Republic of Iraq

Ministry of Higher Education

And Scientific Research

University of Al-Qadisiyah

College of Administration and Economics

Bayesian Estimation For CIR Process With Application To Finance

A thesis submitted to the council of the college of Administration and Economics at the Al-Qadisiyah university, in partial fulfillment of the requirements for the degree of master of science in statistics

By

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1443A.H *Al-Qadisiyah* *2022A.D*

بِسْمِ اللَّهِ الرَّحْمَٰنِ الرَّحِيمِ

﴿ يرفع اللّه الذِين آمنواْ مِنكُم والذِين أُوتُوا
الْعِلم درجاتٍ واللّه بِمَا تعملُونَ خبِير ﴾

صَدقَ اللهُ العَليُّ العَظيـم

(سورة المجادلة ,آية 11)

Dedication

My time as a graduate student is coming to an end and it is time to thank all who have made this thesis possible :
To the soul of who taught me that the world is a struggle and its weapon is science and knowledge … My Father God rest his soul .
To whom you gave me all that she possessed in order to fulfill my dreams , she pushing me forward in order to achieve the desired … My mother.
To my dear Brother and My beloved sisters.
To the companion of the first step and the penultimate step , to that which during the lean years was a rain cloud … My dear Zahra Khaled.
I would like to thank To whom honored me with his supervision of my research memorandum … Dr. Muhannad F. AL-Saadony.

Reyam Abo- AL Hel

Acknowledgements

My deepest thanks to God for providing me the energy, strength, willingness and patience to do this thesis. I would like to express my deep appreciation, sincere thanks and grateful admiration to my supervisor. Prof. Dr. Muhannad F. Al-Saadony for his overseeing, guidance, interests and suggesting the present subject matter and for helping me to present this thesis in a good manner, I am really thankful to his . To who paved the way for science and knowledge all the professors of the statistics department .

Reyam Abo- AL Hell

Abstract

The models of the term structure of interest rates are the hardest part of the modern financial mathematical because of the relative complexity of application techniques. The Cox Ingersoll Ross process has become an important and widely used process, it is a one-factor mathematical formula that depends on a stochastic differential equation in which one factor is a stochastic process, giving a solution known as a stochastic process .This process considered one of the most celebrated models in the financial industry to study and describes the evolution of the term structure of interest rates , determines how the interest rate will develop due to current volatility and the mean spread off rates, then it was the subject of many even recent studies and extensions.

In addition, we present the CIR process in the setting of free probability theory which was introduced by Voicelescu in 1985 which is called Free Cox Ingersoll Ross process (Free CIR).

This thesis has come with the aim of estimating the parameters of CIR and Free CIR process have been used which are the Maximum Likelihood Estimation method (MLE), and Metropolis-hasting method which consider one of the popular types of Markov Chain Monte Carlo (MCMC) method. We use real data Iraq's Stock index (ISX60), the study data were analyzed using programming language R.

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List of Symbols and Abbreviations

Notations	*Name Notations*
(Ω, P, F)	**probability space**
Ω	**Sample space**
F	**σ- algebra**
P	**Probability Measure**
X	**random variable**
X_t	**stochastic process**
t	**time**
T	**set of stochastic process**
W_t	**Brownian motion**
MLE	**Maximum Likelihood Estimation**
B_t	**Free Brownian motion**
SDE	**stochastic differential equation**
inf	**infimum**
∅	**empty set**
(A, φ)	**Non-commutative probability space**
I	**Free Stochastic integral**
V(s)	**Von Neumann algebra-valued**
MCMC	**Markov Chain Monte Carlo**
ISX(60)	**Iraq's Stock Market index**

Chapter one

Introduction, Goal of Thesis , Problem of thesis and Literature Review

Chapter one : Introduction

1.1-Introduction

The short term interest rate is an important topic because it is relevant and useful application to understand interest rate dynamics. We present the popular stochastic differential equation process used in finance is one-factor short term interest rate Cox Ingersoll Ross (CIR) process whereas the one-factor models operate under the assumption that there is only one variable that affects the structure of the interest rate. Indeed, it has been applied in a now widely used in the finance field as modelling the interest rates and stochastic volatility of are common applications. furthermore, it has better model than the former Vasicek framework.

After that, we present the CIR process in different form is called " Free CIR Process" where to apply CIR in free probability theory.

The challenge in this thesis lies is to using CIR process in a free setting , computing the properties of this process represented by expected and variance. Thus, estimate the parameters of CIR and Free CIR process using two methods Maximum Likelihood Estimation (MLE) method and Metropolis- Hasting algorithm which general type of Markov Chain Monte Carlo (MCMC) method.

This thesis is structured as follows : chapter one includes the introduction , problem of thesis and literature review. Chapter Two consists of three sections, section one include CIR process, section two include Free CIR process and section three include parameter estimation. In chapter three ,we present the simulation study and we apply the method of estimation by real data application. Finally, In Chapter Four, we introduce the conclusions and recommendations.

1.2- Problem of Thesis

Our problem is to find the best estimates for the parameters in CIR and Free CIR process. Indeed, this process is considered one of the most complex stochastic process at the current time. This work concerning to estimates these parameters by using Bayesian estimation for its high estimation efficiency.

1.3- Goal of Thesis

This thesis goal is to

- use Bayesian estimators to find parameters estimates for CIR process using Markov Chain Monte Carlo (MCMC), where this process is considered a rather complex process because it contains the square root for a random variable whose values cannot be determined and can take negative values.

- use Bayesian estimators to find parameters estimates for free CIR process using Markov Chain Monte Carlo (MCMC), where this process is considered a rather complex process because it contains the square root for a random variable whose values cannot be determined and can take negative values.

1.4- Literature Review

- (Cox et al., 1985) this study present the term structure of interest rates , the purpose is to describe the evolution of interest rates as a diffusion process , the model is composed of two parts: the drift component, $(\theta_1 - \theta_2 r_t)$ to ensure mean reversion at θ_1 towards the long-run mean value θ_2 .The diffusion term consisting of the random component W(t) Brownian motion , scaled by the standard deviation $\theta_3\sqrt{r_t}$, where θ_3 is the volatility. It was worth noting that, the diffusion process r_t is always positive.

- (John Hull et al.,1990) present one state variable interest rate models of Vasicek and CIR . that can be extended so that they are consistent with both the current term structure of interest rates and the current volatilities of all spot interest rates or the current volatilities of all forward interest rates.

- (John hull et al., 1993) offer a numerical procedure that can be used to structure a wide range of one factor models of the short rate that are jointly Markova and fixed with the initial term structure of interest rates. The aim in this paper to compare differential approximation to the development arbitrage-free model of the term structure .

- (Martin Jacobsen, 2001) show examples for multivariate divisions, that are as for time-reversible with an invariant density that can be determined, or, in other hand of the CIR process, has other a good features, include an agglomeration property and a describe in terms of time-changes of a multivariate Brownian motion with a assured drift and covariance.

- (Peter D. tilley, 2002) explain the CIR model. The results include the extended CIR model and easier construction for it, an effective method of price general interest rate derivatives within this model, and price of bonds by the Laplace transform of functional of the initial process, which is squared Bessel.

- (Patrick Georges, 2003) offers two models : CIR and vasicek of the interest rates that have an introduction into a macro-economic stochastic simulating model (SSM).This paper will aim to use the SSM with alternative term structures of interest rates.

- (Ren – Raw Chen et al., 2003) provide a procedure to estimate multi-factor versions of the CIR model. They estimate the fixed parameters in one , two , and three factor by using the approximation maximum likelihood estimation in a state-space model using data for the U.S. treasury market. As well they estimate the unobservable factors by using A nonlinear Kalman filter. He refers to a three factor model would be able to incorporate random variation in short term interest rates.

- (Sigurd dyrting, 2004) presented the three standard ways to evaluate this function by its representation in terms of a series of gamma functions, by two methods : Analytic approximation and asymptotic expansion. He shows that the gamma series impersonation is accurate over a wide range of parameters but has a runtime that increases proportional to the square root of the non-central parameter. He developed a fourth method for evaluating the upper and lower tails of the non-central distribution based on a Bessel function series representation.

- (M. R. Grasselli et al , 2004) reformulated the CIR model into the chaotic representation ,starting with the squared Gaussian representation of the CIR model, finding a simple expression for the fundamental random variable X_∞, also they derive an explicit form for the nth term of the Wiener chaos expansion of the CIR model, for n = 0, 1, 2,.., at last conclude a new expression for the price of a zero coupon bond which reveals a connection between Gaussian measures and ricatti differential equations.

- (Andrew J. G cairn, 2004) present two model the Vasicek and CIR. He cares about the proofs of the Fundamental Theorem of Asset Pricing by using the partial differential equation (PDE) approach and the martingale

approach more heavily favored today, he also mentioned the comparison between the Vasicek and CIR models.

-(Somnath Chatterjee, January, 2005) the aim of this paper is to improve the term structure of interest rate in the sterling and euro treasury bond markets over the interval 1999-2003.

- (Aurelien alfonsi, 2005) the goal in this thesis is to simulate the CIR process by using many methods including the implicit and explicit , as well as the strong and weak convergence of this methods. He also studies numerically their behavior. At last comparing them with the schemes proposed.

- (Damiano Brigoey al., 2006) show the dynamics of one factor short interest rates. Then they illustrate the no-arbitrage condition for one-factor models and the fundamental notion of market price of risk, as well as present the historical one-factor time-homogeneous models of Vasicek, (CIR), Dothan, and the Exponential Vasicek model. They refer to this models used to be calibrate only to the initial yield curve, without taking into account market volatility structures, and that the calibration can be very poor in many situations.

- (S. zeytun et al. ,2007) study two short rate models CIR and Vasicek, The models are described and then the sensitivity of the models with respect to change in the parameters are studied. At last they give the results for the estimation of the model parameters by using two different methods.

- (Victor Reutenauer et al.,2008) generalize the exact simulation algorithm for one-dimensional solution of SDE.as well as applying Malliavin Calculus for simulating exactly the Greeks. At last they present the CIR process with numerical results.

-(Ashley Frey., 2008) presented two models of interest rate :Vasicek model and CIR model. He suggested different type of estimating the parameters of this models, as well as he show the difficulties in create and implement this model by using Microsoft Excel.

- (Yue – Kuen Kwok ,2008) offers the one factor short rate models and extending to multi-factor models. He suggests the Heath–Jarrow–Morton (HJM) approach of modeling the stochastic movement of the forward rates is discussed. At last , offers the formulation of the forward LIBOR (London-Inter-Bank Offered-Rate) process under the Gaussian HJM framework.

- (Jonathan Aquan – Assee, 2009) study the one- factor CIR model for model interest rates. In this thesis he refers to that by using finite difference techniques boundary behavior serves as a boundary condition and guarantees a uniqueness of solutions if the boundary is attainable, as well as the boundary condition is not needed to guarantee uniqueness if the boundary is non-attainable, The finite difference solution is verified by use of positive numerical approximations.

- (Beata Stehlikova et al.,2009) analyzed solutions for the generalized CIR two- factors model describing clustering of interest rate volatilities. This paper aims to derive an asymptotic expansion of the bond

price with respect to a singular parameter representing the fast scale for the stochastic volatility process.

- (Muhammad Naveed Nazir, 2009) this thesis aims to present analysis of bond price by use one factor short rate model.

- (Emile A. L. J et al., 2010) in this thesis he introduces the CIR , vasicek and Nelson-Siegel models and their advantages and disadvantages , estimation theses model based on monthly Canadian zero-coupon yields from 1986 up until and including 2009. Also he simulation the above models. At last he finding the Nelson-Siegel model outperforms the Vasicek and CIR models.

- (Raju Kumar Mishra, June 2010) introduces the efficiency of the existing FIS-α numerical scheme to simulating the CIR model in context of accuracy and the average CPU time to simulate one path of the CIR interest rate model. He also comparing the CIR model discretization with the FIS-α. At last he studying the behavior of the methods with larger time step size.

- (Olesja Zamovska et al., 2011) shows two models of term structure are CIR model and Vasicek model, also investigates the stationary probability distribution of CIR model with Kolmogorov transition equation as a necessary solution for implementation of the mentioned model into MATLAB environment.

- (Antoon Pelsser et al,, 2012) presents the affine interest rate the Vasicek and the CIR models. Also he gives a characterization of affine term

Structure Models and calibration the Vasicek and CIR models to Euribor interest rate data.

- (Gennady A. Medvedev, 2012) offers the affine term structure of interest rates (CIR model) by using yield curves and forward curves. As well as he is analyzing the one- factor and multi-factor, the yield curves and forward curves by using the duration of the risk-free rate as a temporary variable. also he comparing the result depend on the CIR model and vasicek model.

- (Xiaoxue shan ,2012) presents the mathematical models of interest rate products. He studies different basic one-factor models, and then discover multi factor models. As well as explained the Heath-Jarrow Morton model and the LIBOR Market model. His thesis concluded with a discussion of some modified models that involved stochastic volatility.

- (Steffen Dereich et al ., 2012) explain the strong approximation for the CIR process , as we know that the process do not reach to zero by a positive preserve drift-implicit Euler-type method. They used the p-th mean of the maximum distance between the CIR process and its approximation on a finite time interval. showed that under assumptions on the parameters of the CIR process the proposed method attains, up to a logarithmic term, the convergence of order 1/2.

