The Bayesian Reciprocal Bridge for Composite Quantile Regression with Ordinal Longitudinal Data

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Abstract

This paper proposes, a Bayesian reciprocal bridge composite quantile regression is proposed for variable selection and estimation in ordinal Longitudinal data. A new Gibbs sampling algorithm is constructed for sampling from the full conditional posterior distributions. The proposed approach is illustrated using simulation studies. By using the simulation studies example, we show that the performance of the proposed approach is very well compared with the existing approaches.

Key words: Reciprocal Bridge, Composite Quantile Regression, Gibbs sampler, Ordinal Longitudinal data.

1 Introduction

Quantile regression (QR) has attracted much studies on theoretical properties (see e.g ,Koenker,2005). All of this studies point to many benefits of approach . Most attractive, the ability to introduce unusual errors, and thus make it insensitive to covariance and outliers (Koenker and Bassett,Koenker,2005). Moreover, the quantile regression of the other features compared to the mean regression gives more detail to the relationship between the response variable and the predictors, thus the quantile regression presented by (Koenker and Bassett 1978) was an extension of the standard mean regression. These privileges have led to a practical application of interest in a number of fields such as ecology, science, economics, finance, medicine, and genetic studies, and natural phenomena (see,Yu et al., 2003; Koenker,2005; **Alhamzawi et al. 2011**).

The features of quantile regression became attracted when the data does not satisfy the assumptions of the mean regression. complex computational difficulties were dealt with, especially the non-differentiation of the loss function. Quantile regression estimation is done through the use of special algorithms and reliable estimation methods. classical methods used the simple algorithm and the internal point algorithm. Bayesian methods used the technique samples from Markov chain Monte Carlo(MCMC).the challenge for the development of Bayesian quantile regression is the error does not follow any distribution. Koenker and Machado (1999) showed that the objective function is equal to the exponent in the asymmetric Laplace distribution (ALD) (Kotz et al.,2001;Yu and Zhang,2005). this distribution was implemented by Yu and Moyeed (2001). then the algorithm was developed see Tsioonas (2003), (Reed and Yu (2009) . Finally Kozumi and Kobayshi (2011) proposed a Gibbs sampling assuming the exponential natural mixture of (ALD).

Composite quantile regression appeared as a parametric estimation model. It possesses the characteristics of quantile regression (free distribution, variance, and immunity), moreover, it is superior to single quantile regression in efficiency over median regression.

Regularization methods (Koenker 2004) have proven effective in selecting a variable and estimating a coefficient when the model contains a large number of variables that reduce the accuracy of the prediction.

Quantile regression differs when the response is ordinal, in which the dependent variable is an ordinal discrete value. the goal of interest in ordinal quantile regression is to obtain a richer description of the effect of covariates on the results. Ordinal quantile regression in the literature was estimated using simulated annealing by Zhou (2010). The Bayesian estimate of ordinal quantile regression was presented for the first time in Rahman (2016). As a special case we will address the longitudinal ordinal data.

ordinal Longitudinal data appear in many fields, including medicine, economics, and social studies.

Longitudinal data is a set of observations for each variable in different periods of time. Koenker (2004) used quantile regression for longitudinal data. Geraci and Bottai (2007) proposed Bayesian quantile regression for longitudinal data using the (ALD) distribution of errors. Alhamzawi and Yu (2014) suggested a method of regularization with mixed quantile regression . A Bayesian quantile regression method for parameter estimation in longitudinal ordinal data was introduced by Alhamzawi and Ali (2018).

this study, the composite quantile regression approach will be addressed with longitudinal ordinal data using the bridge penalty function. The approach presents a method of variable selection and parameter estimation that is more efficient than the regularization method of single quantile regression with longitudinal ordinal data. In Sect. 2, we describe the considered model and its hierarchical representation. In Sect. 3, the Gibbs samplers of Bayesian bridge-randomized QR for ordinal longitudinal data is presented. In Sect. 4, numerical studies are implemented to illustrate the proposed methods. Section 5 provides a real data example to illustrate the proposed estimation procedure. The last section draws some conclusions.

2. Methods

We define the response variables $\boldsymbol{y_i}$ for $\boldsymbol{n}$ of the samples indexed by $\boldsymbol{i} \in \{\mathbf{1}, \dots, \boldsymbol{n}\}$, with k of the covariates $\boldsymbol{x_i}$.

we begin by defining the continuous response variable $\boldsymbol{y_i}$, starting from the classical model up to the Bayesian approach. Next we present the composite quantile regression approach and variable selection method for longitudinal ordinal $\boldsymbol{y_i}$ and associated inference methods.

