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Bayesian Reciprocal Adaptive Bridge Composite Quantile Regression with Ordinal Data

A Thesis by

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Supervised by

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بسم الله الرحمن الرحيم

وأن ليس للإنسان إلا ما سعى

وأن سعيه سوف يرى

صَدَقَ اللَّهُ الْعَلِيُّ الْعَظِيمُ

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Supervisor's recommendation

I certify that the thesis entitled (**Bayesian Reciprocal Adaptive Bridge Composite Quantile Regression with Ordinal Data**) has been under my supervision for the student **(Zahraa Saad Anber AL-yassiry)** in the Department of Statistics / College of Administration & Economics / University of Al-Qadisiyah as it is part of the requirements for master's degree in statistics sciences.

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Recommendation of the head of the Graduate Studies Committee

Based on the available recommendation, I would like to forward this thesis for discussion.

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Approval of the linguistics expert

This to hereby that the thesis entitled (**Bayesian Reciprocal Adaptive Bridge Composite Quantile Regression with Ordinal Data**) has been reviewed in terms of stylistics and linguistics (grammar and spelling). Therefore after the modification of all recommended notes thesis has become free of all linguistics errors and ready to be defended and used as a scientific method to award the degree of master in statistical sciences.

Signature

Experts Dr. Haider Hadi Mohammed

Date:

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We are the heads and members of the defense committee certify that we have been looked at the thesis entitled **(Bayesian Reciprocal Adaptive Bridge Composite Quantile Regression with Ordinal Data)** and we have debated the student **(Zahraa Saad Anber AL-yassiry)**. As a result , the student his defended her thesis and all its content. So that we have found the thesis is worthy to be accepted to award a (**excellent**) master's degree in statistics science.

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DEDICATION

This thesis is dedicated to the Lady of the Women of the Worlds, Fatima Al-Zahra (peace be upon her). To my dear parents for their prayers. To my brother and my sisters for their support. To all academic stars, especially my supervisor, Prof. Dr. Rahim Al-Hamzawi. And my colleagues and friends.

Abstract

The main goal of estimating model parameters is to obtain the best estimators that give predictions with high accuracy. If all parameters are estimated under high-dimensional data, this leads to poor prediction with high correlations between independent variables, and thus erroneous results are obtained. Therefore, variable selection (V.S) has become one of the fundamental issues for modeling high-dimensional data.

One of the challenges in building a QR regression model is selecting active variables. Appropriate selection of a subset of covariates improves prediction accuracy in many cases. From a scientific point of view, for a better interpretation, it is recommended to choose a smaller subset. Several techniques have been proposed to obtain the active subset.

This study deals with the hierarchical Bayesian approach to variable selection and estimation in linear QR. In particular, we propose a regularization bridge method and ordinal composite sarcomeric regression. In this thesis, Bayesian Adaptive Inverse Bridge Composite Regression referred to as “BrABCQRO” is proposed for selecting and estimating variables in ordinal data. A new Gibbs sampling algorithm was created for sampling from complete conditional posterior distributions. , with comparison to some Bayesian and non-Bayesian methods. The proposed approach is illustrated using large-scale simulation examples supported by real data example showing that the proposed methods often outperform existing methods.

List of Abbreviations

Abbreviations	Meaning
QR	Quantile Regression
CQR	Composite QR
OR	Ordinal
SMR	Standard mean regression
MLE	Maximum Likelihood Estimation
SMN	Scale Mixture Normal
BCQR	Bayesian Composite QR
SSVS	Stochastic Search Variable Selection
MCMC	Markov Chain Monte Carlo
BOCQR	Bayesian Ordinal CQR
BRBOCQR	Bayesian Reciprocal Bridge Ordinal CQR

BRABCQR……………………………………………………… Bayesian Reciprocal Adaptive Bridge Ordinal CQR

ALD ……………………………………………………….. Asymmetric Laplace Distribution

OLS ……………………………………………………… Ordinary Least Squares

TN ……………………………………………………….. Truncated Normal

RSS ……………………………………………………….. Residual Sum of Squares

AIC ……………………………………………………….. Akaike Information Criterion

BIC ………………………………………………………….. Bayesian Information Criterion

SMT ……………………………………………………….. Scale Mixture Triangular

GG …………………………………………………………... Generalized Gaussian

DIC ……………………………………………………… Deviance Information Criteria

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CHAPTER ONE

INTRODUCTION, THESIS PROBLEM, THE OBJECTIVE OF THE THESIS, AND LITERATURE REVIEW

1.1. Introduction

The interest of researchers especially the statisticians, econometrics, and many applied researchers, has increased in the concept of quantile regression(QR), initially proposed by Koenker and Bassett (1978). It has been used in various fields such as economics, agriculture, medicine, genetic studies, sociology, and other fields .

QR has several advantages over the standard mean regression (SMR) (Orsini & Ottai,2011). It can detect different effects of different quantiles of the response variable. This possibility results from the fact that it does not require a special distribution of the data (Liu, Saat, Qin &Barkan,2013). Moreover, its estimators are insensitive to outliers (Koenker, 2005), and most importantly, it can deal with the heterogeneity of the data, without making assumptions (Qin et al.,2010; Qin,2012; Qin and Reyes, 2011). The features of quantile regression became attractive when the data did not satisfy the assumptions of the mean regression. Although it has excellent properties, it has complex computational difficulties that were dealt with, especially the non-differentiation of the loss function. Quantile regression estimation is done using special algorithms and reliable estimation methods. Classical methods used the simple algorithm and the interior point algorithm.

Bayesian methods used Markov chain Monte Carlo(MCMC) technique samples. The challenge for developing Bayesian quantile regression is the error does not follow any distribution. Koenker and Machado (1999) showed that the objective function is equal to the exponent in the asymmetric Laplace distribution (ALD) (Kotz et al.,2001; Yu and Zhang,2005). The Bayesian method was implemented by Yu and Moyeed (2001) using ALS for the error and MCMC method. This algorithm was developed by Tsioonas (2003) and (Reed and Yu (2009). Finally, Kozumi and Kobayshi (2011) proposed a Gibbs sampling assuming the exponential normal mixture representation for the (ALD).

One of the challenges in building a QR model is the selection of the active variables. The appropriate selection of a subset of predictors leads to an improvement in prediction accuracy in many cases. from a scientific point of view, to obtain a better interpretation, it is desirable to choose a smaller subset. Several techniques have been proposed to obtain the active subset (see, Reed et .,(2009) and Ji et al., (2011), among others).

Composite quantile regression (CQR) appeared as a parametric estimation model. It possesses the characteristics of quantile regression (free distribution, variance, and robust). Moreover, it is superior to single quantile regression in efficiency over median regression. The excellent theoretical properties of CQR apply to models where the outcome of interest is ordinal. Ordinal outcomes usually appear as a response to surveys, and applications are common in medicine, ecology, geology, human and social studies. However, CQR with ordinal outcomes is more difficult because quantiles of ordinal data cannot be got by a simple inverse of the distribution function.

The first author of a penalty-based approach to parameter estimation was Koenker (2004) who adopted special effects while probability dependence. Geraci and Bottai (2007) proposed the random intersect quantile regression for Longitudinal data using the ALD for the errors. Yu and Moyeed adopted the ALD with the use of the Monte Carlo algorithm(MCMC).

1.2. Thesis problem

statisticians try to obtain variables related to the response variable to reach accurate predictions. In light of the high-dimensional data, the problem of selecting a partial set of influential variables is to reach reliable results.

1.3. Objective of the thesis

Proposing new regularization method to estimate the parameters of Bayesian Ordinal Composite Quantile Regression by Reciprocal Adaptive Bridge Penalty Function.

1.4. Literature review

The main goal of estimating model parameters is to obtain the best estimators that give predictions and high accuracy. In the case of estimating all the parameters in the high-dimensional data , it leads to obtain a weak prediction with high correlations between the independent variables, and thus erroneous results are obtained. Therefore, variable selection (V.S) has become one of the basic issues for modeling high-dimensional data. There are two types of variable selection methods. Traditional methods and methods of regulation. We will first address some common traditional criteria.

Akaike 1974 suggested is one of the most commonly Akaike Information Criterion (AIC) used traditional criteria for selecting important variables which can be written as :

$$\boldsymbol{AIC = 2K - 2\ln(L)}, \qquad (1.1)$$

Where L is the maximum likelihood function (MLE), K is the number of parameters. The preference is given to the model with the lowest value of (AIC). The defects of this criterion appear when the value of n is large , as the chosen model is not stable.

Schwarz 1978 suggested the consistent model selection when n is large by proposing the Bayesian Information Criterion (BIC).

$$\boldsymbol{BIC} = \boldsymbol{K}\ \mathbf{ln}(\boldsymbol{n}) - \mathbf{2}\boldsymbol{ln}\ (\boldsymbol{L}), \qquad (1.2)$$

Where L is the maximum likelihood function (MLE), K is the number of parameters, and n sample size. However, defects appeared for this criterion, as it does not deal with complex models which have low bias and high variance. George and McCulloch (1993) suggested the Stochastic Search Variable Selection (SSVS) method which is depending on the probabilistic considerations in selecting the subsets of independent variables. This method can be used in the well-known Bayesian algorithm ($MCMC$) which was developed by Alhamzawi for the quantile regression approach. Spiegelhalter et al. (2002) proposed Deviance Information Criteria (DIC). For model selection in Bayesian hierarchical normal linear models, the generalization of AIC and BIC defined as : $\boldsymbol{DIC} = -\mathbf{2}\ \mathbf{ln}(\boldsymbol{L}) - \mathbf{4}\boldsymbol{K}\ \boldsymbol{ln}\ (\boldsymbol{L})$, (1.3)

Like AIC and BIC, models with smaller DIC are better supported by the data. DIC is particularly useful when the MCMC samples are easily available, and is valid only when the joint distribution of the parameters is approximately multivariate normal .