- (R. Gibson et al ., 2012) present general view of the term structure of interest rates models. Make a comparative between them ,also they offer merits and drawbacks in terms of bond and or interest rate contingent claims continuous time valuation or hedging , parameter estimation and calibration at last they suggest a unified approach for model risk assessment .

- (Lingjiong zhu , 2014) study the CIR process with Hawkes jumps .It can be considered as a generation of the CIR process and the Hawkes process with exponential function.

- (Eugenio Saavedra, 2016) introduces estimation the parameters in the CIR model ,and also he creates the stochastic differential equations of the parameters in this model.

- (R. −V. Arevalo et al. , 2016) presents two method to motivate CIR model, after that they show two techniques to estimate the parameters of this model ,the result is applied to model Euribor interest rate from a real sample.

- (Katarzyna Brzozowska − Rup, January 2017) the purpose of this paper is to use the CIR model to estimate the instantaneous Polish. For estimation the CIR model using a state space model in which estimates of the latent variables and model parameters are obtained by applying an Expectation-Maximization algorithm combined with particle filters (PF). The results show the correctness and attractiveness of the method under consideration..

- (Boru Wang et al. ,2017) the aim in this thesis to compare between short-term interest rate models are vasicek model and CIR model and identify the best model within multiple countries. Based on data from the united states, the United Kingdom, and New Zealand. Also they use General Method of Moments (GMM) for estimating three types of data from the above countries. At last ,the result two models can mimic the interest rate dynamic in long term but these both models cannot prediction the same dynamic movement.

- (Teodor Fredriksson , 2017) the purpose in this paper is to present the CIR model and apply the Radial Basis Function (RBF) approximation way to approximate the Pdf which is found by solving a Fokker-Planck Equation. After that estimates the parameters of this model by using the Maximum Likelihood method, also a numerical error analysis procedure is then implemented in which he change the number of points in the discretization ,the variance in the RBFs and the form of the primary condition.

- (Zan Miao, 2018) presents two short-term interest rate models: the CIR and the Vasıcek models . Numerical simulation for the CIR is performed by using the Euler approximation method and an exact algorithm. By using an ordinary least squares method. Then these method are applied to the vasicek model. After that comparing between those models depend on three-month money market rates.

- (Giuseppe Orlando et al. , 2018) the aim of this paper is to show a new methodology, which named # the CIR process, that fits the term structure of short interest rates so that the market volatility structure is kept as well as the analytical tractability of the original CIR process.

- (Mohamad houda, 2019) provide result on Bernstein processes, which is Brownian diffusions that appearing in Euclidean Quantum Mechanics. They express the distributions of these processes with the help of those of Bessel processes. After that he determine two solutions of the dual equation of the heat equation with potential.

- (Holger Fink, Henry Port et al. ,2019) study the CIR process in the framework of free probability theory. by transforming the classical CIR equation and the Feller condition, the challenge in this thesis is transition from a SDE driven by a classical Brownian motion to a SDE driven by a free Brownian motion.

- (Mohamed ben Alaya et al., 2020) present the CIR process whose drift coefficient based on unknown parameters. This paper uses the conditions of high frequency $\Delta_n \to 0$ and infinite horizon $n\Delta_n \to \infty$ as in the case of ergodic diffusions with globally Lipschitz coefficients, However, in the non-ergodic cases, additional assumptions on the decreasing rate of Δ_n are required due to the fact that the square root diffusion coefficient of the CIR process is not regular enough. Indeed, we assume $n{\Delta_n}^3 \to 0$ for the critical case and $\Delta_n^2\, e^{-b_0 n\Delta_n} \to 0$ for the supercritical case.

- (Henry Alvorado Port, 2020) the paper consists of two parts , in part one he offers free CIR process in the context of free probability theory. By use the existence of a global positive solution for a vector-valued version of the CIR equation. From there, transforms the equation step-by-step into an operator-valued SDE driven by a free Brownian motion. Part wo includes The Impact of sovereign yield curve differentials on value-at-risk forecasts for foreign exchange rates.

Our work in this thesis is to estimate the parameters in stochastic process CIR and Free CIR process using Bayesian estimation.

Chapter Two (Section one : CIR process

Section two: Free CIR process

Section three: Parameter Estimation)

Section One: Cox Ingersoll Ross process (CIR)

2.1- Introduction

This section includes, the theoretical part related to the CIR process , we present the important definitions are stochastic process and Brownian motion. We study the stochastic differential equation , called CIR process driven by Brownian motion with some features as expectation ,variance, auto-covariance, moment, mean reversion , numerical simulation.

2.2- A Stochastic Process

The stochastic process define on a common probability space (Ω, P, F) which is a collection of random variables $\{X_t , t \in T\}$, The index t represents time and the set T is the index set of the process .The stochastic process can be discrete-time stochastic processes as $T = \{0, 1, 2,3,\ldots\}$ if the set T is finite or countable , or Continuous-time processes as $T = [0,\infty)$ if the set T is not finite or uncountable [14].

2.3- Brownian Motion

Brownian Motion is one of the most useful tools in the stochastic models. The discovery of Brownian motion is attributed to the scientist Robert Brown who observed through the microscope the random crowded movement of tiny pollen grains in the water. It was presented by Albert Einstein in 1905s. [47].

A stochastic process $W = \{W(t), t \geq 0\}$ is defined on a probability space (Ω, P, F) is called a Brownian motion if it has the following property [68] :

1- For every $W(0) = 0$.

2- W(t) has independent increments. At every $0 \leq s \leq t$, we have that $W(t) - W(s)$ is independent of every W(s).

3- W(t) has stationary increments; that is for any $0 \leq s \leq t$, we have that W(t) - W(s) =W(t - s).

4- W(t + s) - W(t) it has the Normal distributed with mean 0 and variance

s > 0: W(t + s) - W(t) ~ N(0, s) [68].

2.4- Cox Ingersoll Ross (CIR) driven by Brownian motion

The Cox Ingersoll Ross (CIR) is a type of the one factor models and which assumes the interest rate term structure that is driven by a one dimensional Brownian motion. The most widely used model for short term interest rate modeling and it avoids negative interest rates . It was presented by John Carrington Cox, Jonathan Edwards Ingersoll and Stephen Alan Ross in 1985.The stochastic differential equation from (SDE) is [11] :

$$dr_t = (\theta_1 - \theta_2 r_t)\, dt + \theta_3 \sqrt{r_t}\, dW_t \qquad \ldots.(1)$$

We symbol to σ by θ_3 , where $\theta_1 > 0$, $\theta_2 > 0$, $\theta_3 > 0$ and a standard Brownian motion W_t . The parameters have the following explanations : θ_1 is the speed of mean reversion, θ_2 is the long term average value of the process, θ_3 is the volatility, and r_t is a short rate . If ${\theta_3}^2 > 2\theta_1\theta_2$, r_t can reach zero according to the criteria of the boundary classifications, either in the case $2\theta_1\theta_2 \geq {\theta_3}^2$ the upward drift is sufficiently large to make the origin inaccessible. In both cases , the singularity of the diffusion coefficient at the origin implies that an initially nonnegative interest rate can never subsequently become negative. The interest rate in this process it has the following features [11]:

- r_t is non-negative for all t.
- If the interest rate reaches zero, it can subsequently become positive.
- The interest rate has a steady state distribution.

- If the interest rate increases, the absolute variance of interest rate increases

The probability density of the interest rate at time s, conditional on its value at the present time t by (1) as [11] :

$$f(r(s),s\ ;\ r(t),t)= ce^{-u-v}\left(\frac{v}{u}\right)^{\frac{q}{2}} I_q(2(uv)^{1/2})$$

where

$$c = \frac{2\theta_1}{\theta_3{}^2(1-e^{-\theta_1(t-s)})} \qquad ...(2)$$

$$u = cr_t\, e^{-\theta_1(t-s)} \qquad ...(3)$$

$$v = cr_s \qquad ...(4)$$

$$q = \frac{2\theta_1\theta_2}{\theta_3{}^2} - 1 \qquad ...(5)$$

$I_q(x)$ is the modified Bessel function of the first kind of order q [11] .

$$I_q(x) = \left(\frac{x}{2}\right)^q \sum_{k=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{2k}}{k!\Gamma\,(q+k+1)} \qquad ...(6)$$

The distribution function is the non-central chi-square , with 2q+2 degrees of freedom and parameter of non-centrality 2u proportional to the current spot rate.

2.4.1- Itô formula

The Itô formula is an important tool for understanding and calculating stochastic calculus and is also important in simulations. The formula is a special case of Taylor expansion, this process is the stochastic version of the Taylor expansion which stops at 2nd order where (X) represents a process of

diffusion. The process means that, if represents a twice differentiable function for t and X [16]:

$$dX(t) = \mu(t, x(t))\, dt + \sigma(t, x(t))\, dW(t)$$

where : $\mu \in L^1(0, T)$, $\sigma \in L^2\ (0, T)$

suppose u : R×[0, T] ⟶ ¡ is continuous and that $\frac{\partial u}{\partial t}$, $\frac{\partial u}{x}$, $\frac{\partial^2 u}{\partial x^2}$ exist and are continuous, Set:

$$X(t) = u(x(t),t)$$

Then Y has the stochastic differential [30]:

$$dX = \frac{\partial u}{\partial t}\, dt + \frac{\partial u}{x}\, dx + \frac{1}{2}\, \frac{\partial^2 u}{\partial x^2} \sigma^2 dt$$

$$= \left(\frac{\partial u}{\partial t} + \frac{\partial u}{x}\, \mu + \frac{1}{2}\, \frac{\partial^2 u}{\partial x^2} \sigma^2\right)\, dt + \frac{\partial u}{x}\, \sigma\, dw$$

is named the Itô formula.

The Itô formula has the characterizes as follows :

1. $\forall \alpha, \beta \epsilon \mathbb{R}$

$$\int_0^T (\alpha X(t) + \beta Y(t) dW(t)$$

$$= \alpha \int_0^T X(t) dW(t) + \beta \int_0^T Y(t) dW(t)$$

2.

$$\int_0^T X(t) I_{(a,b]} dt = \int_0^T X(t) dW(t)$$

Moreover , if $\int_0^T E[X^2(t)] dt < \infty$

3. Zero mean property

$$E[\int_0^T X(t)dW(t)]=0$$

4. Isometry property

$$E[(\int_0^T X(t)dW(t))^2]=\int_0^T E[X^2(t)]dt$$

2.4.2- The Expected of the CIR process

The CIR process has the following equation

$$dr_t = (\theta_1 - \theta_2 r_t)\, dt + \theta_3\sqrt{r_t}\, dw_t$$

The unique positive solution to the short rate stochastic differential equation (1) as following [5] [11]:

$$r(t) = \frac{\theta_1}{\theta_2} + (r_0 - \frac{\theta_1}{\theta_2})e^{-\theta_2 t} + \theta_3 e^{-\theta_2 t}\int_0^t e^{\theta_2 s}\sqrt{r(s)}\, dw(s)$$

$$E\,[r(t)] = E[\frac{\theta_1}{\theta_2} + (r_0 - \frac{\theta_1}{\theta_2})e^{-\theta_2 t} + \theta_3 e^{-\theta_2 t}\int_0^t e^{\theta_2 s}\sqrt{r(s)}\, dw(s)]$$

$$= \frac{\theta_1}{\theta_2} + (r_0 - \frac{\theta_1}{\theta_2})e^{-\theta_2 t} + \theta_3 e^{-\theta_2 t}E[\int_0^t e^{\theta_2 s}\sqrt{r(s)}\, dw(s)]$$

$$= \frac{\theta_1}{\theta_2} + (r_0 - \frac{\theta_1}{\theta_2})e^{-\theta_2 t} + \theta_3 e^{-\theta_2 t}\int_0^t e^{\theta_2 s}\sqrt{r(s)}\; E[dw(s)]$$

By using zero property $E\,[\int_0^T X(t)dw(t)] = 0$

$$E[r(t)] = \frac{\theta_1}{\theta_2} + (r_0 - \frac{\theta_1}{\theta_2})e^{-\theta_2 t}$$

2.4.3- The variance of the CIR process

The derivation of the variance of the CIR process can be proved as follows [5] [11] :

$$E\,[r(t) - E(r(t))]^2$$

$$= E\,[\frac{\theta_1}{\theta_2} + (r_0 - \frac{\theta_1}{\theta_2})e^{-\theta_2 t} + \theta_3 e^{-\theta_2 t}\int_0^t e^{\theta_2 s}\sqrt{r(s)}\,dw(s) - \frac{\theta_1}{\theta_2} + (r_0 - \frac{\theta_1}{\theta_2})e^{-\theta_2 t}\quad]^2$$

$$= E\,[\theta_3 e^{-\theta_2 t}\int_0^t e^{\theta_2 s}\sqrt{r(s)}\,dw(s)]^2$$

$$= {\theta_3}^2 e^{-2\theta_2 t}\,E\,[\int_0^t e^{\theta_2 s}\sqrt{r(s)}\,dw(s)]^2$$