2.1. Quantile Regression

Quantile regression is concerned with estimating the parameters $\widehat{\boldsymbol{\beta}}$ of the q^{th} quantile of $\boldsymbol{y}|\boldsymbol{x}$.

Quantile regression has emerged as an alternative to the Standard Model. Standard regression estimates parameters that minimize the sum of squares of error as follows

$$\boldsymbol{argmin} \sum_{i=1}^{n} \left(\boldsymbol{y_i} - \boldsymbol{x_i^T \beta}\right)^2. \qquad (1)$$

Quantile inference uses a similar method, but at a conditional quantile. More precisely, the optimization problem depends on $\boldsymbol{\tau}^{th}$ and this work is done by the check function, and the model can be written as follow:

$$\sum_{i=1}^{n} \boldsymbol{\rho_\tau}\left(\mathbf{y_i} - \mathbf{x_i^T} \boldsymbol{\beta}\right). \qquad (2)$$

where $\boldsymbol{\rho_\tau}(.)$is the check function defined by

$$\boldsymbol{\rho_\tau}(\boldsymbol{t}) = \frac{|\boldsymbol{t}| + (\mathbf{2}\boldsymbol{\tau} - \mathbf{1})\boldsymbol{t}}{\mathbf{2}}, \qquad (3)$$

In Bayesian approaches, koenker and D'orey (1987)suggested that miniaturization can be achieved using an algorithm while Koenker and Machado (1999) proposed the estimated of $\boldsymbol{\beta}$ can be through the link between asymmetric Laplace distribution (ALD) and the unknown parameters $\boldsymbol{\beta}$.

$$\boldsymbol{f}_{(y|\mu,\sigma)} = \frac{\tau(1-\tau)}{\sigma} \boldsymbol{exp}\left\{\frac{\rho_\tau(y-\mu)}{\sigma}\right\}, \qquad (4)$$

the model of Bayesian QR take form of :

$$\boldsymbol{Q}_{y_i}(\boldsymbol{\tau}|\boldsymbol{x}_i) = \boldsymbol{b}_\tau + \boldsymbol{x}_i^T\boldsymbol{\beta}, \qquad (5)$$

where $\boldsymbol{b}_\tau$ is the quantile intercept. The regression parameters $\boldsymbol{b}_\tau$ and $\boldsymbol{\beta}$ are estimated by minimizing

$$\boldsymbol{min} \sum_{i=1}^{n} \boldsymbol{\rho}_\tau\left(\boldsymbol{y}_i - \boldsymbol{b}_\tau - \boldsymbol{x}_i^T\boldsymbol{\beta}\right), \qquad (6)$$

Zou and Yuan (2008) proposed composite quantile regression (CQR) as a more efficient and robust approach. The CQR estimators of $\boldsymbol{b}_q$ and $\boldsymbol{\beta}$ can be estimated by minimizing

$$\mathbf{min} \sum_{k=1}^{K}\left\{\sum_{i=1}^{n} \boldsymbol{\rho}_\tau\left(\boldsymbol{y}_i - \boldsymbol{b}_{\tau_k} - \boldsymbol{x}_i^T\boldsymbol{\beta}\right)\right\}, \qquad (7)$$

When the solution to minimize is not differentiable .than will not be close form solution for $\boldsymbol{\beta}$ (Koenker,2005).

Huang and Chen (2015) and Alhamzawi (2016) show that the minimization problem (7) can be cast into a pseudo likelihood setting of a CQR of the form:

$$\mathbf{f}_{(\mathbf{y}|\mathbf{x})} = \prod_{k=1}^{K}\prod_{i=1}^{n}\left[\frac{\tau(1-\tau)}{\sigma}\mathbf{exp}\left\{-\frac{\rho_\tau\left(y_i - b_{\tau_k} - x_i^T\beta\right)}{\sigma}\right\}\right], \qquad (8)$$

One of the attractive properties of the ALD is that it can be viewed as a normal-exponential mixture representation, which brings Gibbs algorithm and hierarchical formulation for Bayesian QR. See the detail from Kozumi and Kobayashi (2011) and Alhamzawi and Yu (2013b). This mixture representation can be written as

$$\boldsymbol{\varepsilon}_i = \boldsymbol{\theta}\mathbf{v}_i + \sqrt{2\sigma\mathbf{v}_i}\,\mathbf{u}_i \ ,$$

Where $\boldsymbol{v}_i$ and $\boldsymbol{u}_i$ are mutually independent , $\boldsymbol{u}_i \sim \boldsymbol{N}(\mathbf{0},\mathbf{1})$,$\boldsymbol{v}_i \sim \boldsymbol{exp}\left(\frac{1}{\tau(1-\tau)}\right)$,and $\boldsymbol{\theta} = \frac{1-2\tau}{\tau(1-\tau)}$.