All these methods take longer to select the important variables, especially when the number of variables is greater than the sample size. These regularization methods (Koenker 2004) have proven effective in selecting a variable and estimating a coefficient

when the model contains a large number of variables that reduce the accuracy of the prediction.

Regularization methods emerged as a type of variable selection method. Where a group of important variables is selected by shrinking, in addition to its ability to estimate parameters with variable selection at the same time. The estimator Ridge Hoerl and Kennard (1970) gives a

better prediction than the estimator of ordinary least squares (OLS) by shrinking the coefficients towards zero. This method adds some bias and reduces the variance of the estimator by minimizing the residual sum of squares (RSS), i.e.,

$$\boldsymbol{RSS}(\boldsymbol{\beta}) + \boldsymbol{\lambda}\|\boldsymbol{\beta}\|_2^2 , \tag{1.4}$$

Where λ is the shrinkage parameter and λ ≥0, the second term is called the penalty function and $\|\boldsymbol{\beta}\|_2^2 = \sum_{k=1}^{p} \boldsymbol{\beta}_k^2$, When λ=0, the function becomes the least squares estimator. Despite these characteristics, the defects of this method appear by keeping all the variables in the model and not achieving the variable selection. Frank and Friedman (1993) proposed a general method of penalties, Bridge regression characterized by desirable statistical properties (unbiased, oracle).

$$\boldsymbol{RSS}(\boldsymbol{\beta}) + \boldsymbol{\lambda} \sum_{k=1}^{p} |\boldsymbol{\beta}_k|^{\alpha}, \tag{1.5}$$

It achieves variable selection and estimation of model parameters. Bayesian bridge overcomes the problem of instability when calculating standard errors by classical methods. Tibshirani 1996 Introduced a Least absolute shrinkage and selection operator (Lasso) regulation method that provides simultaneous regulation and selection of a variable. It is superior to the ridge method by reducing the number of variables and defining an important set of variables. Several failures appeared, including defining

$$\boldsymbol{RSS}(\boldsymbol{\beta}) + \boldsymbol{\lambda} \sum_{\boldsymbol{k}=\mathbf{1}}^{\boldsymbol{p}} |\boldsymbol{\beta}_{\boldsymbol{k}}|, \qquad (1.6)$$

the second term is the penalty function, where $\lambda \geq 0$ is the shrinkage parameter and $\|\boldsymbol{\beta}\|_{\mathbf{1}} = \boldsymbol{\lambda} \sum_{\boldsymbol{k}=\mathbf{1}}^{\boldsymbol{p}} |\boldsymbol{\beta}_{\boldsymbol{k}}|$, is L1−norm.

variables if the number of variables is greater than $\boldsymbol{n}$. It does not have Oracle properties.

Recently Zou and Hastie (2005) proposed the elastic net for variable selection and estimation in the Linear Regression model outperforms

Lasso in cases when the number of variables is greater than the sample size.

$$\boldsymbol{RSS}(\boldsymbol{\beta}) + \boldsymbol{\lambda}_{\mathbf{1}} \|\boldsymbol{\beta}\|_{\mathbf{1}} + \boldsymbol{\lambda}_{\mathbf{2}} \|\boldsymbol{\beta}\|_{\mathbf{2}}^{\mathbf{2}}, \qquad (1.7)$$

If K=1 the condition becomes ridge, and if K=0 the condition becomes lasso. In addition, the variables are either inside or outside the model at the same time.

Zou 2006 proposed Adaptive Lasso (Alasso) methods for estimation and variable selection. Show that the method has oracle characteristics compared to lasso, which is sometimes inconsistent. Alasso adding to weights for different parameters results in consistent and unbiased estimates.

Polson et al. (2014) presented a set of Bayesian estimates for linear models, using two different a scale mixture of triangular (SMT) and (SMN) to represent Generalized Gaussian (GG) prior. The disadvantages of these methods appear in that the mixing variable is stable, inclined, or non-vertical. To cover these shortcomings, Mallick and Yi (2018) presented a flexible Bayesian approach analysis for the regression of the

classical bridge, which lacks a methodology for inference despite its desirable properties. Depending on the (SMU) Bayesian prior bridge, it was shown that the method works as well or better than the one in estimating the model parameters.

Rahim and Haithem (2018) introduced regularization methods for variable selection and parameter estimation in linear regression New

Bayesian elastic net. New hierarchical forms prior model have developed based on the location-scale mixture of normal mixing with gamma density. The simulation results and real data analysis results showed the outperforms of the proposed model.

Flaih et al. (2020) developed new Gibbs sampler algorithms that have been implemented. Simulation and real data analysis have conducted to investigate the prediction accuracy.

Koenker and Basset (1978) proposed the Quantile Regression (QR) model, as an alternative to standard regression to assess the effects of covariates on outcome variables at various quantile levels. It is used in applied studies, medicine, economics, and environmental sciences. The

most important thing that distinguishes it is the free distribution of error. In addition to being unaffected by abnormal values. Moreover, it has high efficiency compared to standard regression. For the θth quantile ($0 < \theta < 1$). Koenker and Dorey (1987) developed and improved an algorithm for the least absolute error estimation of linear regression to calculate the quantile regression statistics of Koenker and Bassett (1978).

In (2001)Yu and moyeed employed the asymmetric Laplace distribution (ALD) as an error term distribution in the Bayesian quantile regression model. They showed that the minimization of the lasso function is equivalent to the maximization of the likelihood-based ALD. Also, they proved that the improper uniform priors for the unknown parameters give a proper full joint posterior distribution.

Li and Zhu (2008)considered the quantile regression with the penalized L1-norm function (lasso). They desire a new efficient algorithm for computing the exact solution for the lasso function with quantile regression. Also, They proposed new selection for the shrinkage parameter is based on an estimate of the effective dimension of the best-fitted quantile model.

Kozumi and Kobayashi (2011)developed a Gibbs sampling method for quantile regression models based on the location-scale mixture representation of the asymmetric Laplace distribution ALD.

Huang and Chen (2015)studied composite quantile regression from a Bayesian standpoint by using the ALD for the errors. In the literature, composite quantile regression approaches that are robust to heavy-tailed errors or outliers in response have been presented.

Alhamzawi (2016) presented a Bayesian method for composite quantile regression using the skewed Laplace distribution for the error distribution. An effective Gibbs sampling algorithm is improved to modify the unknown quantities from the posteriors.

Alhamzawi and Mallick (2020) proposed the reciprocal lasso quantile regression from the Bayesian point of view. A new simple and efficient Gibbs sampler algorithm has been developed based on the hierarchical priors model . with a scale mixture of double Pareto, as well as with a

scale mixture of truncated normal. simulation and real data analysis results showed that the proposed models perform well. Also, they considered that this model can be extended to the adaptive lasso quantile regression.

Alhamzawi (2022) propose a new method for removing unimportant covariates in high dimensional data to improve the prediction accuracy and obtain better interpretation called Bayesian group bridge composite quantile regression .

Alhamzawi (2022) Introduced regularized approach with a bridge penalty is adopted to conduct variable selection in composite quantile regression. An MCMC algorithm was developed for posterior inference using the normal-exponential mixture representation of the asymmetric Laplace distribution. Gamma prior is placed on the regularization parameter.

Quantile regression differs when the response is ordinal, in which the dependent variable is an ordinal discrete value. The goal of interest in

ordinal quantile regression is to obtain a richer description of the effect of covariates on the results.

Young (1975) described ordinal data in standard form using transform Kruskal's least-square monotonic transformation. Then McCullagh

(1980) developed ordinal regression models to describe the nonlinear models. Toledano and Gatsonis (1996) proposed an ordinal regression model with generalized progression equations.

Alhamzawi (2016) introduced a Bayesian method for quantile regression with ordinal models and estimated the parameters of the model by using a Gibbs sampler. showed that the OQR-SSVS method provides a better model fit relative to the Bayesian QReg for ordinal models.

Alhamzawi (2016) introduced a Bayesian stochastic search variable selection (BSSVS) for selecting the significant variables in the quantile regression with ordinal models.

Alhamzawi (2018) discussed the analysis of quantile regression for longitudinal data with the ordinal outcome.

Alhamzawi (2020) suggested new methods in the Bayesian framework Bayesian Bridge Regression for Ordinal Models. And suggest a new Bayesian hierarchical model for all methods based on uniform scale mixture representation for estimating parameters VS.

This thesis proposed a Bayesian composite quantile regression: "Bayesian Reciprocal Adaptive Bridge Composite Quantile Regression with Ordinal Data" for variable selection and estimation in the Linear quantile regression model.

The remainder of this thesis is organized as follows: In Chapter Two, we introduce a Bayesian Ordinal composite quantile regression model. In Chapter Three procedure of the model with bridge penalties as well as a prior setting of model parameters, In addition to using the MCMC algorithm, and outline of prior assumptions, and a sample Gibbs sampler for model selection and the suitability posterel based on ALD. In Chapter Four Conducting simulations to examine the performance of the proposed methods for the selection and assessment model. We conclude the thesis with a brief conclusions and discussions In Chapter Five.

