Analysis of anemia data using robust sparse sliced inverse regression

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Abstract

The sliced inverse regression (SIR) is a technique for lowering the dimensions in regression applications without losing any information about the regression. Although the SIR has been demonstrated to be an effective strategy for dealing with high dimensional situations, it has the drawback of not containing all of the original predictors. By combining variable selection techniques with SIR, many researchers proposed solutions to this problem. One of these techniques combined the Elastic Net penalty with the SIR method (SIR-EN). The SIR-EN is an effective approach that does not rely on a parametric model. When the predictors are highly correlated under sufficient dimension reduction settings. However, SSIR- EN is not robust to outliers because it uses a loss function that is. As a result, we suggested RSSIR-EN as a robust version of SSIR-EN for outliers in both the dependent variable and the covariates.

Key words: Dimension reduction, SIR, Robust estimation, Elastic-Net.

1-Introduction

Due to the explosion of large information in the past decades, high-dimensional regression analysis problems appear in several applications. The sufficient dimension reduction(SDR) theory has received great attention in high – dimensional regression. The basic idea of SDR is to exchange X with d-dimensional orthogonal dropping PsX onto S, where d<p and p is a number of covariates, without losing information about the conditional distribution of Y|X and without assuming any parameter pattern. Assume Y is a response variable, $X = \left(X_1, X_2, \dots., X_p\right)^T$ is a predictors vector. SDR aims to find the central subspace $S_{Y|X}$and $S_{Y|X}$ is the intersection of all subspaces S that achieve $Y \perp\!\!\!\perp X|P_s X$, when $\perp\!\!\!\perp$ it indicates independence. Therefore, $P_\beta X$ extracts that information from X to Y, where β is the basis of $S_{Y|X}$(Cook, 1998).There are several suggested methods to find $S_{Y|X}$. One of the most important of these methods which has proven to be effective in dealing with high dimensions is SIR method (Li, 1991). SIR is one of the useful tools, which help to solve the problem of the "high dimensions". It is applied in different fields,

including economics and bioinformatics. The results of SIR are linear sums of all the original variables, which may cause difficulty in interpreting the results of SIR. For this reason, there a need to reduce the number of non-zero coefficients in the SIR directions. Under least squares settings , there are many procedures of regularization methods that have been suggested . For example, Lasso (Tibshirani, 1996), Smoothly Clipped Absolute Deviation (Fan and Li, 2001), Elastic Net (Zou and Hastie, 2005), group lasso (Yuan and Li, 2006), Adaptive Lasso (Zou, 2006), and others.

Under SIR framework, several procedures have been proposed that combine SIR method with the regularization methods. Like, a free-models method for determining the contribution of variables which has been suggested by (Cook, 2004). Also, Lasso is combined with SIR to produce shrinkage estimator of SIR by (Ni, 2005). Sparse SIR (SPSIR) in which that combined lasso with LARS into SIR that suggested by (Li and Nachtsheim, 2006). As well as, a number of SDR methods that integrate with the shrinkage estimator that proposed by (Li, 2007). To improve SIR to work when the covariates are highly correlated and settings $p > n$, where n is a sample size(Li and Yin, 2008) they suggested that regularization SIR method (RSIR), for multiple index models with settings $p > n$. A lasso is combined with SIR that proposed by (Lin, 2019). Many researchers suggested approaches to dealing with this problem by combining variable selection methods with SIR. Alkenani (2021) proposed RSIR-Lasso method that does not have the ability to select groups of highly correlated predictors. Alkenani and Hassel (2020) proposed SIR-EN method which deals with correlated predictors but this method sensitive to outliers and are not robust because the method uses the least squares loss function which is sensitive to outliers in data . It is necessary to deal with this problem and solve this problem. The squared loss criterion is used between the covariates and response. Also, the classical estimates of the sample mean and the sample variance of X is used within the least squares formula. These are all sensitive to outliers and are not robust. In this research, we proposed robust method of SIR method with Elastic Net(EN) by using Tukey biweight criterion instead the squared loss criterion. If the derivative of the loss function is descending, the loss function is robust and insensitive to outliers in X and Y (Rousseeuw and Yohai, 1984). Tukey biweight function has this property.

2- SIR and SSIR Methods

For finding the central subspace $S_{Y|X}$, SIR method is suggested by (Li, 1991). This method requires $Z = \Sigma^{\frac{-1}{2}}(X - E(X))$, under the linear condition $E(Z/PgZ) = PgZ$, where $\sum_x = Cov(X)$ is a population covariance matrix of X and g is a basis to $S_{Y|Z}$. $S_{Y|Z}$ is the central subspace of regression Y on Z. This condition connects with the inverse regression of Z on Y. The kernel matrix of SIR is M and $M = Cov[E(Z|Y)]$, $span(M) \subseteq S_{Y|Z}$. We took a random sample of size n of (X, Y), which has a joint distribution. Let $\bar{X}$ is the sample mean of X, the sample version of Z is $\hat{Z} = \hat{\Sigma}^{-\frac{1}{2}}(X - \bar{X})$ and $\hat{\sum}$ is the estimated covariance matrix of X. Assume h be the number of slices also n_y is a number of observations in yth slice. Let $\widehat{M} = \sum_{y=1}^{h} \hat{f}_y \hat{Z}_y \hat{Z}_y^T$ is an estimator of M, where $\hat{f}_y = n_y|n$ and $\hat{Z}_y$ is the average of Z in slice y. Let $\hat{\delta}_1 > \hat{\delta}_2 > \cdots > \hat{\delta}_p \geq 0$ are the eigenvalues corresponding to the eigenvectors $\hat{v}_1, \hat{v}_2, \dots, \hat{v}_p$ of $\widehat{M}$. If d of $S_{Y|Z}$ is known and $span(\hat{\beta}) = span(\hat{\beta}_1, \hat{\beta}_2, \dots, \hat{\beta}_d)$ is a consistent estimator of $S_{Y|X}$, where $\hat{\beta}_i = \hat{\sum}^{\frac{-1}{2}} \hat{v}_i$. The SIR method provides the estimator $span(\hat{\beta})$ of $S_{Y|X}$. Generally, $\hat{\beta} \in \mathbb{R}^{p\times d}$ has nonzero elements, when the number of predictions is huge or when the number of predictions is highly correlated, we only take the important predictions that we need to make 'sufficient predictors' combining the regularizations methods with SIR method is the solution to compress number of the coefficients of $\hat{\beta}$ to 0's. The SIR was formulated by (Cook, 2004) as a regression type minimization problem(least squares problem) as follows :

$$F(A, C) = \sum_{y=1}^{h} \left\| \hat{f}_y^{\frac{1}{2}} \hat{Z}_y - AC_y \right\|^2, \qquad (1)$$

