Review of the Xgamma distribution and its derivative forms

Mohammed H. AL- Sharoot[1] and Karrar R. AL-Sayeab[2]

[1] Professor Doctor of Statistic at the Faculty of Administration and Economics of Al-Qadisiyah University, Iraq.
Email: mohammed.alsharoot@qu.edu.iq
[2] Master Student of the Faculty of Administration and Economics of Al-Qadisiyah University, Iraq.
Email: karrar.alsayeab@jmu.edu.iq

Abstract

This paper reviews some of the mathematical and survival properties of the probability distribution resulting from mixing the exponential and gamma distributions with specific weights, which was derived by the researcher (Subhradev Sen et al.) in 2016, who suggested the name of the Xgamma (XG) distribution, and the most important studies and research that Written in the development of this distribution, such as the study of the weighted xgamma distribution (WXGD)) as a generalization of the Xgamma distribution, and the log-xgamma distribution, which is the logarithmic form of the distribution, is reviewed, and finally the inverse distribution IXG that is derived from the XG distribution is mentioned.

Then we go to review the two-parameter shapes derived from this distribution, such as the quasi xgamma distribution, and the two parameters Xgamma distribution (TPXGD) as a generalization of the original distribution after adding another parameter

Finally, three new proposed forms of the Xgamma distribution are mentioned, which were studied by the researcher, namely, Xgamma1, Xgamma2, and Xgamma3, which provided better results in estimating the survival function than the original distribution.

Keywords: Xgamma distribution, weighted Xgamma distribution, log-xgamma distribution, Inverse Xgamma distribution, quasi xgamma, two parameters Xgamma distribution, Xgamma1 distribution, Xgamma2 distribution, Xgamma3 distribution.

INTRODUCTION

In the previous period, many researchers proposed and developed many statistical models to provide a greater degree of accuracy in dealing with mixed data. Several new models were proposed by mixing two previous probabilistic models, and by using the specific mixture density equation, a new distribution was produced that was more effective in dealing with complex data.

Researcher Subhradev Sen et al. In 2016, a new probability distribution with one parameter, which mixed the exponential distribution and the gamma distribution with specific weights, proposed the name Xgamma distribution (XGD). Then derive the mathematical and survival properties for it. He estimated the distribution coefficient. Then he compared the new distribution to the exponential distribution by analyzing a realistic data set and found that the Xgamma distribution provides a better fit to the data than the exponential distribution(Sen et al., 2016). Then, a group of researchers developed this distribution in several ways, some of which we will discuss, namely, in 2017, Supradev Sen et al. studied Weighted XGD (WXGD)(Sen et al., 2017). In 2018, Altun, Imrah, and J.J. New distribution with one parameter called "log-xgamma" (Altun & Hamedani, 2018). In (2019) Yadav et al. They derived a new probability distribution, which was the inverted Xgamma (IXG) distribution (Yadav et al., 2019).

The development did not stoped on this, but a second parameter was added to the distribution by some researchers to produce the distributions that will be mentioned.

In 2017 Subhradev Sen, along with others, combined an additional parameter of the xgamma distribution with one parameter and named it the quasi-xgamma distribution as a generalization or extension of the xgamma distribution with one parameter (Sen & Chandra, 2017). In 2018, the researcher (S.Sen) generalized the Xgamma distribution by adding a parameter, deriving the two-parameter Xgamma distribution, which added additional flexibility in modeling real-life data (Sen et al., 2018). Finally, we derive three new forms for this distribution by changing the original shapes of the mixed distributions and suggesting the name of the Xgamma1 distribution, and by changing the mixing weights we got each new one named Xgamma2, and by applying the two changes together, changing the shapes of the mixed distributions and changing the mixing weights, we got the third form, Xgamma3.

1. Xgamma distribution:

Sen et al., 2016 proposed a special mixture of exponential (θ) and gamma (3,θ) distributions, denoted XGD, an assuming that the two parameters of mixing are $\pi_1 = \frac{\theta}{1+\theta}, \quad \pi_2 = 1-\pi_1 = \frac{1}{1+\theta}$. He derived the survival function, as well as inferences of the XGD (Elshahhat & Elemary, 2021).