CHAPTER TWO

BAYESIAN ORDINAL COMPOSITE QUANTILE REGRESSION MODEL (BOCQR)

2. Bayesian Ordinal Composite Quantile Regression Model

Quantile regression (QR) has attracted many studies on theoretical properties (see e.g., Koenker,2005). All of these studies point to many benefits of the approach. Most attractive, the ability to introduce unusual errors, and thus make it insensitive to covariance and outliers (Koenker and Bassett, Koenker,2005). Moreover, the quantile regression of the other features compared to the mean regression gives more detail to the relationship between the response variable and the predictors, thus the quantile regression presented by (Koenker and Bassett 1978) was an extension of the standard mean regression. These privileges have led to a practical application of interest in several fields such as ecology, science, economics, finance, medicine, genetic studies, and natural phenomena (see, Yu et al .,2003:Koenker,2005).

For any τ^{th} quantile, $(0< \tau <1)$, the τ^{th} quantile regression can be defined as

$$\mathbf{Q}_{\mathbf{y_i}|\mathbf{x_i}}(\boldsymbol{\tau}) = \boldsymbol{x}_i'\boldsymbol{\beta}_{\boldsymbol{\tau}} \,,$$

where $\boldsymbol{y_i}$ is the response variable, $\boldsymbol{x_i'}$ is a K-dimensional vector, $\boldsymbol{\beta_\tau}$ is the coefficient vector of QR. To estimate the coefficient vector (Koenker, and Bassett,1978) proposed this equation :

$$\sum_{i=1}^{n} \rho_\tau(y_i - x_i'\beta_\tau), \qquad (2.1)$$

where $\boldsymbol{\rho_\tau(u) = \tau(1 - I(u < 0)), I(u < 0)}$ is the indicator function . This problem can be minimization by using linear programing algorithm (Koenker, and D'Orey, 1987).

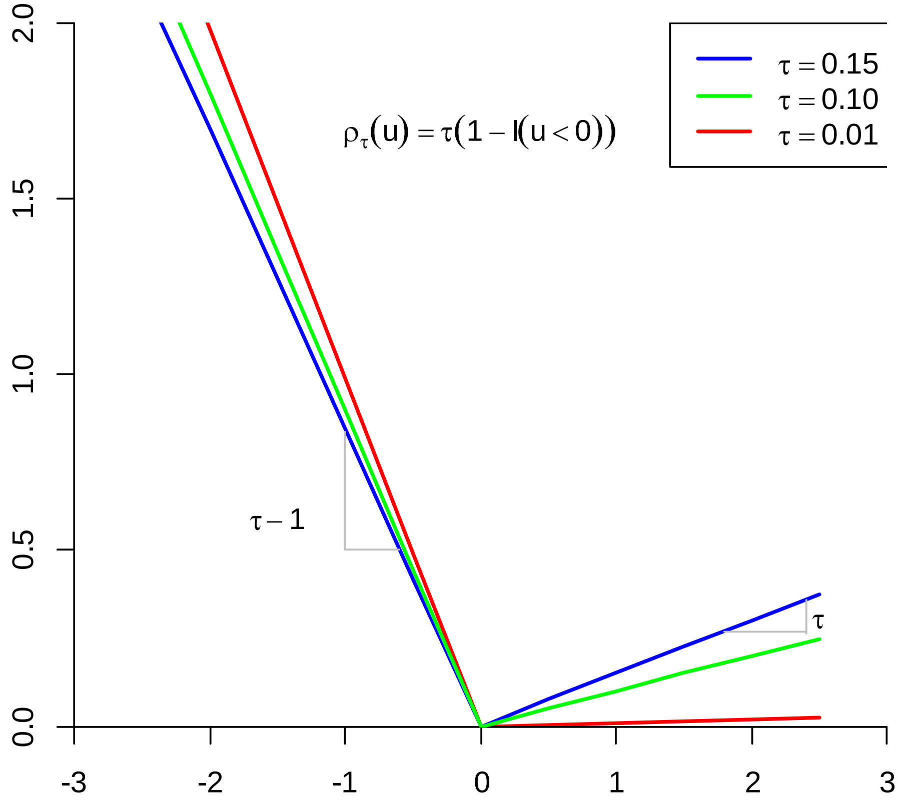

Figure 1 shows the check function at three quantiles 0.01, 0.10 and 0.15 . Since the above check function is not differential at 0 there is no closed form solution. Thus, many researchers used Bayesian methods to find Bayesian estimation for the regression coefficients.

Assuming errors are an asymmetric Laplace distribution (ALD), Bayesian method is used to estimate the parameters of QR (Yu, and Moyeed,2001) . Bayesian approach to QR is accurate in predicting even in cases of small sample sizes and is suitable for ordinal responses due to its characteristics mentioned in (Koenker, 2005) in addition to that there are mathematical indications for the use ALD for the errors in Alhamzawi (2013).

The challenges increase when the response variable is ordinal. Ordinal response models are widely used in many disciplines, particularly in medical contexts where health data outcomes can be written in ordered categories, (e.g., stages of cancer, BMI categories, or grades of disease severity). Following Rahman(2016), in this thesis the $\boldsymbol{\tau}^{\text{th}}$ quantile for the latent variable $\boldsymbol{z_i}$ is simulated according to the regression model from(Rahman 2016).

$$\boldsymbol{z_i = x_i'\beta + \epsilon_i}, \qquad \boldsymbol{i = 1, \ldots, n}, \qquad (2.2)$$

where $\boldsymbol{x_i}$ is a $\boldsymbol{k \times 1}$ vector of explanatory variables of $\boldsymbol{z_i}$,$\boldsymbol{\beta}$ is a $\boldsymbol{k \times 1}$ vector for model parameters,$\boldsymbol{\epsilon_i}$ is error follows ALD. Where a description of the ordinal response variable by a latent variable $\boldsymbol{z_i}$ can be written fellow:

$$\boldsymbol{y_i = c} \quad \boldsymbol{if} \quad \boldsymbol{\delta_{c-1} < z_i \leq \delta_c}\,; \quad \boldsymbol{c = 1, \dots, C}, \qquad (2.3)$$

where $\boldsymbol{\delta_0}, \dots, \boldsymbol{\delta_C}$ are cut-points, that fall within the period

$$-\infty = \boldsymbol{\delta_0} < \boldsymbol{\delta_1} < \cdots < \boldsymbol{\delta_{C-1}} < \boldsymbol{\delta_C} = +\infty$$

From a Bayesian perspective, Rahman (2016) proposed an ordinal Bayesian model for QR, assuming that the error is ALD, and using the Gibbs sampling method to determine the posterior of the parameters.

During the past years, a method was proposed to estimate the parameters that outperform the average regression with an efficiency of more than 70%, called the composite quantile regression (CQR)by Zou, and Yuan, (2008), which is greater than the single QR by taking several quantities at the same time (Zou and Yuan) which is more robust, flexible, and efficient . The advantages of CQR apply to models in which the response variable is ordinal. Whereas, ordinal QR serves as a complement to the classical ordinal model, however, the ordinal QR

Model has been used for the past years for example, see, Hong and Zhou (2013), Goffe et al. (1994), Hong and He (2010).

Regularization methods (Koenker 2004) have proven effective in selecting a variable and estimating a coefficient when the model contains a large number of variables that reduce the accuracy of the prediction.

In the classical literature, quantile regression estimators have been used with ordinal data, depending on different methods (Kirkpatrick et al., 1983; Goffe et al., 1994). Despite the development of these methods over the years, their use with the Bayesian method has not been addressed, Hong and He (2010).

Rahman (2016)showed a quantile ordinal model that provides a better fit than the classical methods using the Bayesian method (Alhamzawi ,R ,Bayesian model selection in ordinal quantile regression). In this chapter we will introduce The Bayesian Composite Quantile Regression Model with Ordinal data .

2.1 Methods

2.1.1 Bayesian composite quantile regression model (BCQR)

Consider the following model

$$\mathbf{y_i} = \mathbf{b_\tau} + \boldsymbol{x_i'\beta} + \boldsymbol{\epsilon_i}\ , \qquad \mathbf{i}=1, \ldots\ldots, \boldsymbol{n}\ , \qquad (2.4)$$

where $\boldsymbol{y_i}$ is response variable, $\boldsymbol{b_\tau}$the parameter for the quantile intersection where (0< τ <1), $\boldsymbol{x_i'}$ is the vector of explanatory variables، $\boldsymbol{\beta}$ is a vector for model parameters, $\boldsymbol{\epsilon_i}$ is the error of the quantile regression model and $\boldsymbol{n}$ is the number of observations. ، the parameters of the composite regression can be estimated by solving the following equation:

$$\left(\widehat{\boldsymbol{b}}_{\tau_1}, \widehat{\boldsymbol{b}}_{\tau_2}, \ldots \ldots, \widehat{\boldsymbol{b}}_{\tau_K}, \widehat{\boldsymbol{\beta}}\right) = \underbrace{\mathbf{arg}\, \boldsymbol{min}}_{\boldsymbol{b_\tau}, \boldsymbol{\beta}} \sum_{\boldsymbol{k=1}}^{\boldsymbol{K}} \left\{\sum_{\boldsymbol{i=1}}^{\boldsymbol{n}} \boldsymbol{\rho_{\tau_k}}\left(\boldsymbol{y_i} - \boldsymbol{b_{\tau_k}} - \boldsymbol{x_i'\beta}\right)\right\} \quad (2.5)$$

where $\boldsymbol{\rho_{\tau_k}(t) = t\big(\tau_k - I(t < 0)\big)}$, is the check function, $I(.)$ is indicator function and $\boldsymbol{\tau_k = \frac{k}{K+1}},\ \ \boldsymbol{where\ k = 1, 2, \ldots, K}$.