Over $A \in \mathbb{R}^{p\times d}$ and$C_y \in \mathbb{R}^d$ with $C = (C_1, \dots, C_h)$. Let $\hat{A}$ and $\hat{C}$ are the values of A and C that minimize F. Then $span(\hat{A})$ equals the space spanned by the d largest eigenvectors of M. By focusing on the coefficients of X variables, (Ni et.al,2005)reformulate$F(A, C)$as::

$$G(B, C) = \sum_{y=1}^{h} \left(\hat{f}_y^{\frac{1}{2}} \hat{\Sigma}^{\frac{-1}{2}} \hat{Z}_y - BCy \right)^T \hat{\Sigma} \left(\hat{f}_y^{\frac{1}{2}} \hat{\Sigma}^{\frac{-1}{2}} \hat{Z}_y - BC_y \right), \qquad (2)$$

Where $B \in \mathbb{R}^{p\times d}$. The value of B, which minimizes (2) is $\hat{\beta}$ and $span(\hat{\beta}) = span(\hat{\Sigma}^{\frac{-1}{2}} \hat{A})$ is the estimator of $S_{Y|X}$. Ni et al. (2005) suggested shrinkage sliced inverse regression(SSIR) for finding $S_{Y|X}$ as $span(diag(\hat{\alpha})\hat{\beta})$, where the shrinkage indices $\tilde{\alpha} = (\tilde{\alpha}_1, \dots, \tilde{\alpha}_p)^T \in \mathbb{R}^p$ are determined by minimizing

$$\sum_{y=1}^{h} \left\| \hat{f}_y^{\frac{1}{2}} \hat{Z}_y - \hat{\Sigma}^{\frac{1}{2}} diag(\hat{B}\hat{C}_y)\alpha \right\| + \lambda \sum_{j=1}^{p} |\alpha_j|, \qquad (3)$$

Where $\hat{B}$ and $\hat{C} = (\hat{C}_1, \dots, \hat{C}_h)$ minimize(2). The minimization of (3) can be done by using a standard Lasso algorithm, let $\tilde{Y} = vec(\hat{f}_1^{\frac{1}{2}} \hat{Z}_1, \dots, \hat{f}_h^{\frac{1}{2}} \hat{Z}_h) \in \mathbb{R}^{ph}$, and

$\tilde{X} = (diag(\hat{B}\hat{C}_1)\hat{\Sigma}^{\frac{1}{2}}, \dots, diag(\hat{B}\hat{C}_h)\hat{\Sigma}^{\frac{1}{2}})^T \in \mathbb{R}^{ph \times p}$. Where $vec(.)$ is a matrix operator that it puts the columns of the matrix in the single vector. Also, the vector α is the estimator of the lasso in the regression $\tilde{Y}$ and $\tilde{X}$.

3-1-Robust SSIR –EN

-Methodology

SIR use the classical estimates of the sample mean and the sample covariance. Also, it uses the squared loss between the response variable and the covariates. The classical estimates for the mean and covariance and loss squared criterion are very sensitive to outliers and they are not robust .

Gather et.al, (2002) studied SIR's sensitivity to outliers, also suggested a robust version for SIR. Yohai and Sertter(2005) proposed another a robust version of SIR. Prendergast(2005) studied the influence function of SIR. When the derivative of the loss function is redescending, it is robust and insensitive to outliers in Y and X (Rousseeuw and Yohai, 1984). This property is existed in Tukey's biweight loss function (Tukey, 1960). We exchange the loss squared function with Tukey's biweight function in(2-5), that achieve the robustness against outliers in X and Y. Alkenani (2021) suggested robust shrinkage for SIR through combining Lasso with Tukey biweight criterion for SIR. The drawback of this method is that it does not deal with data in groups and also data with high correlations. For this reason, we propose a robust method for variable selection under SDR settings deals with grouped predictors. The proposed method (RSSIR – EN) is a robust version of SSIR-EN (Alkenani and Hassel,2020).

In this study, we replace the classical estimates of sample mean with a robust estimator such as the median and replace the classical estimates of sample covariance matrix with robust covariance matrix estimator as ball covariance. The estimates of suggested RSSIR-EN can be obtained by minimizing the following .

$$\sum_{y=1}^{h} \rho\left(\frac{\hat{f}_y^{\frac{1}{2}}\widehat{ROZ}_y - \widehat{RO\Sigma}^{\frac{1}{2}} diag(\hat{B}\hat{C}_y)\alpha}{\hat{\sigma}}\right) + \lambda_1 \sum_{j=1}^{p} \alpha_j^2 + \lambda_2 \sum_{j=1}^{p} |\alpha_j|, \qquad (4)$$

The minimizing of (4) contains two parts. The first part is robust SIR by using Tukey's biweight function and the second part is Elastic Net penalty function, where, ρ is Tukey's biweight function .

$\hat{\sigma}$ is a robust estimate of σ and MAD is used as an estimate for σ, where MAD is the median absolute deviation .

$\widehat{ROZ}_y$is a robust versions of $\hat{Z}_y$.

$\widehat{RO\Sigma}^{\frac{1}{2}}$ is a robust version of $\hat{\Sigma}^{\frac{1}{2}}$.
$\lambda_1, \lambda_2 \geq 0$ is the tuning parameters of EN .

The function of Tukey's biweight is as follows:

$$\rho_c(u) = \begin{cases} \left(\frac{c^2}{6}\right)\left\{1 - \left[1 - \left(\frac{u}{6}\right)^2\right]^3\right\} & if\ |u| \leq c \\ \frac{c^2}{6} & if\ |u| \leq c \end{cases} \tag{5}$$

where c controls the robustness level .

4-Simulation study

The major goal of this section is to compare the effectiveness and variable selection of the proposed approach (RSSIR-EN) with those of RSSIR-Lasso and SSIR-EN. For the tuning parameter, we used a robust RIC suggested by (Alkenani,2021) in all samples. The SSIR-Lasso, R code is created by (Ni et.al, 2005). The SIR- AL, R code is created by (Alkenani and Salman, 2021). The RSIR-L, R code is created by (Alkenani 2021). The SSIR-EN, R code is created (Alkenani and Hassel, 2020). The RSSIR-EN, R code is created by (Alkenani and Alkim, 2023). In term of variable selection, the average number of zeros coefficients(Ave0's) is reported. In term of prediction accuracy, the mean squared error (MSE) is reported. Four distributions are assumed for ε and X.

Dist.1. The standard normal distribution N(0,1).

Dist.2. $t_3/\sqrt{3}$, t-distribution with 3 degree of freedom.

Dist.3. $(1-\alpha)N(0,1) + \alpha\, N(0, 10^2)$

Dist.4. $(1-\alpha)N(0,1) + \alpha\, U(-50,50)$, $(1-\alpha)$ from standard normal and α from normal with mean 0 and variance 100 for (Dist.3) and uniform(-50,50) for(Dist.4).

