_0)}{3(1 + 2\eta)}, \frac{4\tau F_0 E_2}{3(1 + 2\eta)} + \frac{\tau E_1 (F_1 + F_0)}{3(1 + 2\eta)} \right\}.$$

3.3 Coefficient Bounds of Subclass of Bi-Univalent Functions Associated with Horadam Polynomials

Let $\mathcal{A}$ be the class of all normalized analytic functions f in an open unit disk $U = \{z: z \in \mathbb{C}, |z| < 1\}$ of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (z \in U). \quad (3.93)$$

A function f has an inverse f^{-1} is satisfying $f^{-1}(f(z)) = z, (z \in U)$, and

$$f(f^{-1}(w)) = w, \quad \left(|w| < r_0(f), r_0(f) \geq \frac{1}{4}\right),$$

where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \dots, (w \in U). \quad (3.94)$$

If f and f^{-1} are univalent functions in U , then f is said to be bi-univalent in U and the class of bi-univalent functions defined in U is denoted by Σ . (see [38]).

If f and g are two analytic functions in U , we say that f is subordinate to g or g is said to be superordinate to f , written symbolically as follows (see [49]) :

$$f < g \quad \text{or} \quad f(z) < g(z), \quad (z \in U),$$

if there exists a Schwarz function $w(z)$, with $w(0) = 0$ and $|w(z)| < 1$ ($z \in U$), such that $f(z) = g(w(z)), (z \in U)$.

If the function g is univalent in U , we have the following equivalence

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U).$$

Ma and Minda [49] defined a class of starlike and convex functions by using the method of subordination and studied classes $S^*(\phi)$ and $G^*(\phi)$ which is defined by

$$S^*(\phi) = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \phi(z), z \in U \right\},$$

and

$$G^*(\phi) = \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \phi(z), z \in U \right\}.$$

By $S_{\Sigma}^*(\phi)$ and $G_{\Sigma}^*(\phi)$, we denote to bi-starlike and bi-convex functions f is bi-starlike and bi-convex of Ma-Minda type respectively [49].

However, there are only a few works determining the general coefficient bounds $|a_2|$ and $|a_3|$ ([7,8,9,21,32,36] for the analytic bi-univalent functions in the literature. ([21,29,30]).

Hörcum and Kocer [38] studied the Horadam polynomials $h_n(r)$ which are defined by the following repetition relation (also see [37]):

$$h_n(r) = prh_{n-1}(r) + qh_{n-2}(r) \quad (r \in \mathbb{R}, n \in \mathbb{N}),$$

with

$$h_1(r) = a \quad \text{and} \quad h_2(r) = br, \tag{3.95}$$

The generating function of the Horadam polynomials $h_n(r)$ (see [28]) is given by

$$\Pi(r, z) = \sum_{n=1}^{\infty} h_n(r) z^{n-1} = \frac{a+(b-ap)rz}{1-prz-qz^2}. \quad (3.96)$$

Definition (3.3.1). A function $f \in \Sigma$ defined by (3.93) is said to be in the class $\mathcal{H}_{\Sigma}(\rho, \delta, \beta, r)$, if the following conditions:

$$1 + \left(1 + \frac{zf''(z)}{f'(z)}\right) \left[\frac{1}{\rho} \left\{ (1-\delta) \frac{f(z)}{z} + \delta \left(\frac{zf'(z)}{f(z)} \right) + \beta z f''(z) \right\} - 1 \right] < \Pi(r, z) + 1 - a, \quad (3.97)$$

and

$$1 + \left(1 + \frac{wg''(w)}{g'(w)}\right) \left[\frac{1}{\rho} \left\{ (1-\delta) \frac{g(w)}{w} + \delta \left(\frac{wg'(w)}{g(w)} \right) + \beta w g''(w) \right\} - 1 \right] < \Pi(r, z) + 1 - a, \quad (3.98)$$

where $(\delta \geq 0, \beta \geq 0, \rho \in \mathbb{C} \setminus \{0\}, r \in \mathbb{R}), z, w \in U$ and $g = f^{-1}$ is given by (3.94).

If we put $\rho = 1$ and $\delta = 1$ in Definition (3.3.1), we obtain the following Remark such that $\mathcal{H}_{\Sigma}(\beta, r) = \mathcal{H}_{\Sigma}(\rho, \delta, \beta, r)$.