By assuming the error is asymmetric Laplace distribution $(\boldsymbol{\mu = b_\tau + x_i'\beta, \sigma = 1})$. The probability function of ALD is given by:

$$p(y|x_i, b_\tau, \beta, \tau) = \tau(1-\tau)exp\left(-\rho_{\tau_k}\left(y_i - b_{\tau_k} - x_i'\beta\right)\right), \quad (2.6)$$

The check function is not derivable, the classical methods used computational methods and simulation methods using algorithms to estimate the quantile regression (Madsen and Nielsen ,1993)

The Bayesian methods relied on that minimization of the loss function (2.5) is equal to the maximization of the likelihood of the probability function (2.6). Kozumi and Kobayashi (2011) used a mixture of the standard exponential distribution with the standard normal of the error term, suppose that $u \sim N\,(0, 1)$ and $v \sim exp\left(\frac{1}{\tau(1-\tau)}\right)$. Therefore, the error term in (2.4) can be written as $\epsilon = \vartheta v + \sqrt{\varphi v}\,u$ where $\vartheta = (1-2\tau)$ and $\varphi = 2$. The advantage of using the normal-exponential mixture shows access to the properties of the normal distribution, which will be relied upon in this research.

Then the conditional distribution of the quantile variable is as follows:

$$p(y_i|x, b, \beta, v_i) = exp\left(-\sum_{k=1}^{K}\sum_{i=1}^{n}\frac{1}{4v_i}\left(y_i - b_{\tau_k} - x_i'\beta - \vartheta v_i\right)^2\right)\prod_{i=1}^{n}(4\pi v_i)^{\frac{1}{2}} \quad (2.7)$$

where $\boldsymbol{u} = (\boldsymbol{u_1}, \dots, \boldsymbol{u_n})'$

2.1.2 Bayesian Ordinal composite quantile regression model (BOCQR)

Recently, many researchers have used quantile regression to treat models when the response variable is ordinal (Hong, H. G. and Zhou, J. (2013), Zhou, L. (2010). One of the advantages of composite quantile regression over individual is the immunity and efficiency in case of abnormal distribution of error by (Zou, and M. Yuan, 2008).

The response variable $\boldsymbol{y_i}$ can be modeled through the continuous latent variable $\boldsymbol{z_i}$ and cut-off point $\boldsymbol{\delta} = \{\boldsymbol{\delta_0}, \dots, \boldsymbol{\delta_C}\}$ where we impose $\boldsymbol{y}$ to take C ordered values $\{\boldsymbol{c_1}, \boldsymbol{c_2}, \dots, \boldsymbol{c_C}\}$ to be in the following form:

$$\boldsymbol{y_i} = \begin{cases} \boldsymbol{1} \quad \boldsymbol{if} \quad \boldsymbol{\delta_0} \le \boldsymbol{z_i} < \boldsymbol{\delta_1} \\ \boldsymbol{c} \quad \boldsymbol{if}\ \boldsymbol{\delta_{c-1}} \le \boldsymbol{z_i} < \boldsymbol{\delta_c};\ \ \boldsymbol{c} = \boldsymbol{2}, \dots, \boldsymbol{C-1} \\ \boldsymbol{C} \quad \boldsymbol{if} \quad \boldsymbol{\delta_{C-1}} \le \boldsymbol{z_i} < \boldsymbol{\delta_C} \end{cases} \tag{2.8}$$

A composite quantile regression for ordinal data can be represented using a continuous latent random variable $\boldsymbol{z_i}$ as $\boldsymbol{z_i} = \boldsymbol{b_\tau} + \boldsymbol{x_i'\beta} + \boldsymbol{\epsilon_i}$

$\boldsymbol{i}$=1,……,$\boldsymbol{n}$ where $\boldsymbol{x_i}$ is a $\boldsymbol{k \times 1}$ vector of explanatory variables ,$\boldsymbol{\beta}$ is a $\boldsymbol{k \times 1}$ vector for model parameters, $\boldsymbol{\epsilon_i}$ follows an ALD with pdf (2.6) and $\boldsymbol{n}$ is number of observations.

The equation (2.7) can be rewritten as a hierarchical Bayesian model using ordinal composite quantile regression

$$\boldsymbol{p(z_i|x, b, \beta, v_i) =}$$

$$\boldsymbol{exp\left(-\sum_{k=1}^{K}\sum_{i=1}^{n}\frac{1}{4v_i}\left(z_i - b_{\tau_k} - x_i'\beta - \vartheta v_i\right)^2\right)\prod_{i=1}^{n}(4\pi v_i)^{\frac{1}{2}}} \quad (2.9)$$

CHAPTER THREE

BAYESIAN RECIPROCAL ADAPTIVE BRIDGE FOR ORDINAL COMPOSITE QUANTILE REGRESSION (BRABOCQR)

3. Bayesian Reciprocal Adaptive Bridge for Ordinal CQR

The first regularization method in QR was proposed by Koenker (2004) to shrinkage the random effects to zero. Wang et al. (2007) considered the least absolute deviance (LAD) in QR . From a Bayesian perspective, a Bayesian lasso is defined as a posterior procedure that induces a prior hypothesis of the parameters of the regression that is Laplace-independent (Tibshirani,1996:Park and Casella,2008). The challenges increase when the response variable is ordinal. Ordinal response models are widely used in many disciplines, particularly in medical contexts where health data outcomes can be written in ordered categories, (e.g., stages of cancer, BMI categories, or grades of disease severity).

٣.1. Methods of Bridge penalty with the model

3.1.1 Bayesian Reciprocal Bridge for Ordinal Composite Quantile Regression (BRBOCQR)

The reciprocal bridge estimator can be written by making use of formula with quantile regression in (Alhamzawi and Mallick , 2020) which solves the following:

$$\underbrace{\boldsymbol{argmin}}_{\boldsymbol{b_\tau,\beta}} \sum_{i=1}^{n}\left\{\sum_{k=1}^{K}\boldsymbol{\rho_{\tau_k}}\left(\boldsymbol{z_i}-\boldsymbol{b_{\tau_k}}-\boldsymbol{x_i'\beta}\right)\right\}+\boldsymbol{\lambda}\sum_{g=1}^{G}\frac{1}{|\boldsymbol{\beta_g}|^{\alpha}}\boldsymbol{I}\left\{\boldsymbol{\beta_g}\neq \mathbf{0}\right\} \quad (3.1)$$

where λ is parameter of regularization for α , when it is equal to zero, it corresponds L_0, and when it is equal to one, it shows reciprocal LASSO, and when it is equal to 2, reciprocal ridge appear, where the check function is an alternative to the loss function while preserving the identity of the bridge, where the Bayesian approach solves the problem of miniaturization in cases of small samples as well. Noting the penalty term in (3.1), the bridge estimates can be interpreted as posterior mode estimates when the regression parameters have Inverse Generalized Gaussian (IGG) distribution (Mallick et al., 2020) of the form:

$$\boldsymbol{\pi}(\boldsymbol{\beta}) = \prod_{g=1}^{G} \frac{\lambda^{\frac{1}{\alpha}}}{2\beta_g^2\Gamma\left(\frac{1}{\alpha}+1\right)} \boldsymbol{exp}\left\{-\frac{\lambda}{|\beta_g|^{\alpha}}\right\} \boldsymbol{I}\{\boldsymbol{\beta}_g \neq \boldsymbol{0}\}, \qquad (3.2)$$

The Gibbs sampler for the Bayesian reciprocal bridge exploits the following representation of the scale mixture of normal(SMN) following Armagan ,Dunson and Lee (2013);Mallick , Alhamzawi ,and Svetnik(2020). If we assume that $\boldsymbol{\beta} \sim \boldsymbol{N}(\boldsymbol{0}, \boldsymbol{l})\boldsymbol{I}(|\boldsymbol{\beta}| > \eta)$,$\boldsymbol{l} \sim \boldsymbol{Exp}\left(\xi^2/\boldsymbol{2}\right)$,and $\boldsymbol{\xi} \sim \boldsymbol{Exp}(\eta)$,then the inverse double exponential distribution for $\boldsymbol{\beta}$ with scale parameter $\boldsymbol{\lambda} > \boldsymbol{0}$ arises when η follows Inverse Gamma (2,λ) .