Example . Let d =1, iteration=500, p =40and n=50,100 and 200. Consider the model,

$$Y = 1 + 2(\theta^T X + 3) \times \log(3|\theta^T X| + 1) + \varepsilon\ .$$

$$\theta = \left(\underbrace{3, \dots, 3}_{15}, \underbrace{0, \dots, 0}_{25}\right)^T,$$

$x_i = z_1 + \varepsilon_i,\ i = 1, \dots, 5,$

$x_i = z_2 + \varepsilon_i,\ i = 6, \dots, 10,$

$x_i = z_3 + \varepsilon_i,\ i = 11, \dots, 15,$

$x_i,\ i = 16, \dots, 40,$

For $i = 1, \dots, 15$, five predictors within each group and there are three groups in this model. There are 25 zero predictors.

Table 1: The results of example, based on Ave0's, and MSE when n = 50 and α =0.05, for dist3 and dist4.

Dist	Method	MSE	Ave. 0's
1	SSIR-EN	4.680351e-05	1.58
	RSSIR-Lasso	3.332191e-05	3.03
	RSSIR -EN	1.236556e-05	3.52
2	SSIR-EN	0.04513449	1.52
	RSSIR-Lasso	6.082133e-05	3.01
	RSSIR -EN	3.071832e-05	5.00
3	SSIR-EN	0.04507638	1.37
	RSSIR-Lasso	5.94675e-05	3.01
	RSSIR -EN	2.922632e-05	6.63
4	SSIR-EN	0.04508374	1.36
	RSSIR-Lasso	5.988378e-05	3.02
	RSSIR -EN	2.946017e-05	6.58

Table 2: The results of example, based on Ave0's, and MSE when n = 50 and α =0.10, for dist3 and dist4.

Dist	Method	MSE	Ave. 0's
1	SSIR-EN	0.487712e-04	2.38
	RSSIR-Lasso	6.151865e-05	5.00
	RSSIR –EN	2.524819e-05	6.59
2	SSIR-EN	0.04876865	2.48
	RSSIR-Lasso	6.131084e-05	5.02
	RSSIR –EN	2.540396e-05	6.82
3	SSIR-EN	0.04860849	2.56
	RSSIR-Lasso	5.698737e-05	5.04
	RSSIR –EN	2.264034e-05	6.13
4	SSIR-EN	0.04864073	2.38
	RSSIR-Lasso	5.727573e-05	5.03
	RSSIR –EN	2.268371e-05	6.08

Table 3:The results of example, based on Ave0's, and MSE when n = 50 and α =0.15, for dist3 and dist4.

Dist	Method	MSE	Ave. 0's
1	SSIR-EN	9.752811e-05	2.61
	RSSIR-Lasso	7.430622e-05	5.02
	RSSIR –EN	1.021396e-05	6.53
2	SSIR-EN	0.0752785	2.58
	RSSIR-Lasso	7.430148e-05	5.02
	RSSIR –EN	1.02832e-05	6.42
3	SSIR-EN	0.0750644	2.51
	RSSIR-Lasso	7.091946e-05	6.03
	RSSIR -EN	9.290021e-06	6.59
4	SSIR-EN	0.07510269	2.38
	RSSIR-Lasso	7.112544e-05	6.01
	RSSIR -EN	9.188532e-06	6.42

Table 4:The results of example, based on Ave0's, and MSE when n = 50 and α =0.20, for dist3 and dist4.

Dist	Method	MSE	Ave. 0's
1	SSIR-EN	9.40284e-05	3.47
	RSSIR-Lasso	8.376686e-05	5.02
	RSSIR –EN	7.197939e-06	5.66
2	SSIR-EN	0.09402875	3.04
	RSSIR-Lasso	8.390795e-05	6.01
	RSSIR –EN	7.339105e-06	7.71
3	SSIR-EN	0.0929563	3.42
	RSSIR-Lasso	8.483299e-05	6.02
	RSSIR –EN	6.209121e-06	8.78
4	SSIR-EN	0.09384626	3.48
	RSSIR-Lasso	8.461874e-05	4.02
	RSSIR -EN	6.327946e-06	8.25

Table 5: The results of example, based on Ave0's, and MSE when n = 50 and α =0.25, for dist3 and dist4.

Dist	Method	MSE	Ave. 0's
1	SSIR-EN	1.254659e-05	4.51
	RSSIR-Lasso	6.335259e-06	6.03
	RSSIR –EN	1.930139e-06	7.90
2	SSIR-EN	0.1254666	4.44
	RSSIR-Lasso	0.0001351151	6.01
	RSSIR –EN	7.09998e-06	7.56
3	SSIR-EN	0.1202211	4.53
	RSSIR-Lasso	0.0001171884	6.03
	RSSIR –EN	5.199173e-06	7.88
4	SSIR-EN	0.1254129	5.44
	RSSIR-Lasso	0.0001315369	7.02
	RSSIR -EN	5.102739e-06	7.86

Table 6: The results of example, based on Ave0's, and MSE when n = 50 and α =0.30, for dist3 and dist4.

Dist	Method	MSE	Ave. 0's
1	SSIR-EN	1.293606e-05	5.43
	RSSIR-Lasso	1.300193e-05	6.02
	RSSIR -EN	1.345039e-06	6.96
2	SSIR-EN	0.1293615	5.48
	RSSIR-Lasso	0.0001302165	6.05
	RSSIR -EN	6.393441e-06	7.97
3	SSIR-EN	0.1242821	5.35
	RSSIR-Lasso	0.0001200639	6.02
	RSSIR -EN	4.940732e-06	8.89
4	SSIR-EN	0.1292835	5.46
	RSSIR-Lasso	0.0001247306	7.01
	RSSIR -EN	4.758929e-06	8.51

Table 7:The results of example, based on Ave0's, and MSE when n = 50 and α =0.35, for dist3 and dist4.

Dist	Method	MSE	Ave. 0's
1	SSIR-EN	1.437443e-05	5.48
	RSSIR-Lasso	1.412027e-06	6.04
	RSSIR -EN	1.118897e-06	7.78
2	SSIR-EN	0.1437446	5.6
	RSSIR-Lasso	0.0001411884	6.03
	RSSIR -EN	6.163878e-06	8.24
3	SSIR-EN	0.1397133	5.51
	RSSIR-Lasso	0.0001352369	7.05
	RSSIR -EN	4.341259e-06	9.48
4	SSIR-EN	0.1436688	6.48
	RSSIR-Lasso	0.0001355744	8.02
	RSSIR -EN	4.339982e-06	9.48

Table 8: The results of example, based on Ave0's, and MSE when n = 100 and α =0.05, for dist3 and dist4.