T is a random variable then the XGD is define as follow:

$$f\ (\mathrm{t}) = \pi_1 f_1(t;\theta) + \pi_2 f_2(t;\theta)$$

$$f\ (\mathrm{t}) = \frac{\theta}{1+\theta}\left(\theta e^{-\theta t}\right) + \frac{1}{1+\theta}\left(\frac{\theta^3 t^2}{\rceil 3} e^{-\theta t}\right)$$

$$f\ (\mathrm{t};\theta) = \frac{\theta^2}{(1+\theta)}\left(1+\frac{\theta}{2}t^2\right)e^{-\theta t} \quad ,t>0,\theta>0 \qquad (1)$$

This is represented by T ~ xgamma (θ).

The cumulative density function (CDF) of T is given by

$$F(\mathrm{t}) = 1 - \frac{\left(1+\theta+\theta t+\frac{\theta^2 t^2}{2}\right)}{(1+\theta)} e^{-\theta t} \quad , t>0, \theta>0 \tag{2}$$

The survival function S(t) and the hazard function h(t) are defined a follow

$$S(t) = \frac{\left(1+\theta+\theta t+\frac{\theta^2 t^2}{2}\right)}{(1+\theta)} e^{-\theta t} \quad , t>0, \theta>0 \tag{3}$$

$$h(t) = \frac{\left(1+\frac{\theta t^2}{2}\right)\theta^2}{(1+\theta+\theta t+\frac{\theta^2 t^2}{2})} \tag{4}$$

The moments and the variance are defined a follow:

$$\mu'_r = \frac{r![2\theta+(\mathrm{r}+1)(\mathrm{r}+2)]}{2\theta^r(1+\theta)} \qquad for \quad r=1,2,3,... \tag{5}$$

$$\mu'_1 = \frac{(\theta+3)}{\theta(1+\theta)} \tag{6}$$

$$\mathrm{var(t)} = \frac{\theta^2+8\theta+3}{\theta^2(1+\theta)^2} \tag{7}$$

2. The weighted Xgamma distribution (WXGD):

Supradev Sen and others in 2017 studied the weighted XGD (WXGD), then the length-biased version of the XGD (LBXG) was obtained as a special case of the density function of the XGD distribution. WXGD. Distribution characteristics (LBXG) and survival characteristics were studied. Finally, the (LBXG) distribution was compared to other distributions by applying stress lifetime data for 23 deep groove ball bearings to establish that (LBXG) was the best among the rest of the compared models (Sen et al., 2017).

T is a random variable then the WXGD is define as follow:

$$f\,(\mathrm{t};\theta) = \frac{2\theta^{r+2}}{\mathrm{r}![2\theta+(1+r)(2+r)]}\left(t^r+\frac{\theta}{2}t^{r+2}\right)e^{-\theta t} \quad , t>0, \theta>0, r=1,2,3,... \tag{8}$$

The cumulative density function (CDF) and The survival function S(t) and the hazard function h(t) and The moments are defined a follow:

$$F(t)=\frac{2\theta}{r![2\theta+(1+r)(2+r)]}\left[\gamma(r+1,\theta t)+\frac{1}{2\theta}\gamma(r+3,\theta t)\right] \quad (9)$$

Where $\gamma(a,x)=\int_0^x u^{a-1}e^{-u}du$ is the lower incomplete gamma function.

$$S(t)=\frac{2\theta}{r![2\theta+(1+r)(2+r)]}\left[\Gamma(r+1,\theta t)+\frac{1}{2\theta}\Gamma(r+3,\theta t)\right] \quad (10)$$

$$h(t)=\frac{\theta^{r+1}\left(t^r+\frac{\theta t^{r+2}}{2}\right)e^{-\theta t}}{\left[\Gamma(r+1,\theta t)+\frac{1}{2\theta}\Gamma(r+3,\theta t)\right]} \quad (11)$$

$$\mu'_1=\frac{(r+1)![2\theta+(r+2)(r+3)]}{r!\theta^r[2\theta+(1+\theta)(2+r)]} \quad (12)$$

3. The Log-Xgamma distribution

Alton, Imrah, and J.J in 2018 Derive A new distribution with one parameter called "log-xgamma" by replacing the random variable of the Xgamma distribution with a log-normal function of the random variable. Two sets of real data of the log-xgamma distribution were analyzed in comparison to the Beta, Kumaraswamy and Topp-Leone distributions, and it was concluded that the log-xgamma distribution provided a better fit to the real data (Altun & Hamedani, 2018).