Where $\boldsymbol{u} = \frac{\boldsymbol{1}}{\eta}, \boldsymbol{l} = (\boldsymbol{l_1}, \dots, \boldsymbol{l_G})'$, and $\boldsymbol{\xi} = (\boldsymbol{\xi_1}, \dots, \boldsymbol{\xi_G})'$. To specify prior distribution for $\boldsymbol{\delta}$, we follow Alhamzawi (2016) ,we assign order statistics from uniform $(\boldsymbol{\delta_0}, \boldsymbol{\delta_C})$ distribution for the $\boldsymbol{C} - \boldsymbol{1}$ unknown cut-points :

$$\boldsymbol{P_\delta} = (\boldsymbol{C} - \boldsymbol{1})! \left(\frac{\boldsymbol{1}}{\boldsymbol{\delta_{max}} - \boldsymbol{\delta_{min}}}\right)^{\boldsymbol{C}-\boldsymbol{1}} \boldsymbol{I}(\boldsymbol{\delta} \in \boldsymbol{H}), \qquad (3.3)$$

Where $\boldsymbol{\delta} = (\boldsymbol{\delta_0}, \boldsymbol{\delta_1}, \dots, \boldsymbol{\delta_C})$ and $\boldsymbol{H} = \{(\boldsymbol{\delta_{min}}, \boldsymbol{\delta_1}, \dots, \boldsymbol{\delta_{max}}) | \boldsymbol{\delta_{min}} < \boldsymbol{\delta_1} < \cdots < \boldsymbol{\delta_{C-1}} < \boldsymbol{\delta_{max}}\}$.

To summarize, our Bayesian hierarchical formulation ,we consider the following priors for all parameters and latent variables

$$\boldsymbol{y_i} = \begin{cases} \boldsymbol{1} \quad \boldsymbol{if} \quad \boldsymbol{\delta_0} \le \boldsymbol{z_i} < \boldsymbol{\delta_1} \\ \boldsymbol{c} \quad \boldsymbol{if}\ \boldsymbol{\delta_{c-1}} \le \boldsymbol{z_i} < \boldsymbol{\delta_c};\ \ \boldsymbol{c} = \boldsymbol{2}, \dots, \boldsymbol{C-1} \\ \boldsymbol{C} \quad \boldsymbol{if} \quad \boldsymbol{\delta_{C-1}} \le \boldsymbol{z_i} < \boldsymbol{\delta_C} \end{cases}$$

$$\boldsymbol{z_i} | \boldsymbol{x} \sim \boldsymbol{N_n}\left(\boldsymbol{b_{\tau_k}} + \boldsymbol{x_i'\beta} + \boldsymbol{\vartheta v}, \boldsymbol{2\sigma v}\right),$$

$$\boldsymbol{P(\delta)} = (\boldsymbol{C-1})!\left(\frac{\boldsymbol{1}}{\boldsymbol{\delta_{max}} - \boldsymbol{\delta_{min}}}\right)^{\boldsymbol{C-1}} \boldsymbol{I}(\boldsymbol{\delta} \in \boldsymbol{H}) \text{when } \boldsymbol{H} = \{(\boldsymbol{\delta_0}, \dots, \boldsymbol{\delta_C}) | \boldsymbol{\delta_0} < \cdots < \boldsymbol{\delta_C}\}.$$

$$\boldsymbol{\beta} | \boldsymbol{l} \sim \prod_{g=1}^{G} \boldsymbol{N}(\boldsymbol{0}, \boldsymbol{t^2}) \boldsymbol{I}\left\{\left|\boldsymbol{\beta_g}\right|^{\alpha} > \frac{\boldsymbol{1}}{\boldsymbol{u_g}}\right\},$$

$$\boldsymbol{l} | \boldsymbol{\xi} \sim \prod_{g=1}^{G} \boldsymbol{Exp}\left(\boldsymbol{\xi_g^2}\right),$$

$$\boldsymbol{\xi} | \boldsymbol{u} \sim \prod_{g=1}^{G} \boldsymbol{Exp}\left(\frac{\boldsymbol{1}}{\boldsymbol{u_g}}\right),$$

$$\boldsymbol{u} \sim \prod_{g=1}^{G} \boldsymbol{Gamma}\ (\boldsymbol{2}, \boldsymbol{\lambda}), \tag{3.4}$$

$\boldsymbol{\sigma \sim \sigma^{-1}}$,

$\boldsymbol{\lambda \sim \lambda^{-1}}$,

Then the condition posteriors are:

$$\boldsymbol{\beta | z_i \sim N_P((X'\Omega^{-1}X + T^{-1})^{-1}X'\Omega^{-1}(z - \vartheta v), (X'\Omega^{-1}X + T^{-1})^{-1}) \prod_{g=1}^{G} I\left\{\left|\beta_g\right|^{\alpha} > \frac{1}{u_g}\right\}},$$

$$\boldsymbol{v_i^{-1} | z_i \sim Inverse - Gaussian\left(\frac{1}{2}, \frac{1}{\left|z_i + b_{\tau_k} + x_i'\beta\right|}, \frac{1}{2\sigma}\right)},$$

$$\boldsymbol{l^{-1} | z_i \sim \prod_{g=1}^{G} Inverse - Gaussian\left(\frac{1}{2}, \sqrt{\frac{\xi_g^2}{\beta_g^2}}, \xi_g^2\right)},$$

$$\boldsymbol{\xi | z_i \sim \prod_{g=1}^{G} Gamma\left(\left|\beta_g\right|^{\alpha} + \frac{1}{u_g}\right)},$$

$$\boldsymbol{u | z_i \sim \prod_{g=1}^{G} Exponential(\lambda) I\left\{u_g > \frac{1}{\left|\beta_g\right|^{\alpha}}\right\}},$$

$$\sigma|z_i \sim Inverse - Gamma\left(a + \frac{3n}{2}, b + \frac{1}{4}\left(z_i - b_{\tau_k} - x_i'\beta - \vartheta v\right)' V^{-1}\left(z_i - b_{\tau_k} - x_i'\beta - \vartheta v\right)\right),$$

$$\lambda|z_i \sim Gamma\left(\gamma + 2p, d + \sum_{g=1}^{G} \frac{1}{|\beta_g|^{\alpha}}\right),$$

Where $L = diag(l_1, \dots, l_G)$, $\Omega = diag((2\sigma v_1), \dots, (2\sigma v_n))$,$\gamma, p, and\ d$ are fixed hyper parameters.

Algorithm 1. MCMC sampling for the Bayesian reciprocal Bridge composite quantile regression (SMN)

Input: (z , x)

Initialize: $(b_\tau, \beta, \sigma, v, u, \lambda, \alpha)$

For $t = 1, \dots, (t_{max} + t_{burn-in})$ **do**

1. sample $v^{-1}|. \sim \prod_{i=1}^{n}$ $Inverse\ Gaussian\left(\frac{1}{2\sigma}, \frac{1}{\left|z_i - b_{\tau_k} - x_i'\beta\right|}, \frac{1}{2\sigma}\right)$

2. sample $\boldsymbol{u}|. \sim \prod_{g=1}^{G} \boldsymbol{Exponential}\,(\lambda)\boldsymbol{I}\left\{\boldsymbol{u}_g > \frac{1}{|\beta_g|^{\alpha}}\right\}$

3. sample $\boldsymbol{l}^{-1}|.\sim \prod_{g=1}^{G} \boldsymbol{Inverse - Gaussin}\left(\frac{1}{2}, \sqrt{\frac{\xi_k^2}{\beta_k^2}}, \xi_k^2\right)$

4. sample$\boldsymbol{\xi}\Big|. \sim \prod_{g=1}^{G} \boldsymbol{Gamma}\left(2, \left(|\boldsymbol{\beta}_g|^{\alpha} + \frac{1}{u_g}\right)\right)$

5. sample $\boldsymbol{\beta}|$. From a truncated multivariate normal proportionl to

$\boldsymbol{N_P}((\boldsymbol{X}'\Omega^{-1}\boldsymbol{X} + \boldsymbol{T}^{-1})^{-1}\boldsymbol{X}'\Omega^{-1}(\boldsymbol{z} - \boldsymbol{\vartheta v}), (\boldsymbol{X}'\Omega^{-1}\boldsymbol{X} + \boldsymbol{T}^{-1})^{-1}) \prod_{k=1}^{p} \boldsymbol{I}\left\{|\boldsymbol{\beta}_k| > \frac{1}{u_k}\right\},$

$\widehat{\boldsymbol{\beta}} = \left(\sum_{i=1}^{n}\sum_{k=1}^{K} \frac{x'x}{2\sigma v_i}\right)$ and $\widehat{\boldsymbol{\beta}} = \widehat{\boldsymbol{B}}\left(\sum_{i=1}^{n}\sum_{k=1}^{K} \frac{\left(x_i\left(z_i - b_{\tau_k} - x_i'\beta - \vartheta v_i\right)\right)}{2\sigma v_i}\right)$

6. sample$\boldsymbol{b_\tau}|.\sim \boldsymbol{N}\left(\frac{\sum_{i=1}^{n}\sum_{k=1}^{K}\left(z_i - b_{\tau_k} - x_i'\beta - \vartheta v_i\right)}{\sum_{i=1}^{n} 1/2\sigma v_i}, \frac{1}{\sum_{i=1}^{n} 1/2\sigma v_i}\right)$

7. sample $\boldsymbol{\sigma}|. \sim \boldsymbol{Inverse\ Gamma}\ \left(\boldsymbol{a}, \frac{3n}{2}, \boldsymbol{b} + \frac{1}{4}\sum_{i=1}^{n}\sum_{K=1}^{k}\ \left(\boldsymbol{z_i} - \boldsymbol{b}_{\tau_k} + \boldsymbol{x}_i'\boldsymbol{\beta} - \boldsymbol{\vartheta v_i}\right)'\boldsymbol{V}^{-1}\sum_{i=1}^{n}\sum_{k=1}^{K}\ \left(\boldsymbol{z_i} - \boldsymbol{b}_{\tau_k} + \boldsymbol{x}_i'\boldsymbol{\beta} - \boldsymbol{\vartheta v_i}\right)\right)$

8. sample $\boldsymbol{\lambda}|. \sim \boldsymbol{Gamma}\ \left(\boldsymbol{\gamma} + \boldsymbol{2p}, \boldsymbol{d} + \sum_{g=1}^{G} \frac{1}{|\beta_g|^{\alpha}}\right)$

9. sample $\boldsymbol{\delta_c}$, with $\boldsymbol{c}$ from 1 to $\boldsymbol{C-1}$, from a uniform distribution over the interval

$(\boldsymbol{min}\,\{\boldsymbol{min}(\boldsymbol{z_i}|\boldsymbol{y_i}=\boldsymbol{c+1}),\boldsymbol{\delta_{c+1}},\boldsymbol{\delta_C}\},\boldsymbol{max}\{\boldsymbol{max}(\boldsymbol{z_i}|\boldsymbol{y_i}=\boldsymbol{c})\},\boldsymbol{\delta_{c-1}},\boldsymbol{\delta_0})$.