Dist	Method	MSE	Ave. 0's
1	SSIR-EN	2.190735e-04	7.49
	RSSIR-Lasso	3.586474e-05	8.02
	RSSIR –EN	1.969126e-05	8.62
2	SSIR-EN	0.027081	7.43
	RSSIR-Lasso	4.570905e-05	8.01
	RSSIR –EN	2.274569e-05	9.20
3	SSIR-EN	0.02700096	7.54
	RSSIR-Lasso	4.292843e-05	8.02
	RSSIR –EN	2.086527e-05	10.61
4	SSIR-EN	0.02700723	7.48
	RSSIR-Lasso	4.308358e-05	9.02
	RSSIR –EN	2.086641e-05	10.62

Table 9:The results of example, based on Ave0's, and MSE when n = 100 and α =0.10, for dist3 and dist4.

Dist	Method	MSE	Ave. 0's
1	SSIR-EN	5.491451e-05	7.47
	RSSIR-Lasso	3.618804e-05	9.03
	RSSIR –EN	1.194799e-06	10.53
2	SSIR-EN	0.05491358	8.43
	RSSIR-Lasso	6.639158e-05	9.02
	RSSIR –EN	1.200572e-05	10.34
3	SSIR-EN	0.05476601	8.69
	RSSIR-Lasso	5.890493e-05	10.04
	RSSIR –EN	9.989967e-06	10.09
4	SSIR-EN	0.05478319	8.58
	RSSIR-Lasso	5.934513e-05	10.02
	RSSIR –EN	1.003302e-05	11.70

Table10:The results of example, based on Ave0's, and MSE when n = 100 and α =0.15, for dist3 and dist4.

Dist	Method	MSE	Ave. 0's
1	SSIR-EN	8.379288e-05	10.55
	RSSIR-Lasso	8.319258e-05	11.02
	RSSIR –EN	4.459176e-06	12.31
2	SSIR-EN	0.08379099	10.57
	RSSIR-Lasso	8.321388e-05	11.02
	RSSIR –EN	4.502566e-06	12.31
3	SSIR-EN	0.08337906	10.50
	RSSIR-Lasso	8.196286e-05	11.02
	RSSIR –EN	3.986228e-06	12.27
4	SSIR-EN	0.08361779	10.52
	RSSIR-Lasso	8.16096e-05	11.02
	RSSIR –EN	3.979309e-06	12.16

Table11:The results of example, based on Ave0's, and MSE when n = 100 and α =0.20, for dist3 and dist4.

Dist	Method	MSE	Ave. 0's
1	SSIR-EN	1.198281e-05	10.46
	RSSIR-Lasso	1.138074 e-05	11.02
	RSSIR –EN	1.058396e-06	12.11
2	SSIR-EN	0.1128276	11.47
	RSSIR-Lasso	0.0001142325	12.04
	RSSIR –EN	3.109118e-06	13.10
3	SSIR-EN	0.1104485	11.53
	RSSIR-Lasso	0.0001048415	12.02
	RSSIR –EN	2.526569e-06	13.18
4	SSIR-EN	0.1123821	11.49
	RSSIR-Lasso	0.0001133031	13.04
	RSSIR -EN	2.636017e-06	13.12

Table12:The results of example, based on Ave0's, and MSE when n = 100 and α =0.25, for dist3 and dist4.

Dist	Method	MSE	Ave. 0's
1	SSIR-EN	2.184619e-05	10.45
	RSSIR-Lasso	1.243701 e-05	12.03
	RSSIR –EN	2.116956e-06	13.03
2	SSIR-EN	0.118463	11.39
	RSSIR-Lasso	0.0001248949	12.02
	RSSIR –EN	2.158885e-06	13.02
3	SSIR-EN	0.1147694	11.55
	RSSIR-Lasso	0.000112617	13.04
	RSSIR –EN	1.510166e-06	13.60
4	SSIR-EN	0.1175648	11.34
	RSSIR-Lasso	0.0001187808	13.03
	RSSIR –EN	1.552647e-06	14.76

Table13:The results of example, based on Ave0's, and MSE when n = 100 and α =0.30, for dist3 and dist4.

Dist	Method	MSE	Ave. 0's
1	SSIR-EN	1.429294e-05	11.44
	RSSIR-Lasso	1.400087e-05	13.03
	RSSIR –EN	1.572459e-06	13.10
2	SSIR-EN	0.142932	12.58
	RSSIR-Lasso	0.0001395143	13.02
	RSSIR –EN	1.601245e-06	14.00
3	SSIR-EN	0.1399566	12.60
	RSSIR-Lasso	0.0001361497	13.06
	RSSIR –EN	1.233555e-06	14.01
4	SSIR-EN	0.1428769	12.33
	RSSIR-Lasso	0.000141761	14.05
	RSSIR -EN	1.196548e-06	14.99

Table14:The results of exampl2, based on Ave0's, and MSE when n = 100 and α =0.35, for dist3 and dist4.

Dist	Method	MSE	Ave. 0's
1	SSIR-EN	1.998768e-06	11.59
	RSSIR-Lasso	1.47831 e-06	13.02
	RSSIR –EN	1.042574e-06	13.36
2	SSIR-EN	0.1498788	12.62
	RSSIR-Lasso	0.0001485344	13.04
	RSSIR –EN	1.451878e-06	14.53
3	SSIR-EN	0.1497443	12.60
	RSSIR-Lasso	0.0001476943	13.02
	RSSIR –EN	1.140442e-06	14.56
4	SSIR-EN	0.1497785	12.62
	RSSIR-Lasso	0.0001468358	14.04
	RSSIR -EN	1.16248e-06	14.72

Table15:The results of example, based on Ave0's, and MSE when n = 200 and α =0.05, for dist3 and dist4.

Dist	Method	MSE	Ave. 0's
1	SSIR-EN	3.189274e-04	12.39
	RSSIR-Lasso	4.387594e-05	14.08
	RSSIR –EN	1.165695e-05	14.43
2	SSIR-EN	0.03189871	13.28
	RSSIR-Lasso	4.418828e-05	14.05
	RSSIR –EN	1.174325e-05	16.58
3	SSIR-EN	0.03183543	13.42
	RSSIR-Lasso	4.239408e-05	14.06
	RSSIR –EN	1.065352e-05	16.10
4	SSIR-EN	0.03184223	13.47
	RSSIR-Lasso	4.237561e-05	15.02
	RSSIR -EN	1.071299e-05	16.22

Table16:The results of example, based on Ave0's, and MSE when n = 200 and α =0.10, for dist3 and dist4.

Dist	Method	MSE	Ave. 0's
1	SSIR-EN	4.590554e-04	13.41
	RSSIR-Lasso	4.548223e-05	15.02
	RSSIR –EN	3.98697e-06	16.03
2	SSIR-EN	0.04591272	14.52
	RSSIR-Lasso	4.534859e-05	15.02
	RSSIR –EN	4.044684e-06	17.92
3	SSIR-EN	0.0457968	14.40
	RSSIR-Lasso	4.362968e-05	16.02
	RSSIR –EN	3.262841e-06	17.93
4	SSIR-EN	0.04581071	14.42
	RSSIR-Lasso	4.390169e-05	16.02
	RSSIR -EN	3.275948e-06	17.85

Table17: The results of example, based on Ave0's, and MSE when n = 200 and α =0.15, for dist3 and dist4.