T is a random variable then the Log-XGD is define as follow:

$$f(y;\theta)=\frac{\theta^2}{(1+\theta)}\left(1+\frac{\theta}{2}\ln(y)^2\right)y^{\theta-1} \quad ,0<y<1 \quad (13)$$

Where $y=e^{-x}$

The cumulative density function (CDF) and The survival function S(t) and the hazard function h(t) and The moments the variance are defined a follow:

$$F(y)=y^{\theta}(\theta+1)^{-1}\left[1+\theta-\theta\ln y+\frac{\theta^2\ln(y)^2}{2}\right] \quad (14)$$

$$S(t)=1-t^{\theta}(\theta+1)^{-1}\left(1+\theta-\theta\ln(t)+\frac{\theta^2\ln(t)^2}{2}\right) \quad (15)$$

$$h(t)=\frac{\theta^2\left(1+\frac{\theta}{2}\ln(t)^2\right)t^{\theta-1}}{(1+\theta)\left[1-t^{\theta}(\theta+1)^{-1}\left(1+\theta-\theta\ln(t)+\frac{\theta^2\ln(t)^2}{2}\right)\right]} \tag{16}$$

$$\mu'_1=\frac{\theta^2(\theta^2+3\theta+1)}{(1+\theta)^4} \tag{17}$$

$$\text{var(t)}=\frac{\theta^2(\theta^2+5\theta+4)}{(2+\theta)^3(1+\theta)}-\frac{\theta^4(\theta^2+3\theta+1)^2}{(1+\theta)^8} \tag{18}$$

4. The inverse Xgamma distribution (IXG)

Yadav et al. In (2019) derived a new probability distribution, which was the inverted Xgamma (IXG) distribution. Survival characteristics, reversible moments, reversible life expectancy and ordered statistics of the proposed distribution were studied. A data set of failure times (in minutes) for a sample of 15 electronic components was used to demonstrate the real-life applicability of IXGD and to compare it with other inverse distributions. It was concluded that the IXG distribution is the best fit to match the real data used on the basis of comparison criteria (AIC, BIC) (Yadav et al., 2019).

Y is a random variable then the IXGD is define as follow:

$$f\,(y\,;\theta)=\frac{\theta^2}{(1+\theta)\,\text{y}^2}\left(1+\frac{\theta}{2y^2}\right)e^{-\theta/y} \quad ,\text{where}\, y>0, \quad ,\theta>0 \tag{19}$$

Where $y=\frac{1}{x}$

The cumulative density function (CDF) and The survival function S(t) and the hazard function h(t) and The moments are defined a follow:

$$F(y)=\left[1+\frac{\theta}{(1+\theta)\,\text{y}}+\frac{\theta^2}{2(1+\theta)x^2}\right]e^{-\theta/y} \tag{20}$$

$$S(t)=1-\left(\left[1+\frac{\theta}{(1+\theta)t}+\frac{\theta^2}{2(1+\theta)t^2}\right]e^{-\theta/t}\right) \tag{21}$$

$$h(t)=\frac{\left[\frac{\theta^2}{(1+\theta)t^2}\left(1+\frac{\theta}{2t^2}\right)e^{-\theta/t}\right]}{1-\left[\left(1+\frac{\theta}{(1+\theta)t}+\frac{\theta^2}{2(1+\theta)t^2}\right)e^{-\theta/t}\right]} \tag{22}$$

$$\mu'_1 = \frac{\theta + 3}{\theta(1+\theta)} \tag{23}$$

5. The quasi Xgamma distribution (QXGD)

In 2017 Subhradev Sen, along with others, combined an additional parameter of the xgamma distribution with one parameter and named it the quasi-xgamma distribution as a generalization or extension of the xgamma distribution with one parameter. Various mathematical and statistical properties have been studied, along with some important survival properties of the distribution. Estimation of model parameters using the greatest probability method and the moments method. The distribution was compared with some other distributions by application to an unsupervised data set of resting times (in months) of a random sample of 128 patients with bladder cancer, and one generalized model of the Xgamma distribution seemed to provide a better fit than the other. models taking into account the real data set (Sen & Chandra, 2017).