Then the joint distribution of y given by

$$\boldsymbol{p}\left(\boldsymbol{y}\middle|\boldsymbol{x},\boldsymbol{\beta},\boldsymbol{b}_q,\boldsymbol{v},\boldsymbol{\sigma}\right) = \prod_{k=1}^{K}\prod_{i=1}^{n}\left(\frac{1}{\sqrt{4\pi\sigma v_i}}\right)\boldsymbol{exp}\left\{-\frac{\left(y_i - b_\tau - x_i^T\beta - \theta v_i\right)^2}{4\sigma v_i}\right\}, \qquad (9)$$

3.Ordinal Longitudinal Data with Bayesian Composite Quantile Regression method

The response variable $\boldsymbol{y}_{ij}$ at sample $\boldsymbol{ith}$ measured at time $\boldsymbol{jth}$ where $\boldsymbol{i} = \mathbf{1},\dots,\boldsymbol{n}$ and $\boldsymbol{j} = \mathbf{1},\dots,\boldsymbol{J}$, can be modeled through the ordinal latent variable $\boldsymbol{z}_{ij}$ as follows:

$$\boldsymbol{y_{ij}} = \begin{cases} \mathbf{1} & \boldsymbol{if} \quad \boldsymbol{\delta_0} < \boldsymbol{z_{ij}} \le \boldsymbol{\delta_1}; \\ \boldsymbol{c} & \boldsymbol{if} \quad \boldsymbol{\delta_{c-1}} < \boldsymbol{z_{ij}} \le \boldsymbol{\delta_c}; \qquad \boldsymbol{c} = \mathbf{2}, \dots, \boldsymbol{C} - \mathbf{1}; \\ \boldsymbol{C} & \boldsymbol{if} \quad \boldsymbol{\delta_{C-1}} < \boldsymbol{z_{ij}} \le \boldsymbol{\delta_C}; \end{cases} \tag{10}$$

Where $\boldsymbol{\delta_0}, \dots, \boldsymbol{\delta_C}$ are cut-points, that fall with the period $-\infty = \boldsymbol{\delta_0} < \boldsymbol{\delta_1} < \cdots < \boldsymbol{\delta_{C-1}} < \boldsymbol{\delta_C} < +\infty$. Then ,the $\boldsymbol{kth}$ quantile regression model for ordinal longitudinal data using $\boldsymbol{z_{ij}}$ as:

$$\boldsymbol{z_{ij}} = \boldsymbol{b_{\tau_k}} + \boldsymbol{x_{ij}^T \beta} + \boldsymbol{\varepsilon_{ij}} \ , \tag{11}$$

Where $\boldsymbol{x_{ij}}$ is a $\boldsymbol{k} \times \mathbf{1}$ vector of explanatory variables , $\boldsymbol{\beta}$ is a $\boldsymbol{k} \times \mathbf{1}$ vector for model parameters.

Assuming that the error $\boldsymbol{\varepsilon_{ij}}$of the unobserved response $\boldsymbol{y_{ij}}$ has a SLD as in (4), we have $\boldsymbol{\varepsilon_{ij}} = \boldsymbol{\theta v_{ij}} + \sqrt{\boldsymbol{2\sigma v_{ij}}}\boldsymbol{u_{ij}}$, (Kozumi and Kobayashi, 2011). Here, the latent variable $\boldsymbol{v_{ij}}$ follows an exponential distribution, and $\boldsymbol{u_{ij}}$follows the standard normal distribution. Then equation (9) can be rewritten as hierarchical Bayesian model using longitudinal ordinal composite quantile regression

$$\boldsymbol{p}(\boldsymbol{z}|\boldsymbol{x}, \boldsymbol{\beta}, \boldsymbol{b_q}, \boldsymbol{v}, \boldsymbol{\sigma}) = \prod_{\boldsymbol{k=1}}^{\boldsymbol{K}} \prod_{\boldsymbol{i=1}}^{\boldsymbol{n}} \prod_{\boldsymbol{j=1}}^{\boldsymbol{J}} \left(\frac{\mathbf{1}}{\sqrt{\boldsymbol{4\pi\sigma v_{ij}}}}\right) \boldsymbol{exp}\left(-\frac{\left(\boldsymbol{z_{ij}} - \boldsymbol{b_{\tau_k}} - \boldsymbol{x_{ij}^T\beta} - \boldsymbol{\theta v_{ij}}\right)^2}{\boldsymbol{4\sigma v_{ij}}}\right) \ , \tag{12}$$