Dist	Method	MSE	Ave. 0's
1	SSIR-EN	9.671376e-05	14.42
	RSSIR-Lasso	7.012378e-05	16.02
	RSSIR –EN	2.276617e-06	18.07
2	SSIR-EN	0.06671338	15.52
	RSSIR-Lasso	7.02296e-05	17.04
	RSSIR –EN	2.353524e-06	19.94
3	SSIR-EN	0.06652505	15.29
	RSSIR-Lasso	6.79203e-05	17.02
	RSSIR –EN	2.044871e-06	19.33
4	SSIR-EN	0.06655951	15.59
	RSSIR-Lasso	6.756311e-05	17.03
	RSSIR -EN	1.943729e-06	20.12

Table18: The results of example, based on Ave0’s, and MSE when n = 200 and α =0.20, for dist3 and dist4.

Dist	Method	MSE	Ave. 0’s
1	SSIR-EN	8.881025e-05	14.38
	RSSIR-Lasso	8.731137e-05	16.03
	RSSIR –EN	1.861217e-06	19.12
2	SSIR-EN	0.08380834	14.34
	RSSIR-Lasso	8.749299e-05	16.03
	RSSIR –EN	1.930359e-06	20.52
3	SSIR-EN	0.08263936	14.27
	RSSIR-Lasso	8.241334e-05	17.02
	RSSIR –EN	1.575206e-06	20.33
4	SSIR-EN	0.08365682	15.48
	RSSIR-Lasso	8.529328e-05	18.38
	RSSIR -EN	1.592363e-06	20.46

Table19:The results of example, based on Ave0’s, and MSE when n = 200 and α =0.25, for dist3 and dist4.

Dist	Method	MSE	Ave. 0’s
1	SSIR-EN	1.17272e-06	16.38
	RSSIR-Lasso	1.059426e-06	18.08
	RSSIR –EN	9.883827e-07	20.13
2	SSIR-EN	0.103309	17.5
	RSSIR-Lasso	0.0001024433	18.03
	RSSIR –EN	1.628671e-06	21.21
3	SSIR-EN	0.1002983	18.44
	RSSIR-Lasso	9.718615e-05	19.04
	RSSIR –EN	1.107554e-06	21.33
4	SSIR-EN	0.1029608	18.36
	RSSIR-Lasso	0.0001012051	19.02
	RSSIR -EN	1.1501e-06	21.66

Table20:The results of example, based on Ave0's, and MSE when n = 200 and α =0.30, for dist3 and dist4.

Dist	Method	MSE	Ave. 0's
1	SSIR-EN	1.195251e-06	18.46
	RSSIR-Lasso	1.13147e-06	19.03
	RSSIR -EN	1.105524e-06	22.43
2	SSIR-EN	0.1195242	18.65
	RSSIR-Lasso	0.0001126962	19.03
	RSSIR -EN	1.173969e-06	22.57
3	SSIR-EN	0.1122663	18.66
	RSSIR-Lasso	0.0001085499	20.04
	RSSIR -EN	8.777419e-07	22.85
4	SSIR-EN	0.119479	18.51
	RSSIR-Lasso	0.000112262	20.02
	RSSIR -EN	7.939724e-07	22.19

Table21:The results of example, based on Ave0's, and MSE when n = 200 and α =0.35, for dist3 and dist4.

Dist	Method	MSE	Ave. 0's
1	SSIR-EN	1.341499e-06	18.96
	RSSIR-Lasso	1.316776e-06	20.04
	RSSIR -EN	1.064537e-06	21.94
2	SSIR-EN	0.1341508	18.47
	RSSIR-Lasso	0.0001321163	20.08
	RSSIR -EN	1.106291e-06	22.94
3	SSIR-EN	0.1288302	18.58
	RSSIR-Lasso	0.0001216745	20.03
	RSSIR -EN	9.462509e-07	23.52
4	SSIR-EN	0.1341038	18.43
	RSSIR-Lasso	0.0001287023	20.04
	RSSIR -EN	8.133561e-07	24.79

In the results of tables 1,2,3,…., the simulations results show that the RSSIR-EN has better performance than SSIR-EN and RSSIR-Lasso when the outliers exist in X and Y in terms the estimation accuracy and variable selection. Also, the RSSIR-EN gives very close results to SSIR-EN when there are no outliers. It can be seen that there is a slight outperform for the suggested method where it has a lower MSE and it has a bigger values based on Ave.0's. In case of three distributions of x and error, we can note that SIR-EN method was sensitive for the contamination but other methods RSSIR-Lasso and RSSIR-EN were not affected because they have the robustness. Also, we can see that the performance of RSSIR-EN outperformed RSSIR-Lasso method in terms of V.S based on Ave.0's. For the previous example, the MSE values for RSSIR-EN are less than their values for RSSIR-Lasso and SSIR-EN. This means that the suggested RSSIR-EN has the best performance than the rest methods depending on the MSE of simulation studies. It is clear that under various settings

<u>**5-Real data for anemia**</u>

In this section, to check the performance of the suggested RSSIR-EN method, we used the SSIR-EN, RSSIR-Lasso and RSSIR-EN methods in analysis anemia data. Data were collected for 200 samples of anemia patients from Thalassemia Specialist Center in Al-Diwaniyah. We assumed the response variable Y is the level of hemoglobin(HB) in blood, also we assumed twenty-one independent variable X as follows;

X_1 is the age.
X_2 is the gender.
$X_3$is the blood group.
X_4 is the length.
$X_5$is the weight.
X_6 is Academic achievement.
X_7 is living.
X_8 is the income.
X_9 is the nature of food.
X_{10} is having surgeries.
X_{11} is iron percentage.
X_{12} is White blood cells(WBC)
X_{13} is Neutrophils(NE)
X_{14} is Lymphocytes(LY)
X_{15} is Monocytes.(MO)
X_{16} is Eosinophils(EO)
X_{17} is Basophils.(BA)
X_{18} is Platelet count test(PLT)
X_{19} is Mean platelet volume(MPV)
X_{20} is the genetic factor.
X_{21} is the social status.

We made a comparison to evaluate the accuracy of the suggested method RSSIR-EN and SSIR-EN, RSSIR-Lasso methods based on the mean squared error(MSE) and number of zero's coefficient.