= by using the isometry property

$$= {\theta_3}^2 e^{-2\theta_2 t}\int_0^t e^{2\theta_2 s}\,E[r(s)]\,ds$$

$$= {\theta_3}^2 e^{-2\theta_2 t}\int_0^t e^{2\theta_2 s}\,[\frac{\theta_1}{\theta_2} + (r_0 - \frac{\theta_1}{\theta_2})e^{-\theta_2 t}]ds$$

$$= {\theta_3}^2 e^{-2\theta_2 t}\int_0^t\,[e^{2\theta_2 s}\,\frac{\theta_1}{\theta_2} + e^{\theta_2 s}r_0 - e^{\theta_2 s}\,\frac{\theta_1}{\theta_2}]\,ds$$

$$= {\theta_3}^2 e^{-2\theta_2 t}[\frac{e^{\theta_2 t}-1}{\theta_2}r_0 + \frac{e^{2\theta_2 t}-1}{\theta_2}\,\frac{\theta_1}{\theta_2} - \frac{e^{\theta_2 t}-1}{\theta_2}\,\frac{\theta_1}{\theta_2}]$$

$$= {\theta_3}^2 e^{-2\theta_2 t}\,\frac{e^{\theta_2 t}-1}{\theta_2}r_0 + {\theta_3}^2 e^{-2\theta_2 t}\,\frac{e^{2\theta_2 t}-1}{2\theta_2}\,\frac{\theta_1}{\theta_2} - {\theta_3}^2 e^{-2\theta_2 t}\frac{e^{\theta_2 t}-1}{\theta_2}\,\frac{\theta_1}{\theta_2}$$

$$= {\theta_3}^2\left(\frac{e^{-\theta_2 t}-e^{-2\theta_2 t}}{\theta_2}\right)r_0 + \theta_1{\theta_3}^2[\frac{e^{-2\theta_2 t}-e^{-2\theta_2 t}}{2{\theta_2}^2} - \frac{e^{-\theta_2 t}}{\theta_2}]$$

$$= {\theta_3}^2\left(\frac{e^{-\theta_2 t}-e^{-2\theta_2 t}}{\theta_2}\right)r_0 + \frac{\theta_1{\theta_3}^2}{2{\theta_2}^2}[e^{-2\theta_2 t} - e^{-2\theta_2 t} - e^{-\theta_2 t}]$$

$$= {\theta_3}^2\left(\frac{e^{-\theta_2 t}-e^{-2\theta_2 t}}{\theta_2}\right)r_0 + \frac{\theta_1{\theta_3}^2}{2{\theta_2}^2}(1 - e^{-2\theta_2 t})$$

The characteristics of the distribution of the future interest rates are those expected , The mean goes to θ_2 and the variance to zero , if θ_1 approaches infinity .The conditional mean goes to the current interest rate and the variance to $\theta_3{}^2$r(t) *(s - t) if θ_1 approaches zero . If the interest rate does display mean reversion ($\theta_1, \theta_2 > 0$), then as s becomes large its distribution will approaching a gamma distribution. The steady state density function is (1):

$$f\,[r(\infty), \infty; r(t), t\,] = \frac{\omega^v}{\Gamma(v)}\, r^{v-1} e^{-\omega r} \quad \text{, where } \omega = \frac{2\theta_1}{\theta_3{}^2} \quad , v = 2\theta_1 \frac{\theta_2}{\theta_3{}^2}$$

The steady state mean is θ_2 and variance is $\frac{\theta_3{}^2\theta_2}{2\theta_1}$.

2.4.4- Auto covariance of the CIR process

To computing the (auto)-covariance function for the CIR process, we need consider two time instants, t and s, t < s. have for t < s ,that C_X (t, s) is given by : C_X (t, s) = cov (X (t) , X (s)) : [5] [11]

$$COV(X(t),X(s)) = E[r(t) - E(r(t))(r(s) - E(r(s))]$$

$$= \theta_3{}^2 cov(\int_0^t e^{-\theta_2 t}\sqrt{r(t)}\, dW(t)\, ,\int_0^s e^{-\theta_2(s-u)}\sqrt{r(t)}\, dW(t))$$

$$=\theta_3{}^2 cov(\int_0^t e^{-\theta_2(t-u)}\sqrt{r(u)}\, dW(u)\, ,\int_0^t e^{-\theta_2(s-u)}\sqrt{r(u)}\, dW(u) + \int_t^s e^{-\theta_2(s-u)}\sqrt{r(u)}\, dW(u)$$

$$=\theta_3{}^2 cov(\int_0^t e^{-\theta_2(t-u)}\sqrt{r(u)}\, dW(u)\, ,\int_0^t e^{-\theta_2(s-u)}\sqrt{r(u)}\, dW(u))$$

$$=\theta_3{}^2 e^{-\theta_2(t+s)} cov(\int_0^t e^{\theta_2 u}\sqrt{r(u)}\, dW(u), \int_0^t e^{\theta_2 u}\sqrt{r(u)}\, dW(u))$$

$$=\theta_3{}^2 e^{-\theta_2(t+s)} var(\int_0^t e^{\theta_2 u}\sqrt{r(u)}\, dW(u))$$

$$=\theta_3{}^2 e^{-\theta_2(t+s)} \int_0^t e^{2\theta_2 u}\, E\big(r(u)\big) du$$

$$= \theta_3{}^2 e^{-\theta_2(t+s)} \int_0^t e^{2\theta_2 u} [\frac{\theta_1}{\theta_2} + \left(r_0 - \frac{\theta_1}{\theta_2}\right) e^{-\theta_2 u}]du$$

$$= \theta_3{}^2\ e^{-\theta_2(t+s)} \int_0^t e^{2\theta_2 u}\ \frac{\theta_1}{\theta_2} + (e^{2\theta_2 u} r_0 e^{-\theta_2 u} - e^{2\theta_2 u} \frac{\theta_1}{\theta_2} e^{-\theta_2 u})]du$$

$$= \theta_3{}^2 e^{-\theta_2(t+s)} \int_0^t e^{2\theta_2 u}\ \frac{\theta_1}{\theta_2} + (r_0 e^{-\theta_2 u} - e^{-\theta_2 u} \frac{\theta_1}{\theta_2})]du$$

$$= \theta_3{}^2\ e^{-\theta_2(t+s)} (\frac{e^{2\theta_2 t}-1}{\theta_2}\ \frac{\theta_1}{\theta_2} + r_0 (\frac{e^{-\theta_2 t}-1}{\theta_2} - \frac{e^{-\theta_2 t}-1}{\theta_2} \frac{\theta_1}{\theta_2}))$$

$$= \theta_3{}^2\ \frac{e^{-\theta_2 t} - e^{-\theta_2(t+s)}}{\theta_2} r_0 + \theta_3{}^2 \frac{\theta_1}{2\theta_2} (e^{-\theta_2(t-s)} - e^{\theta_2(t+s)})$$

2.4.5- The Moments Of The CIR Process

- We denoted by these symbols to facilitate the solution, as

$$\frac{\theta_1}{\theta_2} + \left(r_0 - \frac{\theta_1}{\theta_2}\right) e^{-\theta_2 t} = X(t)$$

$$\theta_3 e^{-\theta_2 t} = Y(t)$$

$$\int_0^t e^{\theta_2 s} \sqrt{r(s)}\ dw(s) = Z(t)$$

- We apply the law that give the non-central moments from the central moments by the Binomial theorem [32].

$$\mu^n = \sum_{k=0}^{n} \binom{n}{k} \theta^k \mu^{n-k}$$

$$r^n(t) = \sum_{k=0}^{n} \binom{n}{k} X^{n-k}(t) Y^k(t) Z^k(t) \quad \ldots.(7)$$

- After we taking the expectation, we getting that following

$$E[r^n(t)]=E[\sum_{k=0}^{n}\binom{n}{k}X^{n-k}(t)Y^k(t)Z^k(t)] \ldots(8)$$

$$=\sum_{k=0}^{n}\binom{n}{k}X^{n-k}(t)Y^k(t)E[Z^k(t)]$$

- By using the Isometry property for k =2j and k∈ N we get

$$E[Z^k(t)] = (E[Z^2(t)])^j \ \ldots.(9)$$

$$= [E(\int_0^t e^{ks}\sqrt{r(t)}\, dw(t))^2]^j$$

$$= [\frac{1}{2k}\ (e^{2kt}-1)]^j$$

- By using Zero mean property for k =2j+1 and k∈ N we have

$$E[Z^k(t)]= (E[Z^2(t)])^j\ E[Z(t)]$$

$$=[\frac{1}{2k}(e^{2\theta t}-1)]^j * 0 \quad \ldots.(10)$$

$$=0$$

- After that , using the equation (7),(10) at the same time, the equation (11) simplify as

$$E[r^n(t)]=\sum_{k=0}^{\left[\frac{n}{2}\right]}\binom{n}{k}X^{n-2k}(t)Y^{2k}(t)[\frac{1}{2k}(e^{2kt}-1)]^{2k} \quad \ldots(11)$$

Where denotes $(\frac{n}{2})$ the greatest integer less than or equal to $\frac{n}{2}$.

-As we mentioned before**,** the distribution function for r(t) is Non-central Chi-square with 2q+2 degrees of freedom and parameter of non-centrality 2u proportional to the current spot rate **:**

$$f(x) = \sum_{k=0}^{\infty}\frac{e^{(-u-\frac{r(t)}{2})}u^k\,(r(t))^{q+k}}{k!\ 2^{q+1+k}\ \Gamma(q+1+k)} \quad \ldots.(12)$$

- By using the equation (11),(12) at the same time , we get

$$E[r^n(t)] = \int_0^{\infty} r(t)^n \left[\sum_{k=0}^{\infty} \frac{e^{\left(-u-\frac{r(t)}{2}\right)} u^k \, (r(t))^{q+k}}{k!\, 2^{q+1+k} \, \Gamma(q+1+k)}\right] dr.$$

2.4.6- Mean reversion of the CIR Process

We will introduce the most important properties of the CIR process. This means that if the interest rate is greater than the long-term mean ($r_t > \theta_2$), Then the coefficient $\theta_1 > 0$ makes the drift become negative so that the rate is pulled down in the direction of θ_2. In a similar way, If the interest rate is less than the long-term mean ($r_t < \theta_2$), then the coefficient $\theta_1 > 0$ makes the drift term become positive so that the rate will be pulled up in the direction of θ_2. So the parameter θ_2 is called the speed of mean reversion, it should be positive in order to maintain stability around the long-term value θ_2 it called the long term average. The parameter θ_3 is the volatility coefficient. We denote (r_t^x ; $t \geq 0$) the CIR process started from an initial point x and the τ_0^x stopping time, is the first time when the process hit 0, and defined by [62]

$\tau_0^x = \inf\{t \geq 0; r_t^x = 0\}$ with, as usual, inf $(\emptyset) = \infty$.

Suggestion [54]:

1. if $2\theta_1\theta_2 \geq \theta_3^2$, we have $P(\tau_0^x = \infty) = 1$ for all $r_t > 0$

2. if $2\theta_1\theta_2 < \theta_3^2$, we have $P(\tau_0^x < \infty) = 1$ for all $r_t > 0$

We get from The above suggestion that the rate can reach zero if $\theta_3^2 > 2\theta_1\theta_2$, As for If it was $2\theta_1\theta_2 \geq \theta_3^2$, the upward drift is sufficiently large to make the origin inaccessible, this condition is called the Feller condition. In other words, the condition $2\theta_1\theta_2 \geq \theta_3^2$ makes sure that zero is never reached, so that we can grant that r_t remains always positive. In either case, the singularity of the diffusion coefficient at the origin implies that an initially non-negative interest rate can never subsequently become negative. Intuitively, when the interest rate is at a low level (approaches zero), the

volatility term $\theta_3\sqrt{r_t}$ also becomes close to zero. Consequently, when the rate gets close to zero, its evolution becomes dominated by the drift factor, which pushes the rate upwards (towards equilibrium). When the interest rate is high then the volatility is high and this is a desired property [62].

2.4.7- Numerical solution of the CIR process

Numerical solution of the Euler scheme not positive , so we use The Euler approximation method for the CIR process in discrete time to ensure that the solution remains positive [60].