3.1 Bayesian Reciprocal bridge approach of the model

The reciprocal bridge estimator can be written by use the formula in (Alhamzawi ,Mallick . 2020) which following:

$$\boldsymbol{argmin} \sum_{\boldsymbol{k=1}}^{\boldsymbol{K}} \left\{ \sum_{\boldsymbol{i=1}}^{\boldsymbol{n}} \sum_{\boldsymbol{j=1}}^{\boldsymbol{J}} \ \boldsymbol{\rho_{\tau_k}}\left(\boldsymbol{z_{ij}} - \boldsymbol{b_{\tau_k}} - \boldsymbol{x_{ij}^T \beta}\right) \right\} + \boldsymbol{\lambda} \sum_{\boldsymbol{g=1}}^{\boldsymbol{G}} \frac{\mathbf{1}}{|\boldsymbol{\beta_g}|^{\alpha}} \boldsymbol{I}\{\boldsymbol{\beta_g} \neq \mathbf{0}\} \tag{13}$$

where λ is parameter of regularization for α , when it is equal to zero, it corresponds L_0, and when it is equal to one, it shows reciprocal LASSO, and when it is equal to 2, reciprocal ridge appear, where the Bayesian approach solves the problem of miniaturization in cases of small samples as well.

Noting the penalty term in (13), the bridge estimates can be interpreted as posterior mode estimates when the regression parameters have Inverse Generalized Gaussian (IGG) distribution (Mallick et al., 2020) of the form

$$\boldsymbol{\pi}(\boldsymbol{\beta}) = \prod_{\boldsymbol{g=1}}^{\boldsymbol{G}} \frac{\boldsymbol{\lambda}^{\frac{\mathbf{1}}{\alpha}}}{\mathbf{2}\boldsymbol{\beta_g^2}\Gamma\left(\frac{\mathbf{1}}{\alpha}+\mathbf{1}\right)} \boldsymbol{exp}\left\{-\frac{\boldsymbol{\lambda}}{|\boldsymbol{\beta_g}|^{\alpha}}\right\} \boldsymbol{I}\{\boldsymbol{\beta_g} \neq \mathbf{0}\}, \tag{14}$$

The Gibbs sampler for the Bayesian reciprocal bridge exploits the following representation of the scale mixture of normal(SMN) following Armagan ,Dunson and Lee (2013);Mallick , Alhamzawi ,and Svetnik(2020). If we assume that $\boldsymbol{\beta} \sim \boldsymbol{N}(\mathbf{0}, \boldsymbol{l}) \boldsymbol{I}(|\boldsymbol{\beta}| > \eta)$, $\boldsymbol{l} \sim \boldsymbol{Exp}\left({\boldsymbol{\xi}^{2}}/{\mathbf{2}}\right)$,and $\boldsymbol{\xi} \sim \boldsymbol{Exp}(\eta)$,then the inverse double exponential distribution for $\boldsymbol{\beta}$ with scale parameter $\boldsymbol{\lambda} > \mathbf{0}$ arises when η follows Inverse Gamma (2,λ) .

Where $\boldsymbol{u} = \frac{\mathbf{1}}{\eta}$,$\boldsymbol{l} = (\boldsymbol{l}_{\mathbf{1}}, \dots, \boldsymbol{l}_{\boldsymbol{G}})'$ and $\boldsymbol{\xi} = (\boldsymbol{\xi}_{\mathbf{1}}, \dots, \boldsymbol{\xi}_{\boldsymbol{G}})'$. To specify a prior distribution for δ , we follow Alhamzawi (2016), we assign order statistics from uniform $(\boldsymbol{\delta}_{\mathbf{0}}, \boldsymbol{\delta}_{\boldsymbol{C}})$distribution for the $\boldsymbol{C} - \mathbf{1}$unknown cut-points :

$$\boldsymbol{P}_{\boldsymbol{\delta}} = (\boldsymbol{C} - \mathbf{1})! \left(\frac{\mathbf{1}}{\boldsymbol{\delta}_{max} - \boldsymbol{\delta}_{min}}\right)^{\boldsymbol{C}-\mathbf{1}} \boldsymbol{I}(\boldsymbol{I} \in \boldsymbol{H}), \qquad (10)$$