10. Sample $\boldsymbol{z_i}$, $\boldsymbol{for\ i\ from\ 1\ to\ n}$, from truncated normal (TN) distribution

$\boldsymbol{TN_{(\delta_{c-1},\delta_c)}\left(b_{\tau_k}+x_i'\beta+\vartheta v,2\sigma v\right)}$.

end for

3.1.2 Bayesian Reciprocal Adaptive Bridge for Ordinal Composite Quantile Regression (BRABOCQReg)

Through the previous equation (3.1), we show the ordinal composite quantile regression with the reciprocal adaptive bridge penalty function by solving the following:

$$\underbrace{\boldsymbol{argmin}}_{\boldsymbol{b_\tau,\beta}}\sum_{i=1}^{n}\left\{\sum_{k=1}^{K}\boldsymbol{\rho_{\tau_j}}\left(\boldsymbol{z_i}-\boldsymbol{b_{\tau_k}}-\boldsymbol{\acute{x}_i\beta}\right)\right\}+\sum_{g=1}^{G}\frac{\boldsymbol{\lambda_g}}{|\boldsymbol{\beta_g}|^{\alpha}}\boldsymbol{I}\{\boldsymbol{\beta_g}\neq\boldsymbol{0}\}, \quad (3.5)$$

Where $\boldsymbol{\lambda_g\geq 0}$, g=1,…,G . The Gibbs sampler for the Bayesian reciprocal adaptive Bridge is possible by using the scale mixture in (3.2),

$$\frac{\lambda_g^{\frac{1}{\alpha}}}{2\beta_g^2\Gamma\left(\frac{1}{\alpha}+1\right)} e^{-\lambda|\beta_g|^{-\alpha}} = \frac{\lambda_g^{\frac{1}{\alpha}}}{2\beta_g^2\Gamma\left(\frac{1}{\alpha}+1\right)} \int_{u_g>|\beta|^{-\alpha}} \lambda_g e^{-\lambda_g u_g} \qquad (3.6)$$

Under (3.3) , the hierarchical model for the reciprocal adaptive Bridge is the same as (3.4) with λ replaced with $\boldsymbol{\lambda_g}$'s as follows :

$$u_g|\lambda_g \sim Gamma(2, \lambda_g),$$

$$\lambda_g \sim \lambda_g^{-1},$$

Algorithm 2 . MCMC sampling for the Bayesian reciprocal adaptive Bridge composite quantile regression (SMN)

Input: (z ,x)

Initialize: $(\boldsymbol{b_\tau}, \boldsymbol{\beta}, \boldsymbol{\sigma}, \boldsymbol{v}, \boldsymbol{u}, \boldsymbol{\lambda}, \boldsymbol{\alpha})$

For $\boldsymbol{t = 1, \ldots, (t_{max} + t_{burn-in})}$ ***do***

1. Sample $\boldsymbol{v^{-1}}|. \sim \prod_{i=1}^{n}$ ***Inverse Gaussian*** $\left(\frac{1}{2\sigma}, \frac{1}{|z_i - b_{\tau_k} - x_i'\beta|}, \frac{1}{2\sigma}\right)$

2. Sample $\boldsymbol{u}|. \sim \prod_{g=1}^{G} \boldsymbol{Exponential}\,(\lambda_g)\boldsymbol{I}\left\{\boldsymbol{u_g} > \frac{1}{|\beta_g|^{\alpha}}\right\}$
3. Sample $\boldsymbol{l^{-1}}|.\sim \prod_{g=1}^{G} \boldsymbol{Inverse - Gaussin}\left(\frac{1}{2}, \sqrt{\frac{\xi_k^2}{\beta_k^2}}, \xi_k^2\right)$
4. Sample$\boldsymbol{\xi}\Big|. \sim \prod_{g=1}^{G} \boldsymbol{Gamma}\left(2, \left(|\boldsymbol{\beta_g}|^{\alpha} + \frac{1}{u_g}\right)\right)$
5. Sample $\boldsymbol{\beta}|$. From a truncated multivariate normal proportionl to $\boldsymbol{N_P}((\boldsymbol{X}'\Omega^{-1}\boldsymbol{X} + \boldsymbol{T}^{-1})^{-1}\boldsymbol{X}'\Omega^{-1}(\boldsymbol{z} - \boldsymbol{\vartheta v}), (\boldsymbol{X}'\Omega^{-1}\boldsymbol{X} + \boldsymbol{T}^{-1})^{-1}) \prod_{k=1}^{p} \boldsymbol{I}\left\{|\boldsymbol{\beta_k}| > \frac{1}{u_k}\right\}$,

   $\widehat{\boldsymbol{\beta}} = \left(\sum_{i=1}^{n}\sum_{k=1}^{K} \frac{x'x}{2\sigma v_i}\right)$ and $\widehat{\boldsymbol{\beta}} = \widehat{\boldsymbol{B}}\left(\sum_{i=1}^{n}\sum_{k=1}^{K} \frac{\left(x_i\left(z_i - b_{\tau_k} - x_i'\beta - \vartheta v_i\right)\right)}{2\sigma v_i}\right)$
6. Sample$\boldsymbol{b_\tau}|.\sim \boldsymbol{N}\left(\frac{\sum_{i=1}^{n}\sum_{k=1}^{K}\left(z_i - b_{\tau_k} - x_i'\beta - \vartheta v_i\right)}{\sum_{i=1}^{n} 1/2\sigma v_i}, \frac{1}{\sum_{i=1}^{n} 1/2\sigma v_i}\right)$
7. Sample $\boldsymbol{\sigma}|. \sim \boldsymbol{Inverse\ Gamma}\ \left(\boldsymbol{a}, \frac{3n}{2}, \boldsymbol{b} + \frac{1}{4}\sum_{i=1}^{n}\sum_{K=1}^{k}\ \left(\boldsymbol{z_i} - \boldsymbol{b_{\tau_k}} + \boldsymbol{x_i'\beta} - \boldsymbol{\vartheta v_i}\right)'\boldsymbol{V}^{-1}\sum_{i=1}^{n}\sum_{k=1}^{K}\ \left(\boldsymbol{z_i} - \boldsymbol{b_{\tau_k}} + \boldsymbol{x_i'\beta} - \boldsymbol{\vartheta v_i}\right)\right)$
8. Sample $\boldsymbol{\lambda}|. \sim \boldsymbol{Gamma}\ \left(\boldsymbol{\gamma} + \boldsymbol{p}, \boldsymbol{d} + \frac{1}{|\beta_g|^{\alpha}}\right)$
9. Sample $\boldsymbol{\delta_c}$, with $\boldsymbol{c}$ from 1 to $\boldsymbol{C - 1}$, from a uniform distribution over the interval

   $(\boldsymbol{min}\ \{\boldsymbol{min}(\boldsymbol{z_i}|\boldsymbol{y_i} = \boldsymbol{c} + \boldsymbol{1}), \boldsymbol{\delta_{c+1}}, \boldsymbol{\delta_C}\}, \boldsymbol{max}\{\boldsymbol{max}(\boldsymbol{z_i}|\boldsymbol{y_i} = \boldsymbol{c})\}, \boldsymbol{\delta_{c-1}}, \boldsymbol{\delta_0})$.
10. Sample $\boldsymbol{z_i}$, $\boldsymbol{for\ i\ from\ 1\ to\ n}$, from truncated normal (TN) distribution

    $\boldsymbol{TN}_{(\delta_{c-1},\delta_c)}\left(\boldsymbol{b_{\tau_k}} + \boldsymbol{x_i'\beta} + \boldsymbol{\vartheta v}, \boldsymbol{2\sigma v}\right)$.

end for

CHAPTER FOUR

SIMULATION AND REAL DATA

4. Simulation and Real Data

In this section, we carry out simulation studies to investigate the performance of our proposed method "Bayesian reciprocal adaptive bridge composite quantile regression for ordinal data", referred to as "BrABCQRO", with comparison to some Bayesian and non-Bayesian approaches. The approaches in this comparison involve:

- Bayesian QR for ordinal models
- Bayesian model selection Ordinal
- Akaike Information Criterion AIC
- Bayesian Information Criterion BIC

4.1. Simulation Studies

Three simulation studies were conducted to investigate the performance of the proposed approach for the reciprocal adaptive Bridge ordinal composite quantile regression, referred to as "rABOCQR". The proposed approach is compared with Bayesian ordinal quantile regression (Rahman, 2016), referred to as "BOQR" and Bayseian model selection in ordinal quantile regression (Alhamzawi, 2016), referred to as "BMOQR".