Theorem **:** Let $2\theta_1\theta_2 > \theta_3{}^2$, T > 0. Then, for all $1 \le p < \frac{2\theta_1\theta_2}{\theta_3{}^2}$

there found a constant $K_p > 0$ such that

$$(\mathbf{E}\ \max_{t\in[0,T]} |X_t - \bar{x}_t|^p\ ||)^{1/p} \le K_p \sqrt{|\log(\Delta)|}\,\sqrt{\Delta} \quad , \Delta \in (0,\tfrac{1}{2}]$$

We uses Euler method , The CIR process have the following equation :

$$dr_t = (\theta_1 - \theta_2 r_t)\,dt + \theta_3\sqrt{r_t}\,dW_t \qquad R_0 = r_0 \quad , t \ge 0$$

The Euler method for the approximating of the CIR model (1) by [12]is

$$x_{k+1} = |x_k + (\theta_1 - \theta_2 r_t)\Delta + \theta_3\sqrt{x_k}\,\Delta_k W| \quad , \ K=0,1,2$$

And By use the supposition

$$\frac{2\theta_1\theta_2}{\theta_3{}^2} > 1 + \sqrt{8}\ \max\{\frac{\sqrt{k}}{\theta}\sqrt{16\,p - 1}\,, 16\,p\text{-}2\}$$

We find the next

$$E \quad \max_{k=0,\dots,[T/\Delta]} |x_{k\Delta} - x_k|^{2p} \le C_p\,.\,\Delta^p$$

Where $C_p > 0$ is constant depends only p, θ_1, θ_2, θ_3 , x_0 and T .

scheme Both schemes do not preserve positivity, but satisfy

$$x_{k+1} = x_k + (\theta_1 - \theta_2 r_t)\Delta + \theta_3\sqrt{x_k^+}\Delta_k W$$

while the scheme

$$x_{k+1} = x_k + (\theta_1 - \theta_2 r_t)\Delta + \theta_3\sqrt{|x_k|}\Delta_k W$$

The truncated Euler

$$E \max_{k=0,\dots,[T/\Delta]} |x_{k\Delta} - x_k|^2$$

for $\Delta \to 0$ The Euler schemes have a path wise convergence rate of 1/2 - ε for all ε > 0, we get :

$$\frac{1}{\Delta^{\frac{1}{2}-\varepsilon}} . \max_{k=0,\dots,[T/\Delta]} |X_{k\Delta} - x_k| \to 0$$

for $\Delta \to 0$.

The asymptotic error distribution of these schemes for $2\theta_1\ \theta_2 \geq {\theta_3}^2$ it holds that by [44] :

$$\frac{1}{\sqrt{\Delta}} .\max_{k=0,\dots,[T/\Delta]} |X_{k\Delta} - x_k| \to \max_{t\in[0,T]} |\frac{\theta^2}{\sqrt{8}}|\ \Phi_t \int_0^t \frac{1}{\Phi_s} dB_s$$

for $\Delta \to 0$, where $(B_t)_{t\geq 0}$ is a Brownian motion independent of $(W_t)_{t\geq 0}$ and $(\Phi_t)_{t\geq 0}$ is given by :

$$\Phi_t = \exp(-kt - \frac{\theta^2}{8}\int_0^t \frac{1}{X_s}\ ds + \frac{\theta}{2}\int_0^t \frac{1}{\sqrt{X_s}}\ dW_s\ ) \quad , t \geq 0$$

Unite Two: Free Cox Ingersoll Ross process (Free CIR)

2.5- Introduction

CIR process is considered one of the most important models in finance, especially in the last years . In this unite, we present the CIR process stochastics process with noncommutative random variable in a free probability theory framework which call Free CIR process. Free probability is a recent mathematical theory, studies noncommutative random variables. It was introduced at the beginning of the 1980s by Dan Virgil Voiculescu. The subject was relatively slow in its infancy , but it aroused the interest of many later.

We will be transformation the CIR process driven by Brownian motion with Feller condition which guarantees a positive solution of CIR process to Free CIR process which is CIR process driven by Free Brownian motion ,We will present some main definitions related to Free CIR process before we mention it.

This unite contains the theoretical part related to the Free CIR Process. We study the stochastic differential equation , called CIR process driven by Free Brownian motion which is the focus of our thesis, we explain some important definition such as free probability theory, Free Brownian motion and we mention the semicircle distribution with some properties of it.

2.6- Free Probability Theory

Free probability theory was presented in 1980 by Dan Voiculescu and since than it has been the focus of much attention because of its application with large size random matrices, used to calculate the moments or distribution of noncommutative random variable. It based on noncommutative probability space and freeness, it includes huge mathematical concepts and symbols,

and is difficult for the public reader to understand without the following concepts [63] :-

Definition1 **:** Non-commutative probability space is include (A, φ) where A is unite algebra , φ is Linear function φ : A → C , φ(1) =1. Noncommutative probability space which is the element of A , the numbers $\varphi\,(a_{i(1)}, \ldots, a_{i(k)}$ for random variables $a_1, \ldots, a_m \in A$ are named moments, the non-commutative distribution of $a_1, \ldots, a_m$ which is the set of moments [55].

Definition2 **:** let (A, φ) be a non-commutative probability space and let I be an index set [55].

- The sub algebras $(A_i)_{i\in I}$ is called free if

1. $\varphi(a_1 \cdot \cdot \cdot a_k) = 0$, i=1,…,k , where k is a non-negative integer.
2. $a_j \in A_{i(j)}$, where $i(j) \in I$
3. Any two neighboring indices i_1 and i_{1+1} are not equal $i_1 \neq i_2$, $i_2 \neq i_3, \ldots, i_{k-1} \neq i_k$.

- The random variables$(x_i)_{i\in I}$ are free, if their generated unite sub algebras are free, it means if $(A_i)_{i\in I}$ are free, where, for each $i \in I$, A_i is the sub algebra of A which is generated by x_i.

2.7- Free Stochastic integral

Free stochastic integral is a section of mathematics. It is used as a solution to free stochastic differential equations. Indeed, important free stochastic processes to which free stochastic calculus is applied is free Brownian motion , which is used to model free Brownian motion. The free stochastic integral as follows:

$$I = \int_0^1 a_t(dw_t)b_t$$

Where :

a_t and b_t are operator coefficients , B_t is free Brownian motion , suppose that $a_t$and $b_t$are functions of B_τ, $\tau \leq t$. That is, let $a_t$and b_t belong to the sub algebra B_τ. Assume also that $\max\{\|a_t\| ,\| b_t\|\} \leq C$ for all $t \in [0, 1]$ and that $t \to a_t$ and $t \to b_t$ are continuous mappings in the operator norm. Let t_0, . ., t_n and $\tau_1, \ldots, \tau_n$ be real numbers such that [32]:

$$0 = t_0 \leq t_1 \leq \cdots \leq t_n = 1$$

And,

$$0 \leq t_k \leq t_{k-1}$$

We denote the set of t_0, . ., t_n and $\tau_1, \ldots, \tau_n$ as Δ , let

$$d(\Delta) = \max_{1\leq k\leq n}(t_k - \tau_k)$$

Consider the sum

$$I(\Delta) = \sum_{i=1}^{n} a_{\tau i} (B_{t_i} - B_{t_{i-1}})b_{\tau i}$$

It turns out that as $d(\Delta) \to 0$, the sums $I(\Delta)$ converge in operator norm and the limit does not depend on the choice of t_i and τ_i. The limit is called the free stochastic integral and denoted as $I = \int_0^1 a_t(dw_t)b_t$. An important point in the proof of convergence is that the convergence of sums in the operator norm depends on a free analogue of the Burkholder Gundy martingale inequalities .

Norm Estimates for Free Stochastic Integrals:-

1. Ito isometry

For L_2 norm $||a||_2^2 := \varphi(a\, a^*)$, we get

$$||\int A_t ds_t B_t||_2^2 = \int ||A_t||_2^2 \, ||B_t||_2^2 \, dt$$

It is worth nothing that, for a semicircle S of variance dt, which is free from $\{a, a^*, b, b^*\}$, we get

$$\varphi(a\, Sbb^* Sa^*) = \varphi\,(bb^*)\,\varphi\,(aa^*)dt$$

2. free Burkholder-Gundy inequality for p = ∞

For the operator norm , we get the more estimates

$$\| \int A_t ds_t B_t \|_2^2 \leq C \int \|A_t\|^2 \int \|B_t\|^2 \, dt$$

2.8- Free Brownian motion

A free Brownian motion is given by a family $(B_t)_{t\geq 0} \subset (A,\varphi)$ of random variables , where A is unite algebra , φ is linear function , such that :

- $B_t = 0$ [54].
- each increment $B_t - B_s$ (s < t) follows semicircular distribution with mean = 0 and variance = t-s .

$$d\mu B_t - B_s(x) = \frac{1}{2\pi(t-s)}\sqrt{4(t-s) - X^2}dx$$

- disjoint increments are free for $0 < t_1 < t_2 < \cdots < t_n$,

$B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}}$ are free .

Perhaps it is appropriate to present the Semicircle distribution with some properties.

2.8.1- The Semicircle Distribution

The semicircle distribution is one of the important distributions that have a leading role in the stochastic matrices , is the free analogue of the normal

distribution and also called the " Wigner distribution " in recognition of the physicist Eugene who did a great job on stochastic matrices [1] [51].

- The semicircle distribution has probability density function f given by

$$f(X) = \frac{2}{\pi R^2}\sqrt{R^2 - X^2} \qquad X \in [-R, R]$$

and the pdf of the semicircle distribution has the following properties :

1. f is symmetric when X = 0 .
2. f at mode X = 0 is increases and then decreases.
3. f is down concave .

- The cumulative distribution function of the semicircle distribution F is given by :

$$F(X) = \frac{1}{2} + \frac{X\sqrt{R^2 - X^2}}{\pi R^2} + \frac{\arcsin(\frac{X}{R})}{\pi} \qquad -R \leq X \leq R$$

We are done by student as :

$$f(X) = \int_{-R}^{R} \frac{2}{\pi R^2}\sqrt{R^2 - X^2}\, dx$$

$$X = R \sin(t)$$

$$dX = R\cos(t)dt$$

$$X \to -R \;\to \sin t = -1 \;\to t = \frac{3\pi}{2}$$

$$X \to +R \;\to \sin t = +1 \;\to t = \frac{\pi}{2}$$

$$= \int_{\frac{3\pi}{2}}^{\frac{\pi}{2}} \frac{2}{\pi R^2}\sqrt{R^2 - (R\sin(t))^2}\, R\cos(t)\, dt$$

$$= \int_{\frac{3\pi}{2}}^{\frac{\pi}{2}} \frac{2}{\pi R^2}\sqrt{R^2 - (R^2\sin^2(t))}\, R\cos(t)\, dt$$

$$= \int_{\frac{3\pi}{2}}^{\frac{\pi}{2}} \frac{2}{\pi R^2} \sqrt{R^2(1-\sin^2(t))}\ R\cos(t)\,dt$$

$$= \int_{\frac{3\pi}{2}}^{\frac{\pi}{2}} \frac{2}{\pi R^2}\ R\sqrt{1-\sin^2(t)}\ R\cos(t)\,dt$$

$$1-\sin^2(t) = \cos^2(t)$$

$$= \frac{2}{\pi R^2}\ R^2 \int_{\frac{3\pi}{2}}^{\frac{\pi}{2}} \sqrt{\cos^2(t)}\ \cos(t)dt$$

$$= \frac{2}{\pi R^2}\ R^2 \int_{\frac{3\pi}{2}}^{\frac{\pi}{2}} \cos(t)\ \cos(t)dt$$

$$= \frac{2}{\pi R^2}\ R^2 \int_{\frac{3\pi}{2}}^{\frac{\pi}{2}} \cos^2(t)\ dt$$

$$= \frac{2}{\pi R^2}\ R^2 \int_{\frac{3\pi}{2}}^{\frac{\pi}{2}} \frac{1+\cos 2t}{2}\ dt$$

$$= \frac{2}{\pi R^2}\ R^2\ \frac{1}{2} \int_{\frac{3\pi}{2}}^{\frac{\pi}{2}} 1\ dt + \int_{\frac{3\pi}{2}}^{\frac{\pi}{2}} \cos 2t\ dt$$

$$= \frac{2}{\pi R^2}\ R^2\ \frac{1}{2}\ [t + \frac{\sin 2t}{2}]$$

And let,

$$t = \arcsin(\frac{X}{R})$$

$$\sin(2t) = 2\sin(t)\cos(t)$$

$$= \frac{2}{\pi R^2}\ R^2\ \frac{1}{2}\ [\arcsin\left(\frac{X}{R}\right) + \frac{\sin 2\arcsin(\frac{X}{R})}{2}]$$

$$= \frac{2}{\pi R^2}\left[\frac{R^2\arcsin\left(\frac{X}{R}\right)}{2} + \frac{R^2\sin(2\arcsin(\frac{X}{R}))}{4}\right]$$

$$= \frac{2}{\pi R^2}\left[\frac{R^2 \arcsin\left(\frac{X}{R}\right)}{2} + \frac{R^2\, 2 \sin(\arcsin(\frac{X}{R}))\cos(\arcsin(\frac{X}{R}))}{4}\right]$$