Where $\boldsymbol{\delta} = (\boldsymbol{\delta}_{\mathbf{0}}, \boldsymbol{\delta}_{\mathbf{1}}, \dots, \boldsymbol{\delta}_{\boldsymbol{C}})$ and $\boldsymbol{H} = \{(\boldsymbol{\delta}_{min}, \boldsymbol{\delta}_{\mathbf{1}}, \dots, \boldsymbol{\delta}_{max}) | \boldsymbol{\delta}_{min} < \boldsymbol{\delta}_{\mathbf{1}} < \cdots < \boldsymbol{\delta}_{\boldsymbol{C}-\mathbf{1}} < \boldsymbol{\delta}_{max}\}$.

To summarize ,our Bayesian hierarchical formulation :

$$\begin{aligned}
&\boldsymbol{z}_{ij} | \boldsymbol{x} \sim \boldsymbol{N}_{\boldsymbol{n}}\left(\boldsymbol{z}_{ij} + \boldsymbol{b}_{\tau_k} + \boldsymbol{x}_{ij}^{T}\boldsymbol{\beta} + \boldsymbol{\theta v}, \mathbf{2}\boldsymbol{\sigma v}\right),\\
&\boldsymbol{\beta} | \boldsymbol{l} \sim \textstyle\prod_{g=1}^{G} \boldsymbol{N}(\mathbf{0}, \boldsymbol{T}^{2}) \boldsymbol{I}\left\{\left|\boldsymbol{\beta}_{g}\right|^{\alpha} > \frac{\mathbf{1}}{\boldsymbol{u}_{g}}\right\},\\
&\boldsymbol{l} | \boldsymbol{\xi} \sim \textstyle\prod_{g=1}^{G} \boldsymbol{Exp}\left(\boldsymbol{\xi}_{g}^{2}\right),\\
&\boldsymbol{\xi} | \boldsymbol{u} \sim \textstyle\prod_{g=1}^{G} \boldsymbol{Exp}\left(\frac{\mathbf{1}}{\boldsymbol{u}_{g}}\right),\\
&\boldsymbol{u}_{ij} \sim \textstyle\prod_{g=1}^{G} \boldsymbol{Gamma}\,(\mathbf{2}, \boldsymbol{\lambda}), \qquad (15)\\
&\boldsymbol{\sigma} \sim \boldsymbol{\sigma}^{-\mathbf{1}},\\
&\boldsymbol{\lambda} \sim \boldsymbol{\lambda}^{-\mathbf{1}},\\
&\boldsymbol{\beta} \big| \boldsymbol{z}_{ij} \sim \boldsymbol{N}_{\boldsymbol{P}}((\boldsymbol{X}'\Omega^{-\mathbf{1}}\boldsymbol{X} + \boldsymbol{T}^{-\mathbf{1}})^{-\mathbf{1}}\boldsymbol{X}'\Omega^{-\mathbf{1}}(\boldsymbol{z} -\\
&\boldsymbol{\vartheta v}), (\boldsymbol{X}'\Omega^{-\mathbf{1}}\boldsymbol{X} + \boldsymbol{T}^{-\mathbf{1}})^{-\mathbf{1}}) \textstyle\prod_{g=1}^{G} \boldsymbol{I}\left\{\left|\boldsymbol{\beta}_{g}\right|^{\alpha} > \frac{\mathbf{1}}{\boldsymbol{u}_{g}}\right\},\\
&\boldsymbol{v}_{ij}^{-\mathbf{1}} \big| \boldsymbol{z}_{ij} \sim \boldsymbol{Inverse} - \boldsymbol{Gaussian}\left(\frac{\mathbf{1}}{\mathbf{2}}, \frac{\mathbf{1}}{\left|\boldsymbol{z}_{ij} + \boldsymbol{b}_{\tau_k} + \boldsymbol{x}_{ij}^{T}\boldsymbol{\beta}\right|}, \frac{\mathbf{1}}{\mathbf{2}\boldsymbol{\sigma}}\right),\\
&\boldsymbol{l}^{-\mathbf{1}} \big| \boldsymbol{z}_{ij} \sim \textstyle\prod_{g=1}^{G} \boldsymbol{Inverse} - \boldsymbol{Gaussian}\left(\frac{\mathbf{1}}{\mathbf{2}}, \sqrt{\frac{\boldsymbol{\xi}_{g}^{2}}{\boldsymbol{\beta}_{g}^{2}}}, \boldsymbol{\xi}_{g}^{2}\right),\\
&\boldsymbol{\xi} \big| \boldsymbol{z}_{ij} \sim \textstyle\prod_{g=1}^{G} \boldsymbol{Gamma}\left(\left|\boldsymbol{\beta}_{g}\right|^{\alpha} + \frac{\mathbf{1}}{\boldsymbol{u}_{g}}\right),
\end{aligned}$$