Remark (3.3.1). A function $f \in \Sigma$ defined by (3.93) is said to be in the class $\mathcal{H}_{\Sigma}(\beta, r)$, if it is hold the conditions:

$$1 + \left(1 + \frac{zf''(z)}{f'(z)}\right) \left[\left\{ \frac{f(z)}{z} + \beta z f''(z) \right\} - 1 \right] < \Pi(r, z) + 1 - a, \quad (3.99)$$

and

$$1 + \left(1 + \frac{wg''(w)}{g'(w)}\right) \left[\left\{ \frac{g(w)}{w} + \beta w g''(w) \right\} - 1 \right] < \Pi(r, z) + 1 - a, \quad (3.100)$$

where $(\beta \geq 0, r \in \mathbb{R}), z, w \in U$ and $g = f^{-1}$ is given by (3.94).

If we put $\delta = 1$ and $\beta = 0$ in Definition(3.3.1), we obtain the following Remark such that $\mathcal{H}_{\Sigma}(\rho, r) = \mathcal{H}_{\Sigma}(\rho, \delta, \beta, r)$.

Remark (3.3.2). A function $f \in \Sigma$ defined by (3.93) is said to be in the class $\mathcal{H}_\Sigma(\rho, r)$, if it holds the conditions:

$$1 + \left(1 + \frac{zf''(z)}{f'(z)}\right) \left[\frac{1}{\rho} \left\{ \left(\frac{zf'(z)}{f(z)} \right) \right\} - 1\right] < \Pi(r, z) + 1 - a, \quad (3.101)$$

and

$$1 + \left(1 + \frac{wg''(w)}{g'(w)}\right) \left[\frac{1}{\rho} \left\{ \left(\frac{wg'(w)}{g(w)} \right) \right\} - 1\right] < \Pi(r, z) + 1 - a, \quad (3.102)$$

where $(\rho \in \mathbb{C} \setminus \{0\}, r \in \mathbb{R}), z, w \in U$ and $g = f^{-1}$ is given by (3.94).

Theorem (3.3.1). For $\delta \geq 0, \beta \geq 0, \rho \in \mathbb{C} \setminus \{0\}$ and $r \in \mathbb{R}$, let $f \in \mathcal{A}$ be in the class $\mathcal{H}_\Sigma(\rho, \delta, \beta, r)$. Then

$$|a_2| \leq \frac{\rho |br| \sqrt{|br|}}{\sqrt{|\rho b^2 r^2 (2(1+2\beta) + (1+\delta+6\beta)) - (pbr^2 + qa)(1+2\beta)^2|}}, \quad (3.103)$$

and

$$|a_3| \leq \frac{|br|}{1 + \delta + 6\beta} + \frac{\rho^2 b^2 r^2}{(1 + 2\beta)^2}. \quad (3.104)$$

Proof. Let $f \in \mathcal{H}_\Sigma(\rho, \delta, \beta, r)$ and $g = f^{-1}$. Then, there are two analytic functions $u, v: U \rightarrow U$ given by

$$u(z) = u_1 z + u_2 z^2 + u_3 z^3 + \dots, \quad (z \in U) \quad (3.105)$$

and

$$v(w) = v_1 w + v_2 w^2 + v_3 w^3 + \dots, \quad (w \in U) \quad (3.106)$$

with $u(0) = 0$ and $v(0) = 0, |u(z)| < 1, |v(w)| < 1$, such that

$$1 + \left(1 + \frac{zf''(z)}{f'(z)}\right) \left[\frac{1}{\rho} \left\{ (1 - \delta) \frac{f(z)}{z} + \delta \left(\frac{zf'(z)}{f(z)} \right) + \beta zf''(z) \right\} - 1 \right] = \Pi(r, z) + 1 - a,$$

and

$$1 + \left(1 + \frac{wg''(w)}{g'(w)}\right) \left[\frac{1}{\rho} \left\{ (1 - \delta) \frac{g(w)}{w} + \delta \left(\frac{wg'(w)}{g(w)} \right) + \beta wg''(w) \right\} - 1 \right] = \Pi(r, z) + 1 - a.$$