Table22: The results of Real data based on number of zero's and MSE

Method	MSE	Number of zero's
SSIR-EN	0.002985756	8
RSSIR-Lasso	0.003162099	13
RSSIR-EN	0.002700182	15

From the result of table22, it can be seen that there is a slight outperform for the suggested approach where it has a lower MSE and it has a bigger values based on number of zero's coefficients. We can note that SIR-EN method was sensitive for the contamination but other methods RSSIR-Lasso and RSSIR-EN were not affected because they have the robustness. Also, we can see that the performance of RSSIR-EN outperformed RSSIR-Lasso method in terms of V.S based on number of zero's coefficients. For the Real data for anemia, the MSE values for RSSIR-EN are less than their values for RSSIR-Lasso and SSIR-EN. This means that the suggested RSSIR-EN has the best performance than the rest methods depending on the MSE. It is clear that under various settings, the proposed RSSIR-EN has a good performance in terms of variable selection and estimation accuracy.

Table23: The results of Real data based on beta

SSIR-EN	RSSIR-Lasso	RSSIR-EN
2.083801	0.0000000	0.0000000
2.941862	0.0000000	2.2890836
1.304397	1.1615607	2.1744716
0.000000	0.0000000	0.0000000
1.331138	0.0000000	0.0000000
0.000000	0.0000000	0.0000000
2.812370	0.0000000	0.0000000
1.558690	0.5661126	0.0000000
0.000000	0.4323028	0.0000000
55.012939	0.0000000	29.9060514
0.000000	0.0000000	0.0000000
1.014341	2.4093509	0.0000000
2.583451	0.0000000	2.8620446
4.161742	1.2082380	0.0000000
2.571519	1.7851408	1.3786562
3.013674	0.9849392	0.0000000
0.000000	0.0000000	0.0000000
0.000000	0.0000000	0.0000000
4.640600	0.0000000	5.7933282
0.000000	1.8679699	0.0000000
0.000000	0.0000000	0.0000000

From the correlation matrix in table, it is clear that there are high correlations among the variables. High pairwise correlations are found in$(X_1,X_5)(X_1,X_6)(X_4,X_5)(X_4,X_6)(X_5,X_6)(X_6,X_5)(X_{13},X_{14})(X_6,X_4)(X_5,X_4)$ and others as shown in the following table24;

Table24: The results of Real data based on correlation of variables

	y	x1	x2	x3	x4	x5	x6	x7	x8	x9	x10	x11	x12	x13	x14	x15	x16	x17	x18	x19	x20	x21
y	1	0.099009	0.03269	-0.08187	0.052947	0.076646	-0.03159	0.015007	-0.04634	-0.02121	0.059264	0.053919	-0.00341	0.076441	0.07494	0.01491	-0.04989	-0.02562	0.00338	0.03594	-0.00717	-0.08832
x1	0.099009	1	-0.22493	0.044117	0.706896	0.836397	0.547873	-0.14271	0.001766	0.30435	0.379591	0.07619	-0.05413	-0.10423	-0.14547	-0.13241	-0.04399	-0.01546	-0.05396	-0.05957	0.09762	-0.7545
x2	0.03269	-0.22493	1	0.118254	-0.26711	-0.23519	-0.26677	0.03561	-0.0943	-0.10544	-0.04189	-0.11941	0.09865	0.093787	0.142302	0.176342	0.043949	0.047061	0.109557	0.118898	-0.07857	0.33493
x3	-0.08187	0.044117	0.118254	1	-0.05436	-0.025	-0.06068	0.064263	-0.04882	-0.04016	0.129762	0.008999	-0.04134	9.64E-05	0.046909	0.075744	0.024732	-0.00566	-0.01007	0.111751	-0.03526	-0.00718
x4	0.052947	0.706896	-0.26711	-0.05436	1	0.8402	0.758062	-0.1437	-0.06192	0.233008	0.284292	0.115462	-0.05504	-0.18104	-0.3221	-0.27742	-0.07297	-0.05116	-0.04867	-0.03098	0.304392	-0.5365
x5	0.076646	0.836397	-0.23519	-0.025	0.8402	1	0.745113	-0.11523	-0.00354	0.240066	0.302868	0.128584	-0.08219	-0.09836	-0.18168	-0.20932	-0.02318	-0.01503	-0.01541	-0.02346	0.191937	-0.65411
x6	-0.03159	0.547873	-0.26677	-0.06068	0.758062	0.745113	1	-0.08608	0.044776	0.07266	0.133924	0.141047	-0.08335	-0.12675	-0.20006	-0.18181	0.033849	0.008288	-0.05525	-0.04786	0.265309	-0.43871
x7	0.015007	-0.14271	0.03561	0.064263	-0.1437	-0.11523	-0.08608	1	-0.16347	-0.10985	-0.21368	-0.07867	-0.02567	-0.09294	-0.06244	0.058824	0.043206	-0.07641	-0.12348	0.029822	-0.1463	0.040043
x8	-0.04634	0.001766	-0.0943	-0.04882	-0.06192	-0.00354	0.044776	-0.16347	1	0.27672	-0.07753	0.051884	-0.01577	0.150038	0.135724	0.065495	0.124934	0.013174	0.164057	0.082776	3.27E-21	-0.02722
x9	-0.02121	0.30435	-0.10544	-0.04016	0.233008	0.240066	0.07266	-0.10985	0.27672	1	-0.02866	0.09432	-0.06751	0.024284	-0.03951	-0.01865	-0.01153	-0.07201	0.082975	0.145892	0.068768	-0.12486
x10	0.059264	0.379591	-0.04189	0.129762	0.284292	0.302868	0.133924	-0.21368	-0.07753	-0.02866	1	-0.03574	0.064821	-0.08102	-0.03682	0.000902	0.023005	0.04549	0.04684	-0.00364	-0.00718	-0.41145
x11	0.053919	0.07619	-0.11941	0.008999	0.115462	0.128584	0.141047	-0.07867	0.051884	0.09432	-0.03574	1	0.017098	-0.0015	0.002937	0.003736	0.08196	0.084707	0.07472	-0.0376	0.078966	-0.0721
x12	-0.00341	-0.05413	0.09865	-0.04134	-0.05504	-0.08219	-0.08335	-0.02567	-0.01577	-0.06751	0.064821	0.017098	1	0.173883	0.22368	0.277487	0.176519	0.126632	0.129048	0.004752	-0.13477	-0.05022
x13	0.076441	-0.10423	0.093787	9.64E-05	-0.18104	-0.09836	-0.12675	-0.09294	0.150038	0.024284	-0.08102	-0.0015	0.173883	1	0.596196	0.450427	0.084997	0.062102	0.336177	0.301521	0.016293	0.034648
x14	0.07494	-0.14547	0.142302	0.046909	-0.3221	-0.18168	-0.20006	-0.06244	0.135724	-0.03951	-0.03682	0.002937	0.22368	0.596196	1	0.466761	0.171442	0.140717	0.218334	0.100602	-0.08771	0.066185
x15	0.01491	-0.13241	0.176342	0.075744	-0.27742	-0.20932	-0.18181	0.058824	0.065495	-0.01865	0.000902	0.003736	0.277487	0.450427	0.466761	1	0.404269	0.197417	0.200366	0.346201	-0.13767	0.083034
x16	-0.04989	-0.04399	0.043949	0.024732	-0.07297	-0.02318	0.033849	0.043206	0.124934	-0.01153	0.023005	0.08196	0.176519	0.084997	0.171442	0.404269	1	0.430241	0.222033	0.187002	-0.09923	-0.04128
x17	-0.02562	-0.01546	0.047061	-0.00566	-0.05116	-0.01503	0.008288	-0.07641	0.013174	-0.07201	0.04549	0.084707	0.126632	0.062102	0.140717	0.197417	0.430241	1	0.053895	0.047705	0.019042	-0.09463
x18	0.00338	-0.05396	0.109557	-0.01007	-0.04867	-0.01541	-0.05525	-0.12348	0.164057	0.082975	0.04684	0.07472	0.129048	0.336177	0.218334	0.200366	0.222033	0.053895	1	0.096304	0.009564	0.021263
x19	0.03594	-0.05957	0.118898	0.111751	-0.03098	-0.02346	-0.04786	0.029822	0.082776	0.145892	-0.00364	-0.0376	0.004752	0.301521	0.100602	0.346201	0.187002	0.047705	0.096304	1	-0.04241	0.037809
x20	-0.00717	0.09762	-0.07857	-0.03526	0.304392	0.191937	0.265309	-0.1463	3.27E-21	0.068768	-0.00718	0.078966	-0.13477	0.016293	-0.08771	-0.13767	-0.09923	0.019042	0.009564	-0.04241	1	0.025198
x21	-0.08832	-0.7545	0.33493	-0.00718	-0.5365	-0.65411	-0.43871	0.040043	-0.02722	-0.12486	-0.41145	-0.0721	-0.05022	0.034648	0.066185	0.083034	-0.04128	-0.09463	0.021263	0.037809	0.025198	1