$$u|z_{ij} \sim \prod_{g=1}^{G} Exponential(\lambda) I\left\{u_g > \frac{1}{|\beta_g|^{\alpha}}\right\},$$

$$\sigma|z_{ij} \sim Inverse - Gamma\left(a + \frac{3n}{2}, b + \frac{1}{4}\left(z_{ij} - b_{\tau_k} - x_{ij}^{T}\beta - \theta v\right)' V^{-1}\left(z_{ij} - b_{\tau_k} - x_{ij}^{T}\beta - \theta v\right)\right),$$

$$\lambda|z_{ij} \sim Gamma\left(c + 2p, d + \sum_{g=1}^{G} \frac{1}{|\beta_g|^{\alpha}}\right),$$

Algorithm 1. MCMC sampling for the Bayesian reciprocal Bridge composite quantile regression (SMN)

Input: (z , x)

Initialize: $(b_q, \beta, \sigma, v, u, \lambda, \alpha)$

For $t = 1, \dots, (t_{max} + t_{burn-in})$ **do**

1. sample $v|. \sim \prod_{i=1}^{n}$ $Inverse\ Gaussian\left(\frac{1}{2\sigma}, \frac{1}{\left|z_{ij} - b_{\tau_k} - x_{ij}^{T}\beta\right|}, \frac{1}{2\sigma}\right)$
2. sample $u|. \sim \prod_{g=1}^{G} Exponential\,(\lambda) I\left\{u_g > \frac{1}{|\beta_g|^{\alpha}}\right\}$
3. sample $l|. \sim \prod_{g=1}^{G} Inverse - Gaussin\left(\frac{1}{2}, \sqrt{\frac{\xi_g^2}{\beta_g^2}}, \xi_g^2\right)$
4. sample $\xi\Big|. \sim \prod_{g=1}^{G} Gamma\left(2, \left(\left|\beta_g\right|^{\alpha} + \frac{1}{u_g}\right)\right)$
5. sample $\beta|$. From a truncated multivariate normal proportionl to $N_P((X'\Omega^{-1}X + T^{-1})^{-1}X'\Omega^{-1}(z - \vartheta v), (X'\Omega^{-1}X + T^{-1})^{-1}) \prod_{g=1}^{G} I\left\{\left|\beta_g\right| > \frac{1}{u_g}\right\},$

   $\widehat{\beta} = \left(\sum_{i=1}^{n}\sum_{k=1}^{K}\sum_{j=1}^{J} \frac{x'x}{2\sigma v_{ij}}\right)$ and

   $\widehat{\beta} = \widehat{B}\left(\sum_{i=1}^{n}\sum_{k=1}^{K}\sum_{j=1}^{J} \frac{\left(x_i\left(z_{ij} - b_{q_k} - x_{ij}^{T}\beta - \theta v_{ij}\right)\right)}{2\sigma v_{ij}}\right)$
6. sample $b_{\tau}|. \sim N\left(\frac{\sum_{i=1}^{n}\sum_{k=1}^{K}\sum_{j=1}^{J}\left(z_{ij} - b_{\tau_k} - x_{ij}^{T}\beta - \theta v_{ij}\right)}{\sum_{i=1}^{n}\sum_{j=1}^{J} {}^{1}/{}_{2\sigma v_{ij}}}, \frac{1}{\sum_{i=1}^{n}\sum_{j=1}^{J} {}^{1}/{}_{2\sigma v_{ij}}}\right)$