4.1.1 Simulation 1

Consider data generated from the ordinal regression model,

$$z_i = x_i'\beta + \varepsilon_i \qquad i = 1, \ldots, 100, \qquad (4.1)$$

where $x_i = (1, x_{1i})'$and $\beta = (0, 4)'$, including the intercept. The variable x_{1i}

is generated from the standard normal distribution. We added to the model ten noise variables. These variables were simulated independently from $N_{10}(0, \Sigma_x)$with $(\Sigma_x)_{gh} = 0.75^{|g-h|}$, where gh refers to the $(g, h)th$ entry of Σ_x. In this simulation study, $\varepsilon i \sim N(0, 1)$. The outcome of interest y were obtained based on the cut- point vector $\delta = (0.5, 2, 3.5)'$, yielding four categories. 150 data are generated, each with $n = 100$ observations. For our proposed method, we choose $K = 3$. For other methods, we test the other methods with the median. The performance of rABOCQR is also compared with the AIC (Akaike, 1998) and BIC (Schwarz et al.,٢٠٠٨) .

Table 1: Comparing average numbers of correct and wrong zeros for different methods in Simulation example 1, averaged over 150 replications. The standard deviations are listed in parentheses.

	Methods				
	BrABOCQR	BOQR	BMOQR	AIC	BIC
correct	9.22 (0.09)	6.42 (0.14)	6.19 (0.39)	6.81 (0.42)	6.99 (0.08)
wrong	0.04 (0.11)	0.48 (0.45)	0.37 (0.42)	0.18 (0.38)	0.12 (0.13)

(1978). Here, AIC and BIC are respectively given by

$$\mathrm{AIC} = 2\mathrm{k} - 2\ln(\mathrm{L}),$$

and

$$\mathrm{BIC} = \mathrm{k}\ln(\mathrm{n}) - 2\ln(\mathrm{L}),$$

where L is the maximum value of the likelihood function for the subset. Given some models, the model with the lowest AIC or BIC is preferred.

In Table 1 the number of true and false zero regression coefficients for the best model is compared based on 150 generated datasets. The results show that the proposed method performs very well in terms of average numbers of correct and wrong zeros.

4.1.2 Simulation 2

This Simulation example is similar to Simulation 1 except that we set

$$\boldsymbol{z_i = x_i'\beta + \varepsilon_i} \qquad \boldsymbol{i = 1, \dots, 100}, \qquad (4.2)$$

where $\boldsymbol{x_i = (1, x_{1i}, x_{2i}, x_{3i})'}$ and $\boldsymbol{\beta = (1, 4, 2, -2)'}$, including the intercept. The variable x_{1i}, x_{2i}, and x_{3i} are generated from the standard normal distribution. We added to the model ten noise variables. These variables were simulated indepen- dently from N10(0, Σx) with (Σx)gh = $0.75^{|g-h|}$, where gh refers to the (g, h)th entry of Σx. In Table 2 the number of true and false zero regression coefficients is compared based on 150 generated datasets. The results show that the proposed method perform very well in terms of average numbers of correct and wrong zeros.

Table 2: Comparing average numbers of correct and wrong zeros for different methods in Simulation example 2, averaged over 150 replications. The standard deviations are listed in the parentheses.

	Methods				
	BrABOCQR	BOQR	BMOQR	AIC	BIC
correct	8.93 (0.14)	6.19 (0.23)	6.01 (0.42)	6.33 (0.51)	6.02 (0.15)
wrong	0.09 (0.16)	0.35 (0.42)	0.29 (0.33)	0.17 (0.41)	0.29 (0.53)

4.1.3 Simulation 3

This Simulation example is similar to Simulation 1 except that we set

$$\boldsymbol{z_i} = \boldsymbol{x}_i'\boldsymbol{\beta} + \boldsymbol{\varepsilon}_i \qquad \boldsymbol{i} = \mathbf{1}, \dots, \mathbf{100}, \qquad (4.2)$$

where $\boldsymbol{x_i} = (\mathbf{1}, \boldsymbol{x_{1i}}, \boldsymbol{x_{2i}}, \boldsymbol{x_{3i}})'$ and $\boldsymbol{\beta} = (\mathbf{1}, \mathbf{4}, \mathbf{2}, -\mathbf{2})'$, including the intercept. The variable x_{1i}, x_{2i}, and x_{3i} are generated from the standard normal distribution. We added to the model 100 noise variables. These variables were simulated indepen- dently from $N100(0, \Sigma x)$ with $(\Sigma x)gh = 0.90^{|g-h}$. In Table 2 the number of true and false zero regression coefficients is compared based on 150 generated datasets. Again, the results show that our method perform very well in terms of average numbers of correct and wrong zeros.

Table 3: Comparing average numbers of correct and wrong zeros for different methods in Simulation example 3, averaged over 150 replications. The standard deviations are listed in the parentheses.

	Methods				
	BrABOCQR	BOQR	BMOQR	AIC	BIC
correct	95.17 (0.17)	89.23 (0.19)	88.14 (0.53)	79.68 (0.62)	82.18(0.38)
wrong	0.11 (0.19)	0.28 (0.34)	0.38 (0.41)	0.38 (0.34)	0.72 (0.61)

4.2 A real data example

In this section, the performance of the rABOCQR approach is illustrated to those obtained using BOQR and BMOQR on the educational attainment (EA) data from the National Longitudinal Study of Youth (NLSY79), previously analysed in Al- hamzawi (2016) and Rahman (2016). In 1979, the NLSY started annual interviews with more than 12,000 youth on a battery of demographic questions. Alhamzawi (2016) used a subsample of this dataset. This subsample consistent 11 independent variables and one dependent variable, which is the level of education. Regressors include the square root of family income (x1), education of mother (x2), education of father (x3), working status of mother (x4), gender (x5), race (x6), and whether the youth lived in an urban area (x7) or the South at the age of 14 (x8). To control for age cohort affects, three dummy variables are included to indicate an individual's age in 1979 (age cohort 2 (x9), age cohort 3 (x10) and age cohort 4 (x11)). The outcome of interest has four categories:

(**1**)**less than a high school degree**, (**2**)**high schooldegree**, (**3**)**some college degree**

and (**4**) **graduate degree** (Jeliazkov et al., 2008).

The number of observations corresponding to the four categories of the outcome variable were 897, 1392, 876, and 758, respectively. Similar to the simulation studies, we choose K = 3 and compare with other methods in the median level.

Table 4: Estimates of model parameters in the educational attainment application.

Covariate	BrABOCQR DIC=9337.19	BOQR DIC=9781.02	BMOQR DIC=9568.31
Intercept	-3.27	-3.12	-2.01
$x1$	0.31	0.30	0.35
$x2$	0.05	0.15	0.27
$x3$	0.09	0.13	0.00
$x4$	0.00	0.10	0.00
$x5$	0.52	0.33	0.51
$x6$	0.48	0.41	0.22
$x7$	0.00	-0.10	-0.26
$x8$	0.00	0.11	0.00
$x9$	0.00	-0.04	0.00
$x10$	-0.10	-0.05	0.00
$x11$	0.61	0.38	0.33

The results are summarized in Table 4. It shows that the proposed method (rABOCQR) excluded the effect of variables (x4,x7,x8,x9) and compared with the methods used. we notice that the method (BMORQR) excluded variables (x3,x4,x8,x9,x10) while the method (BOQR) did not exclude any variables.

The investigation on model selection based on the DIC reports the following numbers: 9337.19, 9781.02 and 9568.31 for rABOCQR, BOQR and BMOQR, respectively. These results indicates that the proposed approach perform very well. Hence, the results of the simulation studies and real data analysis support the proposed approach.

CHAPTER FIVE

CONCLUSIONS AND FUTURE RESEARCH

5. Conclusions and Future Research

5.1. Conclusions

In this thesis, we propose the Bayesian reciprocal adaptive bridge composite quantile regression for simultaneous estimation and variable selection in ordinal models. This method gives sparse solution and enjoys the computational advantages of reciprocal bridge. We outline the joint posterior distribution, the prior distributions and the conditional distributions. A new Gibbs sampling algorithm is constructed for sampling from the full conditional posterior distributions. The proposed approach is illustrated using extensive simulation examples supported by a real data example shows that the proposed methods often outperform the existing methods.

5.2 Main Contributions

We have made the following contributions:

1- We have summarized the literature review of some Bayesian and non-Bayesian penalized regression approaches.
2- We have proposed the Bayesian approach for composite quantile regression in ordinal models.
3- We have proposed the Bayesian reciprocal adaptive bridge composite quantile regression for simultaneous estimation and variable selection in ordinal models.
4- We have proposed a new Gibbs sampling method for regularization in ordinal models.

5.3 Recommendations for Future Research

The work considered in this thesis can be easily extended to other models such as: the Bayesian reciprocal adaptive bridge composite quantile regression for count data, the Bayesian reciprocal adaptive bridge composite quantile regression for longitudinal data, the Bayesian reciprocal adaptive bridge composite quantile regression for tobit data and so on.

Bibliography

Akaike, H. (1974). A new look at the statistical model identification. IEEE transactions on automatic control, 19(6), 716-723.

Alhamzawi, R. (2016). Bayesian analysis of composite quantile regression. StatisticsinBiosciences8(2),358–373.

Alhamzawi, R. (2016). Bayesian model selection in ordinal quantile regression.