Or, equivalently

$$\begin{aligned} 1 + \left(1 + \frac{zf''(z)}{f'(z)}\right) \left[\frac{1}{\rho} \left\{ (1 - \delta) \frac{f(z)}{z} + \delta \left(\frac{zf'(z)}{f(z)} \right) + \beta zf''(z) \right\} - 1 \right] \\ = 1 + h_1(r) - a + h_2(r)u(z) + h_3(r)u^2(z) + \dots \end{aligned} \quad (3.107)$$

and

$$\begin{aligned} 1 + \left(1 + \frac{wg''(w)}{g'(w)}\right) \left[\frac{1}{\rho} \left\{ (1 - \delta) \frac{g(w)}{w} + \delta \left(\frac{wg'(w)}{g(w)} \right) + \beta wg''(w) \right\} - 1 \right] \\ = 1 + h_1(r) - a + h_2(r)v(w) + h_3(r)v^2(w) + \dots \end{aligned} \quad (3.108)$$

Combining (3.105), (3.106), (3.107) and (3.108), yield

$$\begin{aligned} 1 + \left(1 + \frac{zf''(z)}{f'(z)}\right) \left[\frac{1}{\rho} \left\{ (1 - \delta) \frac{f(z)}{z} + \delta \left(\frac{zf'(z)}{f(z)} \right) + \beta zf''(z) \right\} - 1 \right] \\ = 1 + h_2(r)u_1z + [h_2(r)u_2 + h_3(r)u_1^2]z^2 + \dots \end{aligned} \quad (3.109)$$

and

$$\begin{aligned} 1 + \left(1 + \frac{wg''(w)}{g'(w)}\right) \left[\frac{1}{\rho} \left\{ (1 - \delta) \frac{g(w)}{w} + \delta \left(\frac{wg'(w)}{g(w)} \right) + \beta wg''(w) \right\} - 1 \right] \\ = 1 + h_2(r)v_1w + [h_2(r)v_2 + h_3(r)v_1^2]w^2 + \dots \end{aligned} \quad (3.110)$$

It is quite well-known that if $|u(z)| < 1$, $|v(w)| < 1$, $z, w \in U$, we get

$$|u_i| < 1 \text{ and } |v_i| < 1 \ (\forall i \in \mathbb{N}). \quad (3.111)$$

Comparing the corresponding coefficients in (3.109) and (3.110), after simplifying, yields

$$\frac{1}{\rho}(1 + 2\beta)a_2 = h_2(r)u_1, \quad (3.112)$$

$$\frac{1}{\rho}[(1 + \delta + 6\beta)a_3 + 2(1 + 2\beta)a_2^2] = h_2(r)u_2 + h_3(r)u_1^2, \quad (3.113)$$

$$-\frac{1}{\rho}(1 + 2\beta)a_2 = h_2(r)v_1, \quad (3.114)$$

And

$$\begin{aligned} \frac{1}{\rho}[(1 + \delta + 6\beta)(2a_2^2 - a_3) + 2(1 + 2\beta)a_2^2] = \\ h_2(r)v_2 + h_3(r)v_1^2. \end{aligned} \quad (3.115)$$

From (3.112) and (3.114), we have

$$u_1 = -v_1, \quad (3.116)$$

and

$$2(1 + 2\beta)^2 a_2^2 = \rho^2 h_2^2(r)(u_1^2 + v_1^2). \quad (3.117)$$

If we add (3.113) to (3.115), we deduce that

$$\begin{aligned} \frac{1}{\rho}[4(1 + 2\beta) + 2(1 + \delta + 6\beta)]a_2^2 = h_2(r)(u_2 + v_2) + h_3(r)(u_1^2 + v_1^2). \\ (3.118) \end{aligned}$$

Substituting the value of $u_1^2 + v_1^2$ from (3.117) in the right hand side of (3.118), we find that

$$a_2^2 = \frac{\rho^2 h_2^3(r)(u_2 + v_2)}{\rho h_2^2(r)[4(1 + 2\beta) + 2(1 + \delta + 6\beta)] - 2h_3(r)(1 + 2\beta)^2}. \quad (3.119)$$

Further computations using (3.95), (3.111) and (3.119), we get

$$|a_2| \leq \frac{\rho |br| \sqrt{|br|}}{\sqrt{|\rho b^2 r^2 (2(1 + 2\beta) + (1 + \delta + 6\beta)) - (pbr^2 + qa)(1 + 2\beta)^2|}}.$$

Now, if we subtract (3.115) from (3.113), we get

$$2(1 + \delta + 6\beta)(a_3 - a_2^2) = h_2(r)(u_2 - v_2) + h_3(r)(u_1^2 - v_1^2). \quad (3.120)$$

In view of (3.116) and (3.117), we obtain from (3.120)

$$a_3 = \frac{h_2(r)(u_2 - v_2)}{2(1 + \delta + 6\beta)} + \frac{\rho^2 h_2^2(r)(u_1^2 + v_1^2)}{2(1 + \delta + 6\beta)}.$$