Sin (arcsine (t))=t

$$= \frac{2}{\pi R^2}\left[\frac{R^2 \arcsin\left(\frac{X}{R}\right)}{2} + \frac{R^2\, 2\,\frac{X}{R}\cos(\arcsin(\frac{X}{R}))}{4}\right]$$

$$\cos(\arcsin(t)) = \sqrt{1-t^2}$$

$$= \frac{2}{\pi R^2}\left[\frac{R^2 \arcsin\left(\frac{X}{R}\right)}{2} + \frac{R^2\, 2\,\frac{X}{R}\sqrt{1-(\frac{X}{R})^2}}{4}\right]$$

$$= \frac{2}{\pi R^2}\left[\frac{R^2 \arcsin\left(\frac{X}{R}\right)}{2} + \frac{R^2\,\frac{X}{R}\sqrt{1-(\frac{X}{R})^2}}{2}\right]$$

$$= \frac{2}{\pi R^2}\left[\frac{R^2 \arcsin\left(\frac{X}{R}\right)}{2} + \frac{R\,X\sqrt{1-(\frac{X}{R})^2}}{2}\right]$$

$$= \frac{2}{\pi R^2}\left[\frac{R^2 \arcsin\left(\frac{X}{R}\right)}{2} + \frac{R\,X\sqrt{1-\frac{X^2}{R^2}}}{2}\right]$$

$$= \frac{2}{\pi R^2}\left[\frac{R^2 \arcsin\left(\frac{X}{R}\right)}{2} + \frac{R\,X\,\frac{\sqrt{R^2-X^2}}{R}}{2}\right]$$

$$= \frac{2}{\pi R^2}\left[\frac{R^2 \arcsin\left(\frac{X}{R}\right)}{2} + \frac{X\sqrt{R^2-X^2}}{2}\right]$$

$$= \frac{2}{\pi R^2}\,\frac{R^2 \arcsin\left(\frac{X}{R}\right)}{2} + \frac{2}{\pi R^2}\,\frac{X\sqrt{R^2-X^2}}{2}$$

$$= \frac{1}{2} + \frac{X\sqrt{R^2-X^2}}{\pi R^2} + \frac{\arcsin\left(\frac{X}{R}\right)}{\pi}$$

- The expectation of semicircle distribution is E(x)=0 , We are done by student as :

$$E(X)=\int_{-R}^{R} X\frac{2}{\pi R^2}\sqrt{R^2-X^2}\ d$$

$$=\int_{-R}^{R}\frac{X}{R}\frac{2}{\pi}R\sqrt{1-(\frac{X}{R})^2}\ dx$$

Let,

$$\frac{X}{R}=\sin(t)\ \rightarrow \frac{dX}{dR}=\cos(t)\ dt$$

$$X\rightarrow -R\ \rightarrow \sin(t)=-1\ \rightarrow t=\frac{3\pi}{2}$$

$$X\rightarrow +R\ \rightarrow \sin(t)=+1\ \rightarrow t=\frac{\pi}{2}$$

$$=\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\sin(t)\ \frac{2}{\pi}R\sqrt{1-\sin^2(t)}\ R\cos(t)\ dt$$

$$=\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\sin(t)\ \frac{2R^2}{\pi}\cos^2(t)\ dt$$

$$=\frac{2R^2}{\pi}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\sin(t)\ [1-\sin^2(t)]dt$$

$$=\frac{2R^2}{\pi}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\sin(t)\ -\sin^3(t)\ dt$$

$$=\frac{2R^2}{\pi}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}-\cos(t)\ +\cos(t)-\frac{1}{3}\cos^3(t)\ dt$$

$$=\frac{2R^2}{\pi}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}-\frac{1}{3}\ \cos^3(t)\ dt$$

$$= \frac{2R^2}{\pi} \cdot -\frac{1}{3} \ [\cos^3(t)]_{\frac{3\pi}{2}}^{\frac{\pi}{2}}$$

$$= \frac{2R^2}{\pi} \cdot -\frac{1}{3} \ [\cos^3 \frac{\pi}{2} - \cos^3 \frac{3\pi}{2}]$$

$$= \frac{-2R^2}{3\pi} [0]$$

$$= 0$$

- the variance of semicircle distribution is var (X)= $\frac{R^2}{4}$, We are done by student as :

Var(X) = $E(X^2) - (E(X))^2$

$$E(X^2) = \int_{-R}^{R} X^2 \ \frac{2}{\pi R^2} \sqrt{R^2 - X^2} \, dx$$

$$= \int_{-R}^{R} (\frac{X}{R})^2 \ \frac{2}{\pi} \ R \sqrt{1 - (\frac{X}{R})^2} \, dx$$

Let,

$$\frac{X}{R} = \sin(t) \ , \frac{dX}{dR} = \cos(t) \, dt$$

When,

$$X \to -R \ \to \sin(t) = -1 \, , t = \frac{3\pi}{2}$$

$$X \to +R \ \to \sin(t) = +1 \, , t = \frac{\pi}{2}$$

$$= \int_{\frac{3\pi}{2}}^{\frac{\pi}{2}} (\sin(t))^2 \ \frac{2}{\pi} \ R \sqrt{1 - \sin^2}(t) \ R \cos(t) \, dt$$

$$= \int_{\frac{3\pi}{2}}^{\frac{\pi}{2}} \sin^2(t) \ \cos^2(t) \ \frac{2R^2}{\pi} \, dt$$

$$= \frac{2R^2}{\pi} \int_{\frac{3\pi}{2}}^{\frac{\pi}{2}} \sin^2(t)\ \ (1 - \sin^2(t))dt$$

$$= \frac{2R^2}{\pi} \int_{\frac{3\pi}{2}}^{\frac{\pi}{2}} (\sin^2(t) - \sin^2(t))dt$$

$$= \frac{2R^2}{\pi} \int_{\frac{3\pi}{2}}^{\frac{\pi}{2}} \ (\frac{1}{2}(1 - \cos 2(t)\ ) - \frac{1}{4}(1 - \cos 2(t))^2)$$

$$= \frac{2R^2}{\pi} \int_{\frac{3\pi}{2}}^{\frac{\pi}{2}} \ (\frac{1}{2}(1 - \cos 2(t))dt - \frac{2R^2}{\pi} \int_{\frac{3\pi}{2}}^{\frac{\pi}{2}} \frac{1}{4}\ (1 - \cos 2(t))^2)$$

$$= \frac{2R^2}{\pi} \frac{1}{2} \int_{\frac{3\pi}{2}}^{\frac{\pi}{2}} (1 - \cos 2(t))dt - \frac{2R^2}{\pi} \frac{1}{4} \int_{\frac{3\pi}{2}}^{\frac{\pi}{2}} [1 - 2\cos 2(t) + \cos^2 2(t)\ ]\, dt$$

$$= \frac{R^2}{\pi} [4 - \frac{1}{2}\sin 2(t)]_{\frac{3\pi}{2}}^{\frac{\pi}{2}} - \frac{R^2}{2\pi} [t - \sin 2(t) + \frac{1}{2}t - \frac{1}{8}\cos 4(t)\ ]_{\frac{3\pi}{2}}^{\frac{\pi}{2}}$$

$$= \frac{R^2}{\pi}\ [-\frac{2\pi}{2}] - \frac{R^2}{2\pi}\ [\frac{3\pi}{2}]$$

$$= R^2 - \frac{3R^2}{4}$$

$$= \frac{R^2}{4}$$

$$\text{Var}\ (X) = E(X^2) - (E(X))^2$$

$$= \frac{R^2}{4} - 0$$

$$= \frac{R^2}{4}$$

- The median of semicircle distribution is Zero , We are done by student as :

$$F(X) = \frac{1}{2}$$

$$F(M) = \frac{1}{2}$$

$$F(X)= [\frac{1}{2} + \frac{X\sqrt{R^2-X^2}}{\pi R^2} + \frac{\arcsin(\frac{X}{R})}{\pi}]$$

$$F(M)= [\frac{1}{2} + \frac{m\sqrt{R^2-m^2}}{\pi R^2} + \frac{\arcsin(\frac{m}{R})}{\pi}] = \frac{1}{2}$$

$$= \frac{m\sqrt{R^2-m^2}}{\pi R^2} + \frac{\arcsin\left(\frac{m}{R}\right)}{\pi} = \frac{1}{2} - \frac{1}{2}$$

$$= \frac{m\sqrt{R^2-m^2}}{\pi R^2} + \frac{\arcsin\left(\frac{m}{R}\right)}{\pi} = 0$$

$$= \frac{m\sqrt{R^2-m^2}+R^2 \arcsin\left(\frac{m}{R}\right)}{\pi R^2} = 0$$

$$= \frac{m(R-m)+R^2 \arcsin\left(\frac{m}{R}\right)}{\pi R^2} = 0= 0$$

- The mode of the semicircle distribution is Zero , We are done by student as :

$$f(X) = \frac{2}{\pi R^2} \sqrt{R^2 - X^2}$$

$$= \frac{\partial}{\partial x} \frac{1}{2} \left(\frac{2}{\pi R^2}\right) (R^2 - X^2)^{-\frac{1}{2}} (-2X)$$

$$= \frac{-2X}{\pi R^2 \, (R^2-X^2)^{\frac{1}{2}}}$$

$$= \frac{-2X}{\pi R^2\sqrt{R^2-X^2}} = 0$$

$$-2X= 0$$

$$X = 0$$

2.9- Free Itô

Let a_t,b_t,c_t, d_t be operator-valued functions and $(W(t))_{t\geq 0}$ is a Brownian motion . Then [17][18][48]:

$a_t dt \cdot b_t dt = a_t dt \cdot b_t dW(t)\ c_t = a_t dW(t)\ b_t \cdot c_t\ dt = 0$

$a_t dW(t) bt \cdot c_t dW(t) dt = E(b_t c_t)\ a_t d_t dt$

2.10- Free CIR equations

As we know the CIR equation is of the form

$$dr_t = (k - \theta r_t)\ dt + \sigma\sqrt{r_t}\ dW_t$$

where :

$k, \theta, \sigma > 0$, $t \in [0,\infty[$, and $(W(t))_{t\geq 0}$ is a Brownian motion.

The main challenge here is to prove that the feller condition ensures the positive solution for free SDE.

$$dr_t = (k - \theta r_t)\ dt + \frac{\sigma}{2}\ \sqrt{r_t}\ dB(t) + \frac{\sigma}{2}\ dB(t)\sqrt{r_t}$$

To prove this we do the following steps.

Firstly : [17][18][48]

We start showing existence and uniqueness of these free SDEs by the following Stochastic differential equation with additive Brownian motion term , also called " square root process".

$$dV(t) = \left(k(t) - \frac{\sigma^2(t)}{2}\right) \frac{1}{2}\ V^{-1}(t)) - \frac{1}{2}\theta V(t)dt + \frac{\sigma(t)}{2}\ dW(t)$$

where :

$k, \theta, \sigma > 0$, $t \in [0,\infty[$

$(W(t))_{t\geq 0}$ is a Brownian motion

$(V(t))_{t\geq 0}$ is a Von Neumann algebra-valued or vector-valued .

Secondly : [17][18][48]

Transform this process with two cases :

- first case transform the square root process into the setting of commutative function spaces by the theorem (1).

Theorem (1): Let $k \subset [0,\infty[$ be compact. Let $k, \theta, \sigma: [0,\infty[\rightarrow C(K)_+$ be continuous, feller condition ($2k\theta \geq \sigma^2$) holds and let $\widehat{V}_0 \in C(K)_+$. Then the SDE [17][18][49]:

$$d\widehat{V}(t) = [(k(t) - \frac{\sigma^2(t)}{2})\frac{1}{2}\widehat{V}^{-1}(t) - \frac{\theta(t)}{2}\widehat{V}(t)]dt + \frac{\sigma(t)}{2}\, 1_k dW(t) \ , \widehat{V}(0) = \hat{v}_0$$

for $t \in [0,\infty[$, has a global solution $V \in C([0,\infty[\ , L_2(p_\varphi, C(K)_+))$.

Proof : Consider a Brownian motion $(W(t))_{t\geq 0}$ on the probability space (Ω, f, p_φ). Then using point mass measures δ_k for $k \in K$ and the Feller condition prove the global existence of

$$d\widehat{V}_k(t) = ((k_k(t) - \frac{\sigma_k(t)^2}{2})\frac{1}{2}\hat{V}_k^{-1}(t) - \frac{1}{2}\theta_k(t)\widehat{V}_k(t)dt + \frac{\sigma_k(t)}{2}\, dW(t) \ ,$$
$$\widehat{V}_k(0) = \hat{v}_{0,k} \ , \text{ for } t \in [0,\infty[.$$

Using a countable dense subset $\widetilde{K} \subset K$ and the point mass functional we show that the paths keep positive except at a P-zero set $N := \bigcup \bigcup_{k\in\widetilde{K}} N_k$. So for all $\omega \in \Omega \setminus N$ the paths stay positive on $[0,\infty[$ for all k.

- Second case transform the square root process into the setting of the general non-commutative von Neumann algebra –valued case in the theorem (2). By employing the functional calculus and The spectral theorem we have an isometric homomorphism

$$T : B(\theta_3(V_0)) \to \langle V_0, id\rangle$$

where :

$\langle V_0, id\rangle$: is the von Neumann algebra generated by V_0 and the identity

$B(\theta_3\ (V_0))$: is the function space of bounded, measurable functions on θ_3 (V_0), the spectrum of V_0. If φ is a until, then consider

$$Ep_\varphi = T *(\varphi) \text{ with the identity}$$

$$Ep_\varphi(g) = \int g d\, p_\varphi = \varphi(T(g)) \quad , \text{for all } g \in B(\theta_3(V_0))$$

Theorem (2) Let $V_0 \in A_+$ and let $\theta_1, \theta_2, \theta_3 : [0,\infty[\to \langle V_0, id\rangle_+$ Feller condition $(2\theta_1\theta_2 \geq {\theta_3}^2)$ holds for all $t \in [0,\infty[$. Then the SDE [17][18][48]:

$$d\overline{V}(t) = ((\theta_1(t) - \frac{{\theta_3}^2(t)}{2})\frac{1}{2}\overline{V}^{-1}(t) - \frac{\theta_2(t)}{2}\overline{V}(t))dt + \frac{\theta_3(t)}{2}\, dW(t) \ , \overline{V}(0) = v_0$$

for $t \in [0,\infty[$, has a global solution in $V \in C([0,\infty[, L_2(p_\varphi, A_+))$. Note that id is the unit in the corresponding von Neumann algebra.