T is a random variable then the QXGD is define as follow:

$$f\,(t;\theta,\alpha) = \frac{\theta}{(1+\alpha)}\left(\alpha + \frac{\theta^2}{2}t^2\right)e^{-\theta t} \quad ,\mathrm{t} > 0, \quad ,\alpha,\theta > 0 \tag{24}$$

The cumulative density function (CDF) and The survival function S(t) and the hazard function h(t) and The moments are defined a follow:

$$F(t) = 1 - \frac{\left(1+\alpha+\theta\alpha+\frac{\theta^2 t^2}{2}\right)}{(1+\alpha)}e^{-\theta t} \tag{25}$$

$$S\,(t) = \frac{\left(1+\alpha+\theta\alpha+\frac{\theta^2 t^2}{2}\right)}{(1+\alpha)}e^{-\theta t} \tag{26}$$

$$h(t) = \frac{\theta\left(\alpha + \frac{1}{2}\theta^2 t^2\right)}{\left(1+\alpha+\theta t+\frac{\theta^2 t^2}{2}\right)} \tag{27}$$

$$\mu'_1 = \frac{\alpha + 3}{\theta(1+\alpha)} \tag{28}$$

6. The two parameters Xgamma distribution (TPXGD)

(S.Sen) In 2018 generalized the Xgamma distribution by adding a parameter, and derived the two-parameter Xgamma distribution, which added additional flexibility in modeling real-life data. To compare methods in the estimation process. Two sets of data were analyzed, they studied the failure times of 18 electronic devices and a data set of lifetimes of 50 devices, and compared the proposed model with some other two-parameter lifetime models (Sen et al., 2018).

T is a random variable then the TPXGD is define as follow:

$$f(t;\theta,\alpha)=\frac{\theta^2}{(\theta+\alpha)}\left(1+\frac{\alpha\theta^2}{2}t^2\right)e^{-\theta t} \quad ,\mathrm{t}>0, \quad ,\alpha,\theta>0 \tag{29}$$

The cumulative density function (CDF) and The survival function S(t) and the hazard function h(t) and The moments are defined a follow:

$$F(t)=1-\frac{\left(\alpha+\theta+\alpha\theta t+\frac{\alpha\theta^2t^2}{2}\right)}{(\theta+\alpha)}e^{-\theta t} \tag{30}$$

$$S(t)=\frac{\left(\theta+\alpha+\theta\alpha t+\frac{\alpha\theta^2t^2}{2}\right)}{(\theta+\alpha)}e^{-\theta t} \tag{31}$$

$$h(t)=\frac{\theta^2\left(1+\frac{1}{2}\alpha\theta t^2\right)}{\left(\theta+\alpha+\alpha\theta t+\frac{\alpha\theta^2t^2}{2}\right)} \tag{32}$$

$$\mu'_1=\frac{\theta+3\alpha}{\theta(\theta+\alpha)} \tag{33}$$

7. Xgamma1 distribution (XG1D):

In a previous search we suggest a new formula for Xgamma distribution, this is the new formula proposed by us for the Xgamma distribution, we refer to it as the Xgamma1 distribution, and denoted as XG1D. Suppose that we have gamma distribution with parameter (3, $\lambda=1/\theta$) and exponential distribution with parameter ($\lambda=1/\theta$) and let the mixing equation is:

$$f(t;\theta)=\pi_1 f_1(t)+\pi_2 f_2(t)$$

Where

$$\pi_1=\frac{\theta}{1+\theta}, \quad and \quad \pi_2=1-\pi_1=\frac{1}{1+\theta}$$

$$f_1(t) = \frac{1}{\theta} e^{-\frac{t}{\theta}} \quad and \quad f_2(\text{t}) = \frac{t^2}{\theta^3 \Gamma 3} e^{-\frac{t}{\theta}}$$ then:

$$f\ (t;\theta) = \frac{\theta}{1+\theta}\left(\frac{1}{\theta} e^{-\frac{t}{\theta}}\right) + \frac{1}{1+\theta}\left(\frac{t^2}{\theta^3 \Gamma 3} e^{-\frac{t}{\theta}}\right)$$

$$f\ (t;\theta) = \frac{1}{1+\theta} e^{-\frac{t}{\theta}}\left(1+\frac{t^2}{2\theta^3}\right), \quad t,\theta > 0 \tag{34}$$

This is represented by T ~ xgamma1 (θ), and is denoted a XG1D.