Applying (3.95), we can easily see that

$$|a_3| \leq \frac{|br|}{1 + \delta + 6\beta} + \frac{\rho^2 b^2 r^2}{(1 + 2\beta)^2}.$$

By putting $\rho = 1$ and $\delta = 1$ in Theorem (3.3.1), we get the next Corollary:

Corollary (3.3.1). For $\beta \geq 0$ and $r \in \mathbb{R}$, let $f \in \mathcal{A}$ be in the class $\mathcal{H}_\Sigma(\beta, r)$. Then

$$|a_2| \leq \frac{|br| \sqrt{|br|}}{\sqrt{|b^2 r^2 (2(1+2\beta)+(1+6\beta))-(pbr^2+qa)(1+2\beta)^2|}}, \quad (3.121)$$

and

$$|a_3| \leq \frac{|br|}{1 + 6\beta} + \frac{b^2 r^2}{(1 + 2\beta)^2}. \quad (3.122)$$

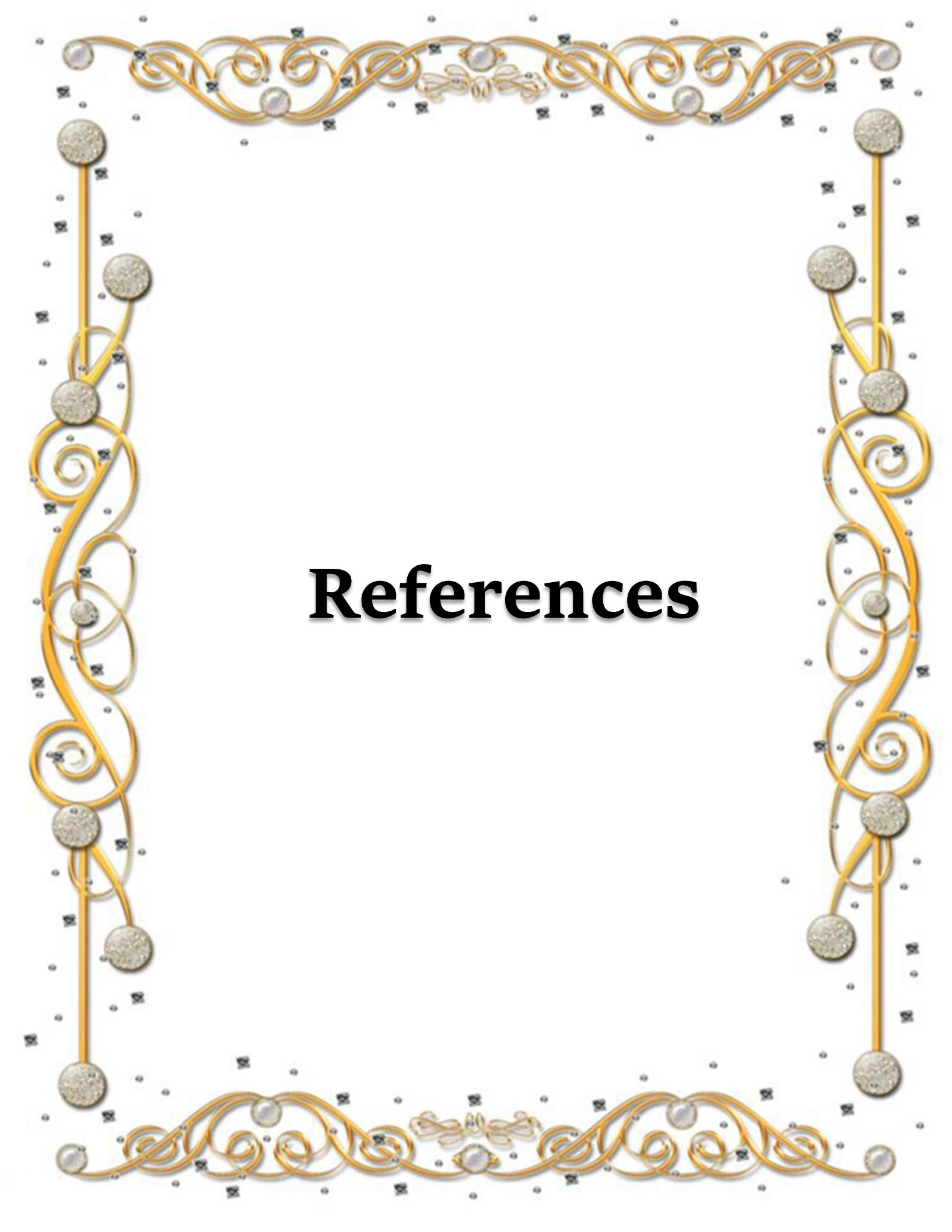
By putting $\delta = 1$ and $\beta = 0$ in Theorem (3.3.1), we get the next Corollary:

Corollary (3.3.2). For $\rho \in \mathbb{C} \setminus \{0\}$ and $r \in \mathbb{R}$, let $f \in \mathcal{A}$ be in the class $\mathcal{H}_\Sigma(\rho, r)$. Then

$$|a_2| \leq \frac{\rho |br| \sqrt{|br|}}{\sqrt{|4\rho b^2 r^2 - (pbr^2 + qa)|}}, \quad (3.123)$$

and

$$|a_3| \leq \frac{|br|}{2} + \rho^2 b^2 r^2. \quad (3.124)$$

A decorative border surrounds the central text. It features gold-colored swirling lines, small silver beads, and larger circular silver ornaments. The border is symmetrical and ornate.

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المستخلص

الغرض من هذه الرسالة هو مناقشة نظرية الدالة الهندسية باستخدام التابعيات وشبه التابعيات التفاضلية. هي دراسة نتائج جديدة للتابعيات والتابعيات العليا التفاضلية للدوال احادية التكافؤ التحليلية المعرفة بواسطة مؤثر تفاضلي. حصلنا على بعض مبرهنات الساندوج لصنف جزئي من الدوال احادية التكافؤ المعرفة بواسطة مؤثر تفاضلي $\mathcal{N}_{\alpha,\lambda}^n f(z)$ ايضاً تم دراسة نتائج الساندوج التفاضلية للدوال متعددة التكافؤ الميرومورفية المعرفة بواسطة المؤثر الخطي حيث قدمنا نتائج الساندوج للدوال متعددة التكافؤ الميرومورفية باستخدام نتائج التابعية والتابعية العليا التفاضلية والمعرفة بواسطة المؤثر الخطي $\mathfrak{M}_{\alpha,\beta,p,\eta}^{\gamma,k,\mu} f(z)$. ناقشت رسالتنا حدود المعامل ومسألة فيكييتي - سيزيكو لأصناف جزئية من الدوال ثنائية التكافؤ وتركز هذه الدراسة على نوعين من الدوال ثنائية التكافؤ $\Psi_{\Sigma}(\delta, \tau; \alpha)$ و $D_{\Sigma}(\delta, \tau; \beta)$ حيث حصلنا على مخمنات معاملات تايلر - ماكلورين $|a_2|$ و $|a_3|$ للدوال في هذه الاصناف الجزئية بالإضافة إلى هذا جهزنا حل المسألة فيكييتي - سيزيكو للدوال في هذه الاصناف المحددة. رسالتنا تعاملت ايضاً مع نتائج حول اصناف جزئية جديدة للدوال ثنائية التكافؤ باستخدام شبه التابعية والدراسة الحالية قدمت صنفين جزئيين محددين $W_{\Sigma}^q(\tau, \mu, \eta, \phi)$ و $M_{\Sigma}^q(\beta, \rho, \lambda)$ من الدوال ثنائية التكافؤ والمعرفة بواسطة شبه التابعية حيث حصلنا على تخمينات لمعاملات تايلر - ماكلورين $|a_2|$ و $|a_3|$ للدوال في هذه الاصناف الجزئية. اشتقينا حدود المعاملات لصنف جزئي من الدوال ثنائية التكافؤ المرتبطة مع متعددات حدود هوردام وهنا استخدمنا متعددات حدود هوردام لتقديم صنف من الدوال ثنائية التكافؤ التحليلية والمعرفة في قرص الوحدة المفتوح U وايجاد الحدود العليا للمعاملات الثاني والثالث للدوال الموجودة في هذا الصنف الجزئي الجديد.



جمهورية العراق

وزارة التعليم العالي والبحث العلمي

جامعة القادسية - كلية العلوم

قسم الرياضيات

حول نظرية الدالة الهندسية باستخدام التابعيات وشبه التابعيات التفاضلية

رساله

مقدمه الى مجلس كليه العلوم في جامعة القادسية كجزء من متطلبات نيل درجه

ماجستير علوم في الرياضيات

من قبل

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