Alhamzawi, R., & Ali, H. T. M. (2018). Bayesian quantile regression for ordinal longitudinal data. Journal of Applied Statistics, 45(5), 815-828.

Alhamzawi, R., & Mallick, H. (2022). Bayesian reciprocal LASSO quantile regression. Communications in Statistics-Simulation and Computation, 51(11), 6479-6494.

Alhamzawi, R., Yu, K., 2013. Conjugate priors and variable selection for Bayesian quantile regression. Comput. Statist. Data Anal. 64, 209–219.

Aljabri, D. H. Q., & Alhamzawi, R. (2019, September). Bayesian bridge regression for ordinal models with a practical application. In Journal of Physics: Conference Series (Vol.1294,No.3,p.032030).IOPPublishing.

Aljasimee, M., & Alhamzawi, R. (2023). Bayesian group bridge composite quantile regression. Journal of Namibian Studies: History Politics Culture, 33, 433-444.

Alsaadi, Z., & Alhamzawi, R. (2022). Bayesian bridge and reciprocal bridge composite quantile regression. Communications in Statistics-Simulation and Computation, 1-18.

Computational Statistics & Data Analysis 103, 68–78.

D. F. Benoit, and D. Van den Poel,” Binary quantile regression: A Bayesian approach based on the asymmetric Laplace distribution.” Journal of Applied Econometrics, vol. 27, no.7, p. 1174-1188, 2012.

doi: http://dx.doi.org/10.1080/02664763.2013.785489.

Frank, L. E., & Friedman, J. H. (1993). A statistical view of some chemometrics regression tools. Technometrics, 35(2), 109-135.

George, E. I., & McCulloch, R. E. (1993). Variable selection via Gibbs sampling. Journal of the American Statistical Association, 88(423), 881-889. Bayesian Analysis (2009) 4, Number 1, pp. 85–118

H. Zou, and M. Yuan, “Composite quantile regression and the oracle model selection theory.” The Annals of Statistics, vol. 36, no. 3, 1108-1126, 2008.

Hoerl, A. E., & Kennard, R. W. (1970). Ridge regression: Biased estimation for nonorthogonal problems. Technometrics, 12(1), 55-67.

Hong, H. G. and Zhou, J. (2013). “A Multi-Index Model for Quantile Regression

Jeliazkov, I., J. Graves, and M. Kutzbach (2008). Fitting and comparison of models for multivariate ordinal outcomes. In Bayesian econometrics. Emerald Group Publishing Limited.

K. Khare, and J. P. Hobert, “Geometric ergodicity of the Gibbs sampler for Bayesian quantile regression.” Journal of Multivariate Analysis, vol. 112, p. 108-116, 2012.

K. Yu, and R. A. Moyeed, “Bayesian quantile regression.” Statistics & Probability Letters, vol. 54, no. 4, p. 437-447, 2001.

K. Yu, and R. A. Moyeed, “Bayesian quantile regression.” Statistics & Probability Letters, vol. 54, no. 4, p. 437-447, 2001

Koenker, R. (2005). Quantile Regression. Cambridge Books. Cambridge University Press.

Koenker, R. W., & d'Orey, V. (1987). Algorithm AS 229: Computing regression quantiles.Appliedstatistics,383-393.

Koenker, R., & Bassett Jr, G. (1978). Regression quantiles. Econometrica: journal of the Econometric Society, 33-50.

Kozumi, H. and Kobayashi, G. (2011). “Gibbs Sampling Methods for Bayesian Quantile Regression.” Journal of Statistical Computation and Simulation,81(11):1565–1578.MR2851270.doi:

http://dx.doi.org/10.1080/00949655.2010.496117. 1, 4, 6, 10

Liu, X., Saat, M.R., Qin, X., Barkan, C.P.L. (2013). Analysis of U.S. freight-train derailment severity using zero-truncated negative binomial regression and quantile regression. Accident Analysis and Prevention 59, 87-93.

Mallick, H., & Yi, N. (2018). Bayesian bridge regression. Journal of applied statistics,45(6),988-1008.

McCullagh, P. (1980). Regression models for ordinal data. Journal of the Royal Statistical -Society: Series B (Methodological), 42(2), 109–127.

Orsini, N., Bottai, M. (2011). Logistic quantile regression in Stata. The Stata Journal 11(3), 327-344.

Polson, N. G., Scott, J. G., and Windle, J. (2014). The bayesian bridge. Journal of the Royal Statistical Society: Series B (Statistical Methodology), 76(4), 713–733

Qin, X. (2012). Quantile effects of casual factors on crash distributions. Transportation Research Record 2219, 40-46.

Qin, X., Ng, M., Reyes, P.E. (2010). Identifying crash-prone locations with quantile regression. Accident Analysis and Prevention 42(6), 1531-1537.

Qin, X., Reyes, P.E. (2011). Conditional quantile analysis for crash count data. Journal of Transportation Engineering 137(9), 601-607.

R. Koenker, and J. A. Machado, "Goodness of fit and related inference processes for quantile regression." Journal of the american statistical association, vol. 94, no. 448, p. 1296-1310, 1999.

R. Koenker, and JR. Bassett," Regression quantiles." Econometrica: journal of the Econometric Society, vol. 46, no. 1, p. 33-50, 1978

R. Koenker, and V. D'Orey, "Algorithm AS 229: Computing regression quantiles." Journal of the Royal Statistical Society. Series C (Applied Statistics), vol. 36, no. 3, p. 383-393, 1987.

Rahman (2016). Bayesian quantile regression for ordinal models. Bayesian Analy- sis 11 (1), 1–24.

Royal Statistical Society: Series B (Statistical Methodology), 76(4), 713–733.

Schwarz, G. et al. (1978). Estimating the dimension of a model. The annals of statistics 6(2),461–464.

Spiegelhalter DJ, Best NG, Carlin BP, Der Linde AV (2002) Bayesian measures of model complexity and fit. J R Stat Soc Series B Stat ethodol 64: 583-639.

Tibshirani, R. (1996). Regression shrinkage and selection via the lasso. Journal of the Royal Statistical Society. Series B (Methodological), 267–288.

Toledano, A. Y., and Gatsonis, C. (1996). Ordinal regression methodology for ROC curves derived from correlated data. Statistics in Medicine, 15(16), 1807–1826.

Young, F. W. (1975). Methods for describing ordinal data with cardinal models. Journal of Mathematical Psychology, 12(4), 416–436.

Yu, K., & Moyeed, R. A. (2001). Bayesian quantile regression. Statistics & ProbabilityLetters,54(4),437-447.

Yu, K., Lu, Z., Stander, J., 2003. Quantile regression: applications and current research area. The Statistician 52, 331–350

Zhou, L. (2010). “Conditional Quantile Estimation with Ordinal Data.” Ph.D. thesis,

Zou, H. (2006). The adaptive lasso and its oracle properties. Journal of the American statistical association 101 (476), 1418–1429 .

الخلاصة

الهدف الرئيسي من تقدير معلمات النموذج هو الحصول على أفضل المقدرات التي تعطي تنبؤات ودقة عالية. وفي حالة تقدير جميع المعلمات في ظل البيانات عالية الأبعاد، يؤدي ذلك إلى الحصول على تنبؤ ضعيف مع ارتباطات عالية بين المتغيرات المستقلة، وبالتالي يتم الحصول على نتائج خاطئة. ولذلك، أصبح اختيار المتغير (V.S) واحدة من القضايا الأساسية لنمذجة البيانات عالية الأبعاد.

أحد التحديات في بناء نموذج الانحدار القسيمي QR هو اختيار المتغيرات النشطة . يؤدي الاختيار المناسب لمجموعة فرعية من المتغيرات المشتركة إلى تحسين دقة التنبؤ في كثير من الحالات. من وجهة نظر علمية، للحصول على تفسير أفضل، فمن المستحسن اختيار مجموعة فرعية أصغر. تم اقتراح العديد من التقنيات للحصول على المجموعة الفرعية النشطة.

تتناول هذه الدراسة المنهج الهرمي البايزي لاختيار المتغير والتقدير في QR الخطي. على وجه الخصوص، نقترح طريقة جسر التنظيم والانحدار القسيمي المركب الترتيبي. في هذه الرسالة، تم اقتراح الانحدار القسيمي المركب للجسر التكيفي العكسي البايزي والمشار إليها باسم "BrABCQRO"لاختيار المتغيرات وتقديرها في البيانات الترتيبية. تم إنشاء خوارزمية أخذ عينات Gibbsجديدة لأخذ العينات من التوزيعات الخلفية الشرطية الكاملة. ، مع المقارنة ببعض الأساليب البايزية وغير البايزية. يتم توضيح النهج المقترح باستخدام أمثلة محاكاة واسعة النطاق مدعومة بمثال بيانات حقيقي يوضح أن الأساليب المقترحة غالبًا ما تتفوق على الأساليب الحالية.

جمهورية العراق
وزارة التعليم العالي والبحث العلمي
جامعة القادسية
كلية الادارة والاقتصاد
قسم الاحصاء
الدراسات العليا

الانحدار القسيمي المركب البيزي بمقدرات Reciprocal Adaptive Bridge

رسالة مقدمة
الى مجلس كلية الادارة والاقتصاد – جامعة القادسية
جزءا من متطلبات نيل درجة الماجستير في علوم الاحصاء

تقدمها
زهراء سعد عنبر الياسري

اشراف
ا.د.رحيم جبارالحمزاوي

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