As well as testing the presence of outliers through the method ($mean_{+}^{-} 3\sigma$) in variables real data.

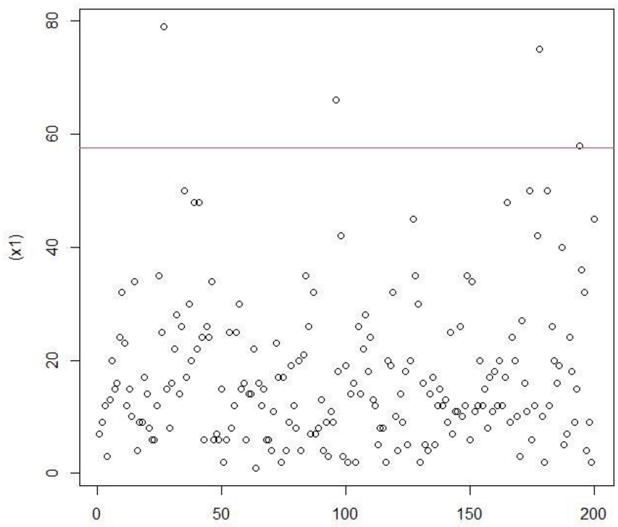

Figure-1: Test for the presence of outliers in X_1

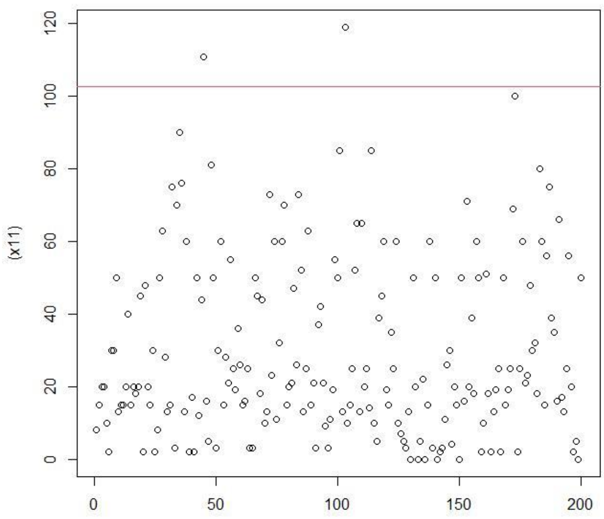

Figure-2: Test for the presence of outliers inX_{11}

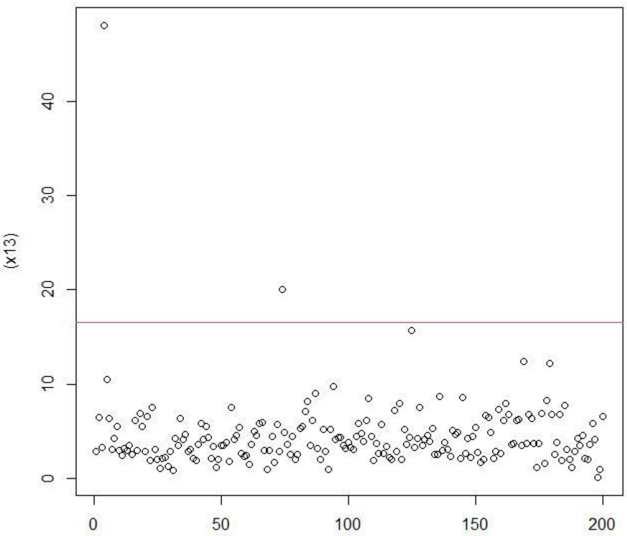

Figure-3: Test for the presence of outliers inX_{13}

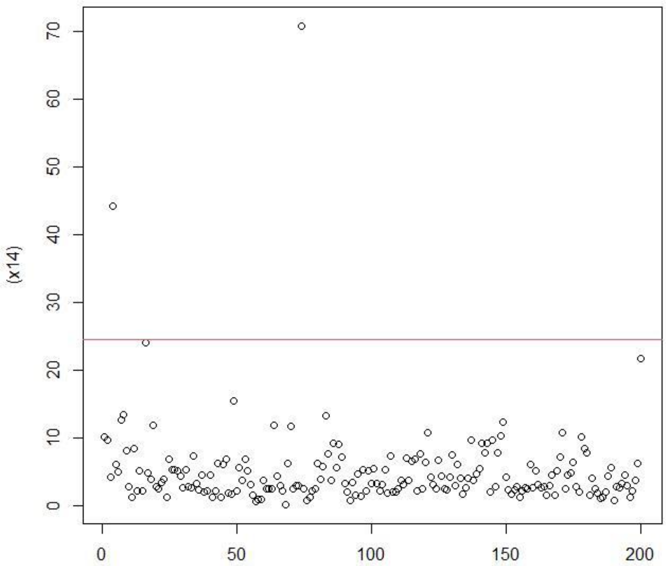

Figure-4: Test for the presence of outliers inX_{14}

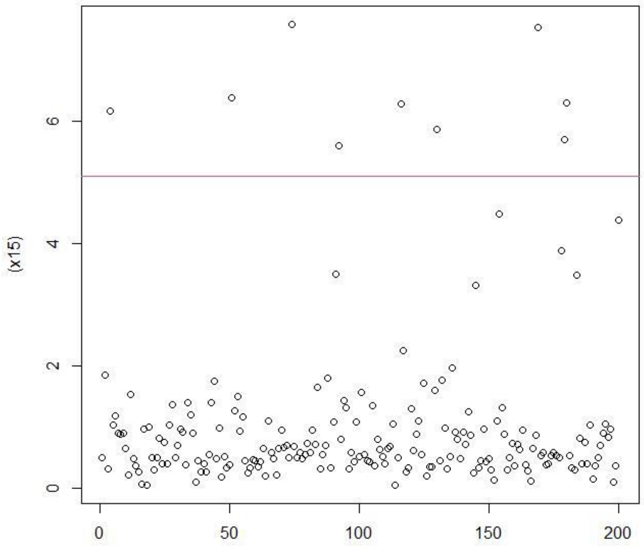

Figure-5: Test for the presence of outliers inX_{15}

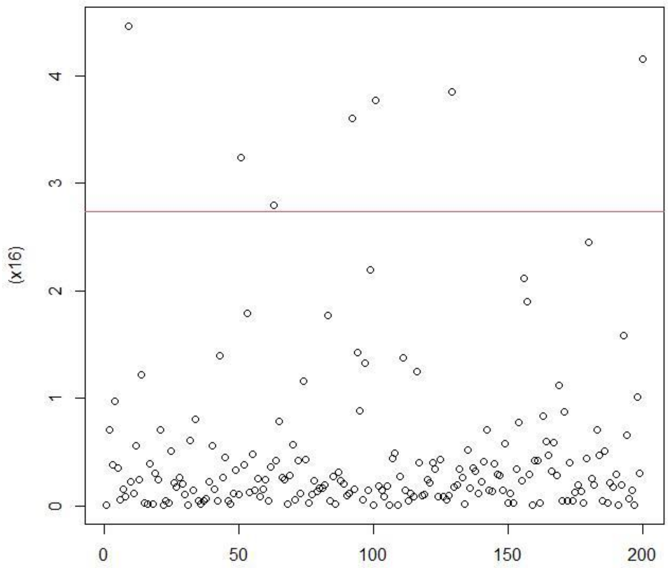

Figure-6: Test for the presence of outliers inX_{16}

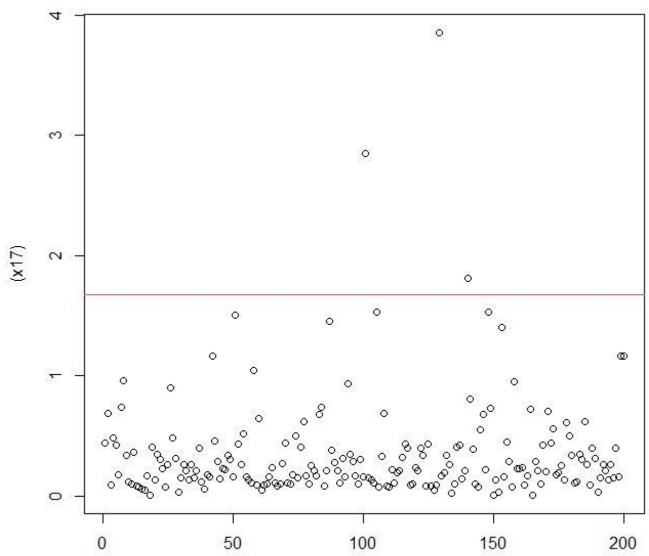

Figure-7: Test for the presence of outliers inX_{17}