The corresponding cumulative distribution function (C.D.F) of the XG1 is:

$$F(t;\theta) = 1 - e^{-\frac{t}{\theta}}\left(1+\frac{(2\theta+t)t}{2\theta^2(1+\theta)}\right), \quad t,\theta > 0 \tag{35}$$

The survival S(t) function and hazard function h(t) are given by

$$S(t) = \left(1+\frac{t(2\theta+\text{t})}{2\theta^2(1+\theta)}\right) e^{-\frac{t}{\theta}} \quad ,t > 0, \theta > 0 \tag{36}$$

$$h(t) = \frac{2\theta^3 + t^2}{2\theta^3(1+\theta) + \theta t(2\theta + t)} \tag{37}$$

The rth moments of x about zero is:

$$\mu_r' = \frac{r!\theta^r}{(1+\theta)}\left(\theta + \frac{(\text{r}+2)(\text{r}+1)}{2}\right) \quad for \quad r = 1,2,... \tag{38}$$

The mean and the variance are defined a follow:

$$\mu_1' = \frac{\theta(3+\theta)}{(1+\theta)} \tag{39}$$

$$\text{var}(x) = \frac{\theta^2(\theta^2+8\theta+3)}{(1+\theta)^2} \tag{40}$$

8. Xgamma2 distribution (XG2D):

In another search we propose a new form for the Xgamma distribution, by mixing the exponential distribution with the parameter (θ) and the gamma distribution with the two parameters (3,θ) and switching the mixing weights in the mixing equation for the XG distribution, one in place of the other, to become $\pi_1 = \frac{1}{1+\theta}$ and $\pi_2 = 1 - \pi_1 = \frac{\theta}{1+\theta}$; Let's get the proposed new format as follows:

$$f(t;\theta) = \pi_1 f_1(t) + \pi_2 f_2(t)$$

Where

$$\pi_1 = \frac{1}{1+\theta}, \quad and \quad \pi_2 = 1 - \pi_1 = \frac{\theta}{1+\theta}$$

$$f_1(t) = \theta e^{-\theta t} \quad and \quad f_2(\text{t}) = \frac{t^2\theta^3}{\Gamma 3} e^{-\theta t}$$ then:

$$f(t;\theta) = \frac{1}{1+\theta}\left(\frac{1}{\theta} e^{-\theta t}\right) + \frac{\theta}{1+\theta}\left(\frac{t^2\theta^3}{\Gamma 3} e^{-\theta t}\right)$$

$$f(\text{t}) = \frac{\theta}{1+\theta} e^{-\theta t}\left(1 + \frac{\theta^3 t^2}{2}\right), \quad t > 0, \quad \theta > 0 \tag{41}$$

This is represented by T ~ Xgamma2 (θ), and is denoted a XG2D.

The corresponding cumulative distribution function (C.D.F) of the XG2D is:

$$F(t;\theta) = 1 - \frac{e^{-\theta t}}{(1+\theta)}\left[1 + \frac{t^2\theta^3}{2} + t\theta^2 + \theta\right] \tag{42}$$

The survival S(t) function and hazard function h(t) are given by

$$S(t) = \frac{e^{-\theta t}}{(1+\theta)}\left(1 + \theta + \theta^2 t + \frac{\theta^3 t^2}{2}\right) \quad , t > 0, \theta > 0 \tag{43}$$

$$h(t) = \frac{\theta(1 + \frac{\theta^3 t^2}{2})}{1 + \theta + \theta^2 t + \frac{\theta^3 t^2}{2}} \tag{44}$$