Proof : Let $\widehat{V}(t)$ be a global solution in $C(K)_+$ by Theorem 1. By the functional calculus we see that $T\ (\widehat{V}\ (t))$ is a positive solution to the above equation under the Feller condition, where $T : C(K) \to \langle V0, id\rangle$ is the functional calculus mapping for the self-ad joint element V_0. In particular we have that $\overline{V}(t) \in \langle V0, id\rangle_+$ for all $t \in [0, \infty[$.

After these two cases the process still with a classical Brownian motion as driving process.

Proposition. Let T > 0 and let V ∈ C([0,T [,A) ∩ C([0,T], $L_2(\varphi)$) be a free Itô process and let $\overline{V}$ ∈ C([0,T],$L_2(P_\varphi$,A)) be a vector-valued Itô processes. Suppose for all t ∈ [0,T [and all orthogonal projections p free of V (t) that we have

$$\| pV(t)^2p \|_1 = \| p\overline{V}(t)^2p \|_1$$

1. If V (t) ∈ A and V (T) ∈ $L_2(\varphi)$ then V (T) ∈ A.

2. If $\overline{V}$(t) is invertible for all t ∈ [0,T], V (t) is invertible for all t ∈ [0,T [and V (T) ∈ A, then V (T) ∈ A is also invertible.

Theorem (3) Let $V_0 \in A_+$be given. Let $\theta_1 : [0,\infty] \to A_+$, $\theta_3 : [0,\infty] \to \langle V_0, \mathrm{id}\rangle_+$ and $\theta_1, \theta_2, \theta_3 > 0$ a constant , Feller condition ($2\theta_1\theta_2 \geq {\theta_3}^2$) holds. Then the free SDE [17][18][48]:

$$dV(t) = \left(\left(\theta_1(t) - \frac{{\theta_3}^2(t)}{2}\right)\frac{1}{2}V^{-1}(t) - \frac{\theta_2}{2}V(t)\right)dt + \frac{\theta_3(t)}{2}\,dB(t) \quad , V(0) = v_0$$

for t ∈ [0,∞[, has a global solution V∈ C([0,∞[, A_+).

Proof:

1- In the first step we choose a maximal interval [0,T [, where a solution of the equation

$$dV(t) = \left(\left(\theta_1(t) - \frac{{\theta_3}^2(t)}{2}\right)\frac{1}{2}V^{-1}(t) - \frac{\theta_2}{2}V(t)\right)dt + \frac{\theta_3(t)}{2}\,dB(t) \quad , V(0) = v_0$$

for t ∈ [0,T [, By Theorem (2) we know that the solution for

$$d\overline{V}(t) = \left(\left(\theta_1(t) - \frac{{\theta_3}^2(t)}{2}\right)\frac{1}{2}\overline{V}^{-1}(t) - \frac{\theta_3}{2}\overline{V}(t)\right)dt + \frac{\theta_3(t)}{2}\,dB(t) \quad , \overline{V}(0) = v_0$$

for t ∈ [0,∞[, exists globally.

2- we know that the process V (t) exists for t < T . The reference process V (t) exists globally with values in A. Because of the isometry we can define

$V(T) = L_2\text{–}\lim_{t\to T} V_0 + \int_0^T (\frac{1}{2}(\theta_1 - \frac{\theta_3{}^2}{2})V^{-1}(t) - \theta_2 V(t)dt + \theta_3{}^2 B(T)$

Using Proposition 1(1.), we can deduce that $V(T) \in A$

3- Step 2 told us that $V(T) \in A$. We want to extend the (unique) solution V (t) beyond T . we need the inevitability of V (T). This would allow us an extension and the solution is global. Again by Proposition 1 (2.), we see that V (T) is invertible. Therefore the solution of square root process is global.

Let $r_t = V(t)^2$. Then according to the free Itô formula:

$dr_t = (V(t) + dv(t))^2 - (V(t))^2$

$= (V(t))^2 + 2\left(V(t)\right) dV(t) + (dV(t))^2 - (V(t))^2$

$= (dV(t))^2 + 2\left(V(t)\right) dV(t)$

$= [2\,(\frac{\theta_1 - \frac{\theta_3{}^2}{2}}{2V(t)}.\,2\,V(t) - \frac{\theta_2}{2}V(t))V(t) + \frac{\theta_3{}^2}{2}]\,dt + \frac{\theta_3}{2}V(t)\,dB(t) + \frac{\theta_3}{2}\,dB(t)V(t)$

$= [2(\frac{\frac{2\theta_1 - \theta_3{}^2}{2}}{2V(t)}.\,2V(t) - \frac{\theta_2}{2}V(t))V(t) + \frac{\theta_3{}^2}{2}]\,dt + \frac{\theta_3}{2}V(t)\,dB(t) + \frac{\theta_3}{2}\,dB(t)V(t)$

$= [\,\frac{4\theta_1 - 2\theta_3{}^2}{4}\,4\,V(t)V(t) - \frac{\theta_2}{2}\,2\,V(t)V(t) + 2\frac{\theta_3{}^2}{2}]dt + \frac{\theta_3}{2}V(t)\,dB(t) + \frac{\theta_3}{2}$
$dB(t)V(t)$

$= [\theta_1 - \frac{\theta_3{}^2}{2}4\,V(t)^2 - \theta_2\,V(t)^2 + 2\frac{\theta_3{}^2}{2}]dt + \frac{\theta_3}{2}V(t)\,dB(t) + \frac{\theta_3}{2}\,dB(t)V(t)$

$= [\theta_1 - \theta_2 V(t)^2]\,dt + \frac{\theta_3}{2}V(t)\,dB(t) + \frac{\theta_3}{2}\,dB(t)V(t)$

According to the above hypothesis replace every $V(t)^2 = r_t$ to get the final form of free CIR process as following [17][18][48]:

$$dr_t = (\theta_1 - \theta_2 r_t)dt + \frac{\theta_3}{2}\sqrt{r_t}\,dB(t) + \frac{\theta_3}{2}\,dB(t)\sqrt{r_t}$$

2.10.1- The expected of the Free CIR process

$$r_t = \frac{\theta_1}{\theta_2} + (r_0 - \frac{\theta_1}{\theta_2})e^{-\theta_2 t} + \frac{\theta_3}{2} e^{-\theta_1 t} \int_0^t e^{\theta_1 s} \sqrt{r_s}\, dw_s + \frac{\theta_3}{2} \sqrt{r_s}\, dw_s$$

$$E[r_t] = E[\frac{\theta_1}{\theta_2} + (r_0 - \frac{\theta_1}{\theta_2})e^{-\theta_2 t} + \frac{\theta_3}{2} e^{-\theta_1 t} \int_0^t e^{\theta_1 s} \sqrt{r_s}\, dw_s + \frac{\theta_3}{2} \sqrt{r_s}\, dw_s]$$

$$= \frac{\theta_1}{\theta_2} + (r_0 - \frac{\theta_1}{\theta_2})e^{-\theta_2 t} + \frac{\theta_3}{2} e^{-\theta_1 t}\, E[\int_0^t e^{\theta_1 s} \sqrt{r_s}\, dw_s + \frac{\theta_3}{2} \sqrt{r_s}\, dw_s]$$

$$= \frac{\theta_1}{\theta_2} + (r_0 - \frac{\theta_1}{\theta_2})e^{-\theta_2 t} + \frac{\theta_3}{2} e^{-\theta_1 t} \int_0^t e^{\theta_1 s} \sqrt{r_s}\, E[dw_s] + \frac{\theta_3}{2} \sqrt{r_s}\, E[dw_s]$$

$$= \frac{\theta_1}{\theta_2} + (r_0 - \frac{\theta_1}{\theta_2})e^{-\theta_2 t}$$

2.10.2- The variance of the Free CIR process

The Variance of the Free CIR process

$$r_t = \frac{\theta_1}{\theta_2} + (r_0 - \frac{\theta_1}{\theta_2})e^{-\theta_2 t} + \frac{\theta_3}{2} e^{-\theta_1 t} \int_0^t e^{\theta_1 s} \sqrt{r_s}\, dw_s + \frac{\theta_3}{2} \sqrt{r_s}\, dw_s$$

$$Var[r_t] = E[r_t - E(r_t)]^2$$

$$= E\,[\frac{\theta_1}{\theta_2} + (r_0 - \frac{\theta_1}{\theta_2})e^{-\theta_2 t} + \frac{\theta_3}{2} e^{-\theta_1 t} \int_0^t e^{\theta_1 s} \sqrt{r_s}\, dw_s + \frac{\theta_3}{2} \sqrt{r_s}\, dw_s - \frac{\theta_1}{\theta_2} + (r_0 - \frac{\theta_1}{\theta_2})e^{-\theta_2 t}]^2$$

$$= E[\frac{\theta_3}{2} e^{-\theta_1 t} \int_0^t e^{\theta_1 s} \sqrt{r_s}\, dw_s + \frac{\theta_3}{2} \sqrt{r_s}\, dw_s\,]^2$$

$$= [(E\frac{\theta_3}{2} e^{-\theta_1 t} \int_0^t e^{\theta_1 s} \sqrt{r_s}\, (dw_s)) + \frac{\theta_3}{2} \sqrt{r_s}\, E(dw_s\,)]^2$$

$$= \frac{{\theta_3}^2}{4} e^{-2\theta_1 t} \int_0^t e^{2\theta_1 s}\, E[r_s] ds$$

$$= \frac{{\theta_3}^2}{4} e^{-2\theta_1 t} \int_0^t e^{2\theta_1 s}\, [r_0 e^{-\theta_1 t} + \theta_2 (1 - e^{-\theta_1 t})] ds$$

$$= \frac{\theta_3{}^2}{4} e^{-2\theta_1 t} \int_0^t e^{\theta_1 s} r_0 + \theta_2 \left(e^{2\theta_1 s} - e^{-\theta_1 s}\right)] ds$$

$$= \frac{\theta_3{}^2}{4} e^{-2\theta_1 t} \frac{e^{\theta_1 t}-1}{\theta_1} r_0 + \frac{\theta_3{}^2}{4} e^{-2\theta_1 t} \theta_2 \left(\frac{e^{2\theta_1 t}-1}{2\theta_1} - \frac{e^{-\theta_1 t}}{\theta_1}\right)$$

$$= \frac{\theta_3{}^2}{4} \left(\frac{e^{\theta_1 t}-e^{-2\theta_1 t}}{\theta_1}\right) r_0 + \frac{\theta_3{}^2}{4} \frac{\theta_2}{2\theta_1} (1 - e^{-\theta_1 t})^2$$

We conclude from the above expectation and variance the distribution function of Free CIR process is non- central chi-square distribution with and parameters of non- centrality 2u and 2q+2 degree of freedom.

Unite Three : Parameter Estimation

2.11- Introduction

There are several ways in which good estimates can be obtained for one or more parameters in the statistical community. In this unit ,we are going to present overview of Maximum Likelihood Estimation (MLE) method, Bayesian Inference Method and Markov Chain Monte Carlo (MCMC). We first provides the MLE method, Bayesian Inference method and then we show the MCMC method in the CIR and Free CIR process.

2.12- Maximum Likelihood Estimate (MLE)

The method of Maximum Likelihood estimation is One of the most widely used statistical estimation method, it is presented by fisher 1992. This method has best features in estimation as sufficiency, consistency, efficiency and parameterization invariance. In this processes we can be present as[34][41] :-

the conditional density function is follows :-

$$f(r(s),s\ ;\ r(t),t) = c e^{-u-v} \left(\frac{u}{v}\right)^{\frac{q}{2}} I_q(2(uv)^{1/2}) \qquad \ldots(14)$$

As we mention the equation (2),(3),(4),(5) and $I_q(.)$ is the modified Bessel function of the first kind of order q. The likelihood function for the interest rate r_t with n observations is :

$$L(\theta_1,\theta_2,\theta_3) = \prod_{i=1}^{n-1} ce^{-u-v}\left(\frac{u}{v}\right)^{\frac{q}{2}} I_q(2(uv)^{1/2}) \qquad \ldots(15)$$

The log-likelihood function is as following :

$$\text{Ln } L(\theta_1,\theta_2,\theta_3) = (n-1)\ln(c) + \sum_{i=1}^{n-1}(-u_{t_i} - v_{t_{i+1}}) + \frac{1}{2}\, q \ln\left(\frac{v_{t_{i+1}}}{u_{t_i}}\right) + \ln[I_q(2\sqrt{u_{t_i} v_{t_{i+1}}})]) \quad \ldots.(16)$$

where $u_{t_i} = cx_{t_i}e^{-\alpha\Delta t}$ and $v = cx_{t_{i+1}}$.