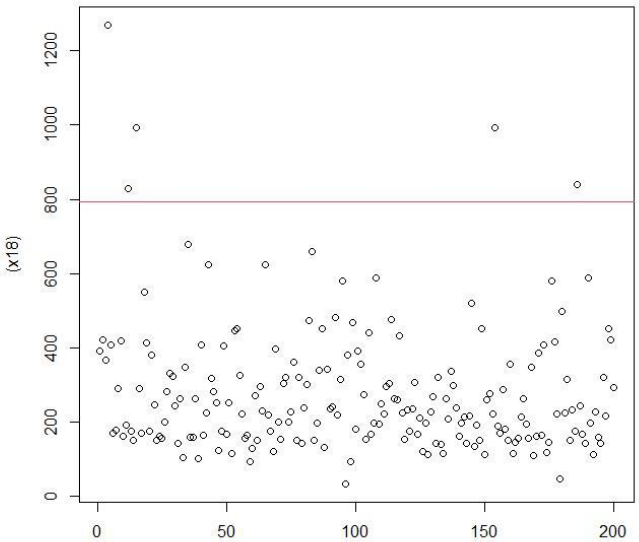

Figure-8: Test for the presence of outliers inX_{18}

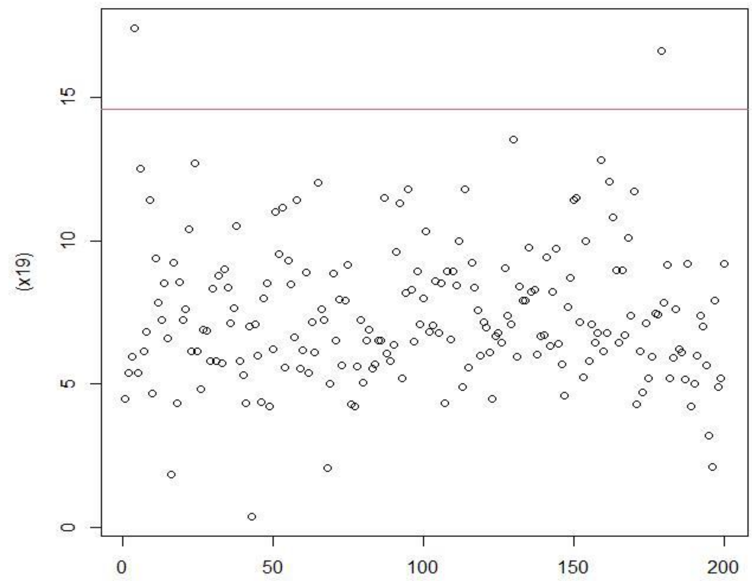

Figure-9: Test for the presence of outliers inX_{19}

6. Conclusion

In this article, the simulations results and the real data analysis show that the RSSIR-EN has best performance than SSIR-EN and RSSIR-Lasso when the outliers exist in Y and X in terms the estimation accuracy and variable selection. Also, the RSSIR-EN gives very close results to SSIR-EN when there are no outliers in (Dist.1). RSSIR-EN method is proposed. It is a robust variable selection method under SDR settings. Computationally, Simulations and real data analysis showed that the RSSIR-EN has favorable predictive accuracy.

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