7. sample$\boldsymbol{\sigma}|. \sim \boldsymbol{Inverse\ Gamma}\left(\boldsymbol{a}, \frac{\mathbf{3n}}{\mathbf{2}}, \boldsymbol{b} + \frac{\mathbf{1}}{\mathbf{4}}\sum_{i=1}^{n}\sum_{k=1}^{K}\sum_{j=1}^{J}\left(\boldsymbol{z}_{ij} - \boldsymbol{b}_{\tau_k} + \boldsymbol{x}_{ij}^{T}\boldsymbol{\beta} - \boldsymbol{\vartheta v}_{ij}\right)'\boldsymbol{V}^{-1}\sum_{i=1}^{n}\sum_{k=1}^{K}\sum_{j=1}^{J}\left(\boldsymbol{z}_{ij} - \boldsymbol{b}_{\tau_k} + \boldsymbol{x}_{ij}^{T}\boldsymbol{\beta} - \boldsymbol{\vartheta v}_{ij}\right)\right)$
8. sample $\boldsymbol{\lambda}|. \sim \boldsymbol{Gamma}\left(\boldsymbol{\gamma} + \mathbf{2p}, \boldsymbol{d} + \sum_{g=1}^{G}\frac{\mathbf{1}}{|\boldsymbol{\beta}_g|^{\alpha}}\right)$
9. sample $\boldsymbol{\delta_c}$, with $\boldsymbol{c}$ from 1 to $\boldsymbol{C-1}$, from a uniform distribution over the interval

$(\boldsymbol{min}\,\{\boldsymbol{min}(\boldsymbol{z_i}|\boldsymbol{y_i} = \boldsymbol{c}+\mathbf{1}), \boldsymbol{\delta_{c+1}}, \boldsymbol{\delta_C}\}, \boldsymbol{max}\{\boldsymbol{max}(\boldsymbol{z_i}|\boldsymbol{y_i} = \boldsymbol{c})\}, \boldsymbol{\delta_{c-1}}, \boldsymbol{\delta_0})$.

10. Sample $\boldsymbol{z_i}$, $\boldsymbol{for\ i\ from\ 1\ to\ n}$, from truncated normal (TN) distribution $\boldsymbol{TN}_{(\boldsymbol{\delta_{c-1}}, \boldsymbol{\delta_c})}(\boldsymbol{z_i} + \boldsymbol{b}_{\tau_k} + \boldsymbol{x}_i'\boldsymbol{\beta} + \boldsymbol{\vartheta v}, \mathbf{2}\boldsymbol{\sigma v})$.

end for

4. Simulation studies

Simulation study 1

In this section, we set z_i as follows:

$z_{ij} = \beta_1 x_{1ij} + \beta_2 x_{2ij} + \beta_3 x_{3ij} + \beta_4 x_{4ij} + \beta_5 x_{5ij} + \beta_6 x_{6ij} + \varepsilon_i$,($i = 1, \cdots, 40$; $j = 1, \cdots, 10$), where x_{1ij}, x_{2ij} and x_{3ij} were sampled independently from uniform distribution on the interval $[-1, 1]$, x_{4ij}, x_{5ij} and x_{6ij} were sampled independently from standard normal distribution $(\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6) = (-4, -8, 12, 0, 0, 0)$ and ε_i are sampled from a logistic distribution with location parameter $\mu = 0$ and scale parameter $s = 1$. The response variable was sampled according the cut-points (-0.50, 0, 0.50). The performance of the proposed approach for the reciprocal adaptive Bridge ordinal longitudinal composite quantile regression, referred to as "rABOLCQR" approach is compared with Bayeian ordinal quantile regression (**?**), referred to as "BOQR" and Bayeian model selection in ordinal quantile regression (**?**), referred to as "BMOQR". In Table 1 the number of true and false zero regression coefficients is compared based on 100 generated datasets. The results show that the proposed method perform very well in terms of average numbers of correct and wrong zeros. Convergence of the proposed Gibbs sampler was conducted using the multivariate potential scale reduction factor (MPSRF) (Brooks and Gelman, 1998) which is given by (Alhamzawi, 2016):

Table 1: Comparing average numbers of correct and wrong zeros for different methods in Simulation example 1, averaged over 100 replications. The standard deviations are listed in the parentheses.

			Methods		
	rABOLCQR	BOQR	BMOQR	AIC	BIC
correct	2.45 (0.22)	1.33 (0.14)	1.29 (0.53)	1.01 (0.39)	1.05 (0.23)
wrong	0.07 (0.21)	0.47 (0.56)	0.48 (0.46)	0.31 (0.31)	0.18 (0.49)