The mean and the variance are defined a follow:

$$\mu_1' = \frac{(1+3\theta)}{\theta(1+\theta)} \tag{45}$$

$$\mathrm{var}(x) = \frac{(3\theta^2 + 8\theta + 1)}{\theta^2 (1+\theta)^2} \qquad (46)$$

8. Xgamma3 distribution (XG3D):

The new formula we propose in another paper referred to as the Xgamma3 distribution and denoted as XG3D. We call it the new formula the Xgamma3 distribution. We suppose we have exponential (1/θ) and gamma (3,λ=1/θ) distributions, and by making the first weight the place of the second and versa in the mixing equation of the XG distribution, we get the new form of the following mixing equation:

$$f(t;\theta) = \pi_1 f_1(t) + \pi_2 f_2(t)$$

Where

$$\pi_1 = \frac{1}{1+\theta}, \quad and \quad \pi_2 = 1 - \pi_1 = \frac{\theta}{1+\theta}$$

$$f_1(t) = \frac{1}{\theta} e^{-\frac{t}{\theta}} \quad and \quad f_2(\mathrm{t}) = \frac{t^2}{\theta^3 \Gamma 3} e^{-\frac{t}{\theta}}$$ then:

$$f(t;\theta) = \frac{\theta}{1+\theta}\left(\frac{1}{\theta} e^{-\frac{t}{\theta}}\right) + \frac{1}{1+\theta}\left(\frac{t^2}{\theta^3 \Gamma 3} e^{-\frac{t}{\theta}}\right)$$

$$f(\mathrm{t}) = \frac{1}{\theta(1+\theta)} e^{-\frac{t}{\theta}}\left(1 + \frac{t^2}{2\theta}\right), \quad t > 0, \quad \theta > 0 \qquad (47)$$

The corresponding cumulative distribution function (C.D.F) of the XG3D is:

$$F(\mathrm{t}) = 1 - e^{\frac{-t}{\theta}}\left(1 + \frac{(2\theta + t)t}{2\theta(1+\theta)}\right) \qquad (48)$$

The survival S(t) function and hazard function h(t) are given by

$$S(t) = e^{\frac{-t}{\theta}}\left(1 + \frac{(2\theta + t)t}{2\theta(1+\theta)}\right) \qquad (49)$$

$$h(t)=\frac{(2+\frac{t^2}{\theta})}{2\theta(1+\theta)+(2\theta+t)t} \tag{50}$$

The rth moments of x about zero is:

$$\mu_r^{'}=\frac{r!\theta^r}{(1+\theta)}\left(1+\frac{\theta(r+2)(r+1)}{2}\right) \quad for\ r=1,2,... \tag{51}$$

The mean and the variance are defined as follows:

$$\mu_1^{'}=\frac{\theta(1+3\theta)}{(1+\theta)}=Mean(t) \tag{52}$$

$$\text{var(t)}=\frac{\theta^2(3\theta^2+8\theta+1)}{(1+\theta)^2} \tag{53}$$

REFERENCES

Altun, E., & Hamedani, G. (2018). The log-xgamma distribution with inference and application. *Journal de la Société Française de Statistique*, *159*(3), 40-55 .

Elshahhat, A., & Elemary, B. R. (2021). Analysis for Xgamma parameters of life under Type-II adaptive progressively hybrid censoring with applications in engineering and chemistry. *Symmetry*, *13*(11), 2112 .

Sen, S., & Chandra, N. (2017). The quasi xgamma distribution with application in bladder cancer data. *Journal of data science*, *15*(1), 61-76 .

Sen ،S., Chandra, N., & Maiti, S. S. (2017). The weighted xgamma distribution: Properties and application. *Journal of Reliability and Statistical Studies*, 43-58 .

Sen, S., Chandra, N., & Maiti, S. S. (2018). On Properties and Applications of a Two-Parameter Xgamma Distribution. *J. Stat. Theory Appl.*, *17*(4), 674-685 .

Sen, S., Maiti, S. S., & Chandra, N. (2016). The xgamma distribution: statistical properties and application. *Journal of Modern Applied Statistical Methods*, *15*(1), 38 .

Yadav, A. S., Maiti, S. S & ،.Saha, M. (2019). The inverse xgamma distribution: statistical properties and different methods of estimation. *Annals of Data Science*, *8*(2), 275-293 .