The log-likelihood function must be maximization by taking partial derivatives of equation (1) with respect to θ_1, θ_2 and θ_3 make them equal to zero yield three equations:

$$\frac{\partial \ln L(r_t;\theta_1,\theta_2,\theta_3)}{\partial\theta_1}\Big|_{\widehat{\theta_1}} = 0 \qquad \ldots(17)$$

$$\frac{\partial \ln L(r_t;\theta_1,\theta_2,\theta_3)}{\partial\theta_2}\Big|_{\widehat{\theta_2}} = 0 \qquad \ldots(18)$$

$$\frac{\partial \ln L(r_t;\theta_1,\theta_2,\theta_3)}{\partial\theta_3}\Big|_{\widehat{\theta_3}} = 0 \qquad \ldots(19)$$

By solving these above equations will yield the maximum likelihood estimates:

$$\hat{\mu} = (\widehat{\theta_1}, \widehat{\theta_2}, \widehat{\theta_3}) = \arg\max_{\mu} \ln L(\mu).$$

2.13- The Bayesian Inference

Bayesian Inference is considered one of the estimation methods in statistic, the key to Bayesian inference is the posterior distribution which consists of three components, the likelihood function p(Y |X,θ), the state variable specification, p (X|θ), and the prior distribution p (θ). In this method, we begin with an appropriate model to describe the problem we wish to analyze It. After that, we suppose the prior distribution for the unknown parameters of the model. and Multiple the likelihood function by the prior distribution we getting the posterior distribution. We will introduce the basic elements of Bayesian Estimation [31].

2.13.1- The Prior Distribution

The prior is a probability distribution that is formulated before any information is obtained for the phenomenon under study. This distribution was characterized by essential economic and statistical roles. The prior is multiplied by the likelihood function, then normalization to estimate the posterior probability distribution p(θ|X), In general. The prior probability distribution be either informative or uninformative prior [31] [57].

2.13.2- The Likelihood Function

The Likelihood function is depend on the information provided by the sampling. The likelihood function in the following form:-

$$p(Y|\theta) = \prod_{i=1}^{n} p(Y_i|\theta)$$

The data Y impact the posterior distribution p(θ|Y) only out of the likelihood function. And therefore, Bayesian inference is subject to the probability which states that for a given sample of data, any two probability function p(Y| θ) with the same likelihood function result the same inference for θ [57].

2.13.3- The posterior distribution

In order to complete the definition of Bayesian inference, we present the posterior distribution. It is the distribution of a collection of unknown parameters conditional on the current data, which is the distribution depend on the Likelihood function and a prior distribution. The posterior distribution has the following equation [31] [57] :-

$$p(\theta \mid X) = p(\theta, X \mid Y) \propto p(Y \mid X, \theta)\, p(X \mid \theta)\, p(\theta)$$

Where :-

θ are the parameters.

X is the Current data.

$p(\theta)$ is the distribution of the parameters.

$p(X \mid \theta)$ is the distribution of the state variables.

$p(Y \mid X, \theta)$ is the likelihood function.

2.13.4- Markov Chain Monte Carlo (MCMC)

MCMC method has theoretical foundations, the Hammersley Clifford theorem provides a description of $p(\theta, X \mid Y)$ in full conditional distributions. The MCMC method iteratively samples from these conditional distributions. When these full conditionals can be sampled directly, the MCMC algorithm is called as a Gibbs sampler. When cannot be directly sampled, we discuss a number of different approaches based on the widely applicable Metropolis-Hastings algorithm. MCMC algorithms, in general, possess attractive limiting properties.

We are going to apply a Markov chain Monte Carlo (MCMC) algorithm to sample from the posterior distribution of the parameter**s.** There are many different types of MCMC algorithms. The two most basic and widely used

are the Metropolis-Hastings algorithm and the Gibbs sampler which we will now review [31].

2.13.5- Gibbs Sampler

It is the simplest type of MCMC method, it was presented in 1984 by the Geman and Geman .Gibbs sampling is applicable in general when the joint parameter distribution is not known explicitly but the conditional distribution of each parameter given the others is known, the Sampling Algorithm is [31] [57] :-

It then samples randomly from the conditional densities p $(\theta|\theta_{\sim i})$ for i=1,…k successively as follows:

Sample θ_1^1 from $p(\theta_1|\theta_2^0, \theta_3^0, \dots \theta_k^0)$

Sample θ_2^1 from $p(\theta_2|\theta_2^1, \theta_3^0, \dots \theta_k^0)$

…….

Sample θ_k^1 from $p(\theta_k|\theta_1^1, \theta_2^1, \dots \theta_{k-1}^1)$

Indeed, the Gibbs sampler is significantly faster to calculate than the Metropolis-Hastings algorithm. Thus, to use Gibbs Sampler method we have to know how to directly sample from the conditional posterior distributions for each parameter.

2.13.6- Metropolis-Hastings algorithm

The Metropolis-Hasting algorithm is a basic method of Markov Chain Monte Carlo method, it is one of the most strong and best algorithm .It is used to generate samples from a probability distribution when samples cannot be taken directly from the distribution, it include simulating a

candidate value from a proposal distribution. After that, decide if or not to accepted the candidate value.

The Metropolis-Hastings algorithm can draw samples from any target probability density for the uncertain parameters, then sample iteratively similar to the Gibbs Sampler. But it first draws a candidate point that will be accepted or rejected depend on the acceptance probability [57] .

This algorithm can be described as follows **:-**

1. We use Gibbs Sampler for sample, in our models, we generate a new value from the probability density function.

2. Compute the acceptance probability

$$p(r_t^{(g)}, \hat{r}_t) = \min\{1, \frac{p(\hat{r}_t|\theta,Y)q(r_t^{(g)}|\widehat{r_t})}{p(r_t^{(g)}|\theta,Y)q(\hat{r}_t|r_t^{(g)})}\}$$

Chapter Three

(The Simulation and Applied Part)

3.1- Introduction

To study the behavior of any estimates distributions stochastic statistical models, we use a simulation. Analysis by using simulation is one of the important tools that can be employed in formulating and solving mathematical and statistical models , especially during the last decades. The basic idea of simulation is to build a model that simulates the studied phenomenon and then, conducting experiments.

In this chapter, a simulation experiment will be described for the purpose of achieving the research, Programming R was used for the purpose of the application. Analytically, the CIR process ensures an analytical non-negative solution, which makes the CIR process interest rate model more in line with the actual behavior of interest rates. This makes the CIR model popular among existing interest rate models. We estimate the parameters of the mean-reverting model for CIR process driven by Brownian motion and Free Brownian motion. We present and analyze simulation results for Maximum Likelihood estimation method and Bayesian estimation method of CIR process and Free CIR process. We apply our methods to real data (Homepage www.isx-iq.net). We have (ISX60) for the dinar, the period from 1/1/2017 to 1/1/2019. We used analyzed results of real data for the maximum likelihood estimation and MCMC Bayesian Estimation.

[1]

[1] Iraq's Stock Market index (ISX60) it is an economic market with financial and administrative independent. It was established on the 25th of June 2004 , not associated to any party and managed by nine members representing the various economic segments of the investment sector (called the Board of Governors) and aims to facilitate and regulate dealing in securities.

3.2- Simulation study

In this section, we perform a simulation method to generate data , then we estimate the CIR and Free CIR process by using the estimation methods mentioned in the second chapter, then the comparison is made between CIR and Free CIR process.

Simulation the CIR process driven by Brownian. In the main model the initial values of the parameters are $\theta_1 = 1, \theta_2 = 0.5$ and $\theta_3 = 1$ also that the initial value of the interest rate r_t =10. In simulation side ,we will simulate realizations $\theta_1, \theta_2, \theta_3$ and from their posterior distribution by using 50000 iterations of the Metropolis Hastings algorithm. shows the CIR process in a simulation state . Figure 3.1 represents the CIR process driven by Brownian motion using discrete-time. As it is clear from that the random process is clear in this figure and that the process is always positive.

And the simulation the CIR process driven by Free Brownian motion and In the main model the initial values of the parameters are $\theta_1 = 1, \theta_2 = 0.5$ and $\theta_3 = 1$ also that the initial value of the interest rate r_t =1.2. In simulation side ,we will simulate realizations $\theta_1, \theta_2, \theta_3$ and from their posterior distribution using 50000 iterations of the Metropolis Hastings algorithm. shows the CIR process in a simulation state . Figure 3.3 represents the CIR process driven by Free Brownian motion using discrete-time.

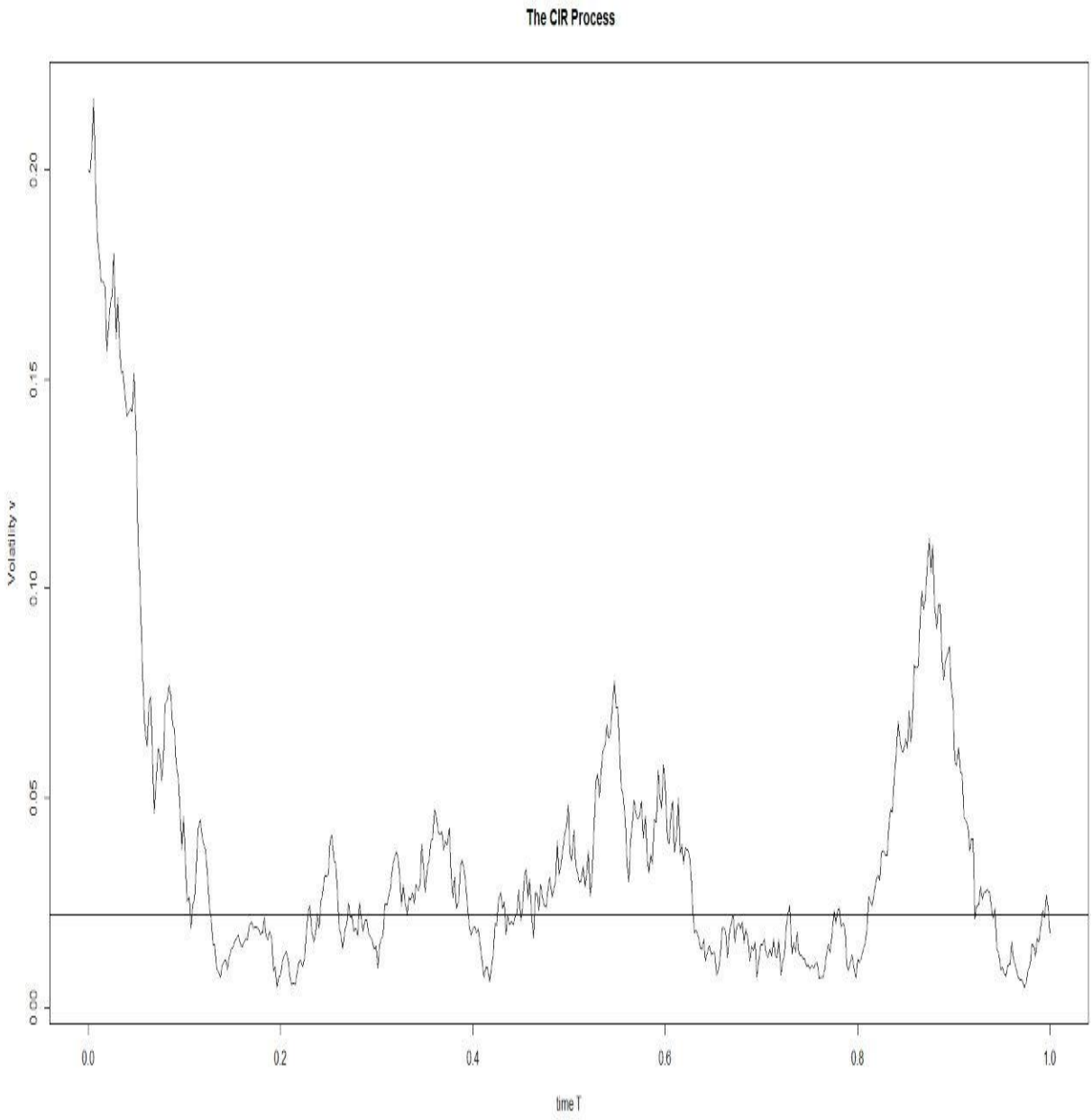

Figure 3.1 : shows the CIR process in a simulation state when the initial value of parameters are $\theta_1 = 1, \theta_2 = 0.5$ and $\theta_3 = 1$, initial value of r_t =10.

Table 3.1: shows estimation of parameters by using MLE for CIR process

Parameters	Initial value	Parameters estimator	Standard Error
θ_1	1	1.5757	0.955
θ_2	0.5	0.47013	0.21534
θ_3	1	1.98333	0.0444

-Log .Likelihood = -1213.961

Table 3.1 shows estimation of parameters by using MLE estimation , noting that the initial value for parameters are $\theta_1 = 1, \theta_2 = 0.5$ and $\theta_3 = 1$. Thus, the MLE estimation is so closer to the initial value. Therefore, it means that the MLE was able to estimate the parameters well, and the standard error was low especially since the θ_3 was a very good estimate because the S.D is its value 0.0444.