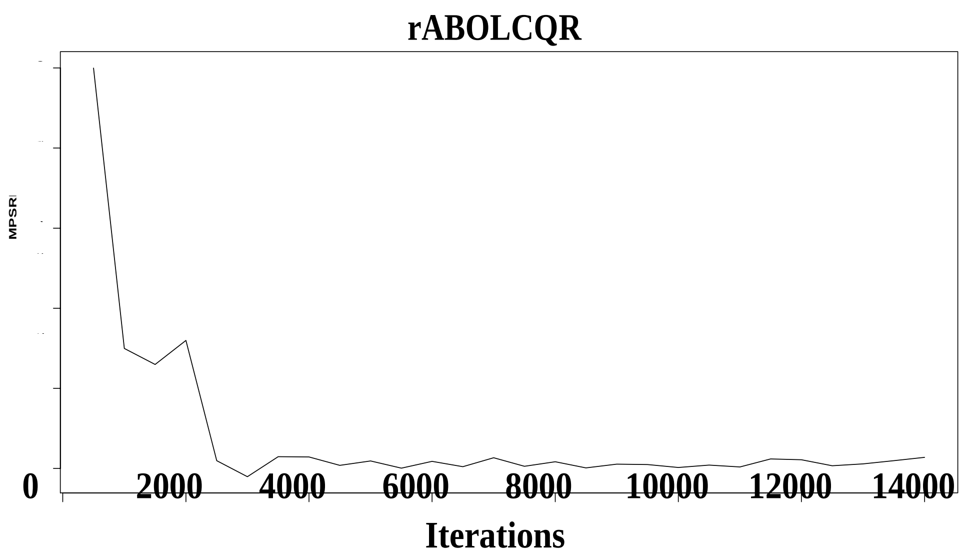

Figure 1: MPSRF for the Simulation study 1.

Figure (1) shows that the MPSRF for the proposed methods becomes stable andclose to 1 after about 2000 iterations.

Simulation study 2

This simulation study follows the same setup in simulation study 1 except that, we add 10 dummy variables. The results are summarized in Table 2, which presented the number of true and false zero regression coefficients is compared based on 100 generated datasets. The results show that the proposed method perform very well in terms of average numbers of correct and wrong zeros.

Table 2: Comparing average numbers of correct and wrong zeros for different methods in Simulation example 2, averaged over 100 replications. The standard deviations are listed in the parentheses.

			Methods		
	rABOLCQR	BOQR	BMOQR	AIC	BIC
correct	11.99 (0.17)	6.42 (0.35)	6.92 (0.61)	5.42 (0.44)	8.17 (0.34)
wrong	0.11 (0.33)	0.53 (0.47)	0.58 (0.68)	2.17 (0.45)	1.19 (0.76)

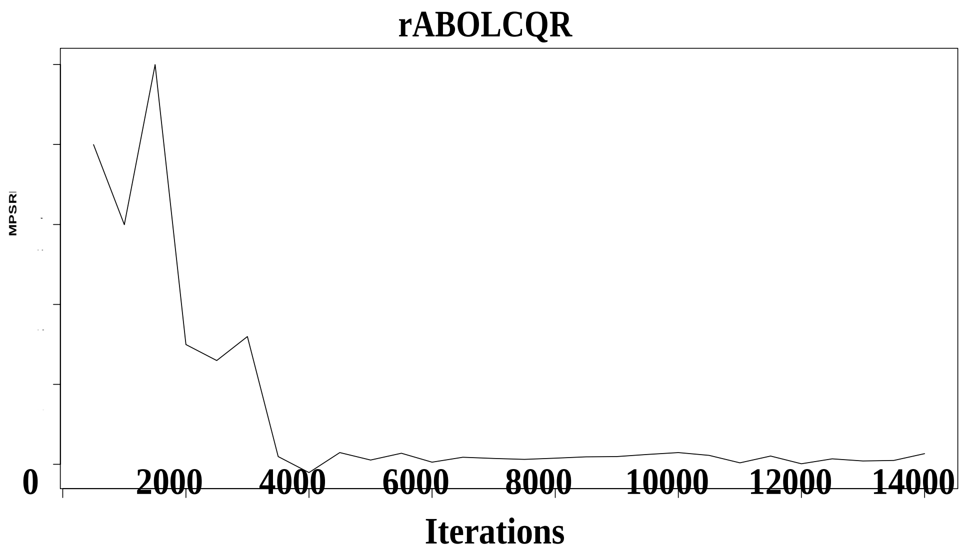

Figure 2: MPSRF for the Simulation study 2.

Figure (2) shows that the MPSRF for the proposed methods becomes stable and close to 1 after about 2000 iterations.

Conclusion and Discussion

In this paper, we propose the Bayesian reciprocal bridge composite quantile regression for simultaneous estimation and variable selection in ordinal longitudinal data. This method gives sparse solution and enjoys the computational advantages of reciprocal bridge. A new Gibbs sampling algorithm is constructed for sampling from the full conditional posterior distributions. The proposed approach is illustrated using extensive simulation examples shows that the proposed methods often outperform the existing methods.

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