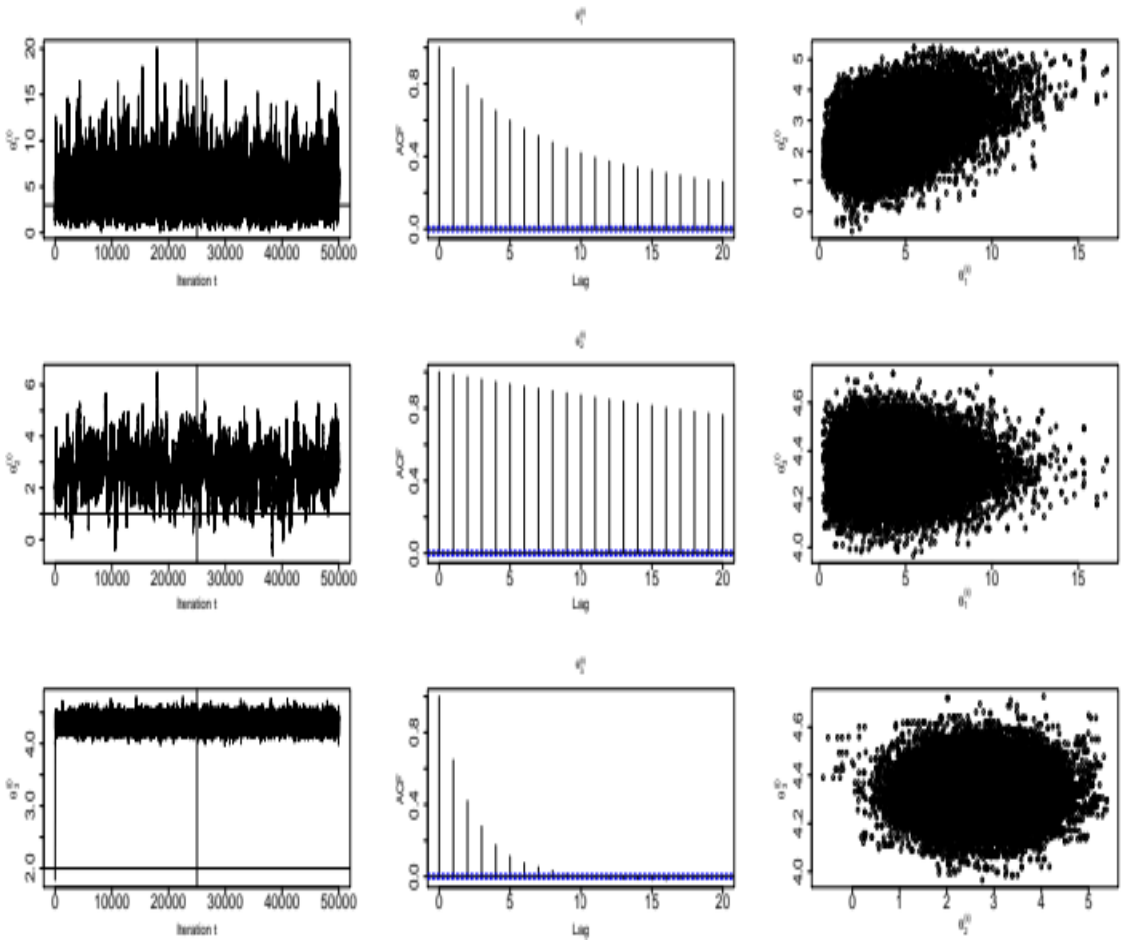

Figure 3.2: shows the results obtained from the MCMC algorithm where the first column represent the posterior distribution. The second column represent Auto-correlation for our parameters. Finally , the third column represent the spread of values between $\theta_1, \theta_3,$ and θ_3.

Figure 3.2 we assume that, the θ_1, θ_2 distributed as log normal and θ_3 distributed as inverse gamma as the prior distributions. It is good estimating using MCMC algorithm especially θ_1, θ_3. Also, the Autocorrelation for θ_1, θ_3 is very good comparing to θ_2. Therefore, it can be said that the Bayesian estimate of the CIR process is good. Finally, we can say that the posterior distribution of these parameters is stable. In addition, the MCMC estimation is better than MLE.

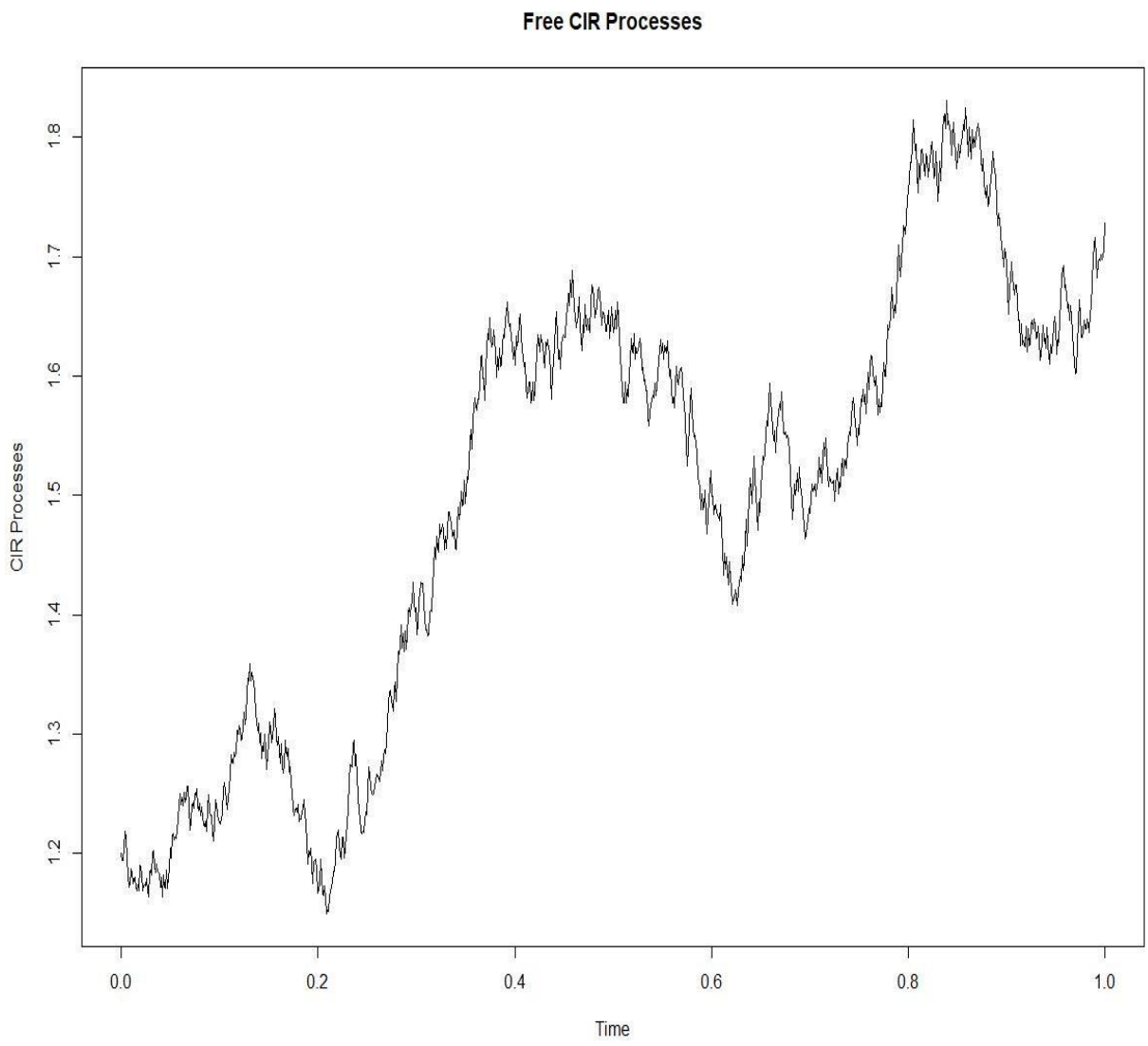

Figure 3.3: shows the Free CIR process in a simulation state when the initial value of parameters are $\theta_1 = 1, \theta_2 = 0.5$ and $\theta_3 = 1$, initial value of r_t =1.2

Table 3.2: shows estimation of parameters by using MLE for free CIR process

Parameters	Initial value	Parameters estimation	Standard Error
θ_1	1	0.7575	0.855
θ_2	0.5	0.27013	0.455
θ_3	1	1.583	0.135

-2 Log Likelihood= - 1023496

Table 3.2: shows estimation of parameters by using MLE estimation , Thus, the MLE estimation is so closer to the initial value. Therefore, it means that the MLE was able to estimate the parameters well, and the standard error was low especially since the θ_3 was a very good estimate because the S.D is its value 0.135.

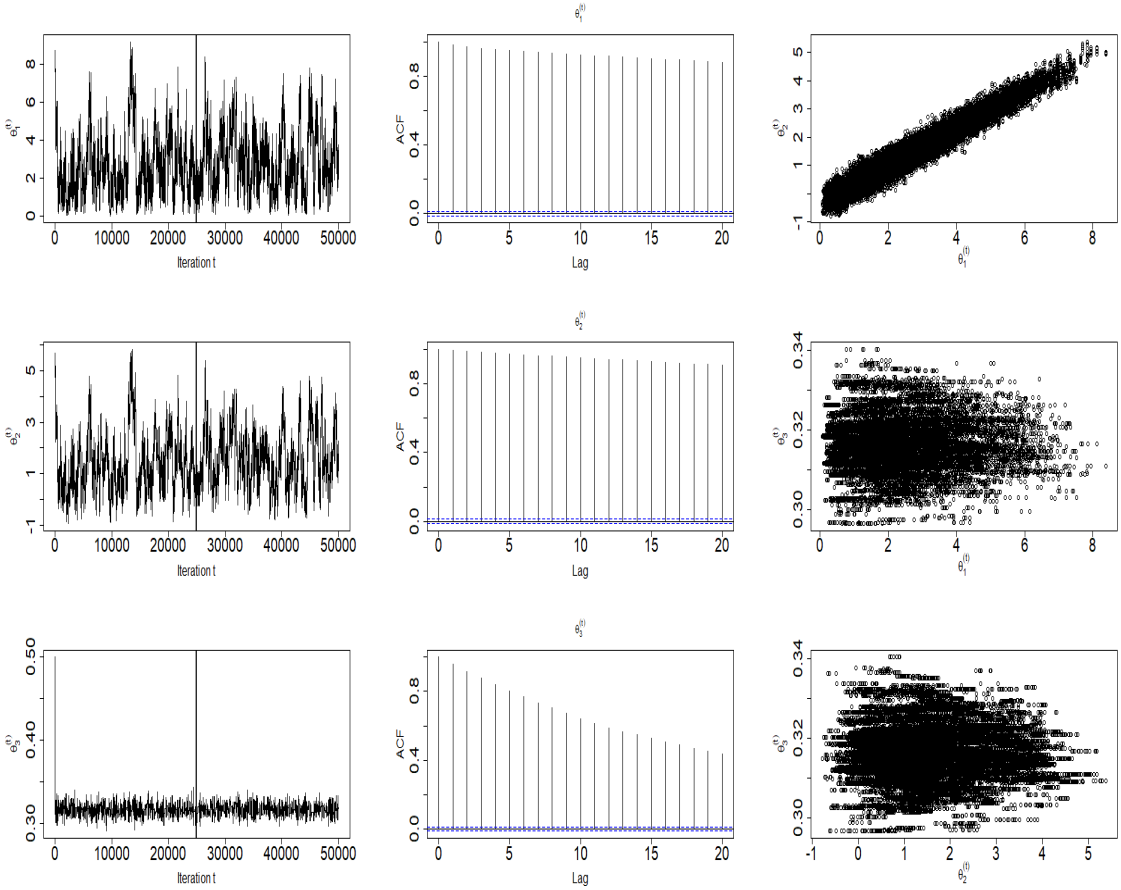

Figure 3.4: shows the results obtained from the MCMC algorithm where the first column represent the posterior distribution. The second column represent Auto-correlation for our parameters. Finally , the third column represent the spread of values between $\theta_1, \theta_3,$ and θ_3.

Figure 3.4 we assume that, the θ_1, θ_2 distributed as log normal and θ_3 distributed as inverse gamma as the prior distributions. It is good estimating using MCMC algorithm especially θ_1, θ_3. Also, the Autocorrelation for θ_3 is very good but the parameters θ_1, θ_2 is not good comparing to θ_3. Therefore, it can be said that the Bayesian estimate of the free CIR process is good. Finally, we can say that the posterior distribution of these parameters is stable. In addition, the MCMC estimation is better than MLE.

Applied Part

3.3- Two Sample Kolmogorov-Smirnov test

Two- Sample Kolmogorov- Smirnov test is one of the most popular distribution in statistics. it was named after its discoverer "Andrey Kolmogorov and Nikolai Smirnov", is a Non-parametric test of the equality of continuous and non-continuous that can be used to compare two samples. Assuming that the null hypothesis H_0 assume a normal distribution of data, as for the alternative hypothesis H_1 assumes that the data is semicircle distribution.

Table 3.3: present the test Two- Sample Kolmogorov- Smirnov of normality distribution.

Data	d.f	p-value	The decision
X and Y	**1**	$\mathbf{2.2e^{-16} \approx 0}$	**Rejected** H_0

Table 3.3 present the Two- Sample Kolmogorov-Smirnov test of normal distribution of data. We notice through P-value is less than 5% , we and rejected the null hypothesis H_0 and accepted the alternative hypothesis H_1. That's mean the data are semicircle distribution.

3.4- The Result

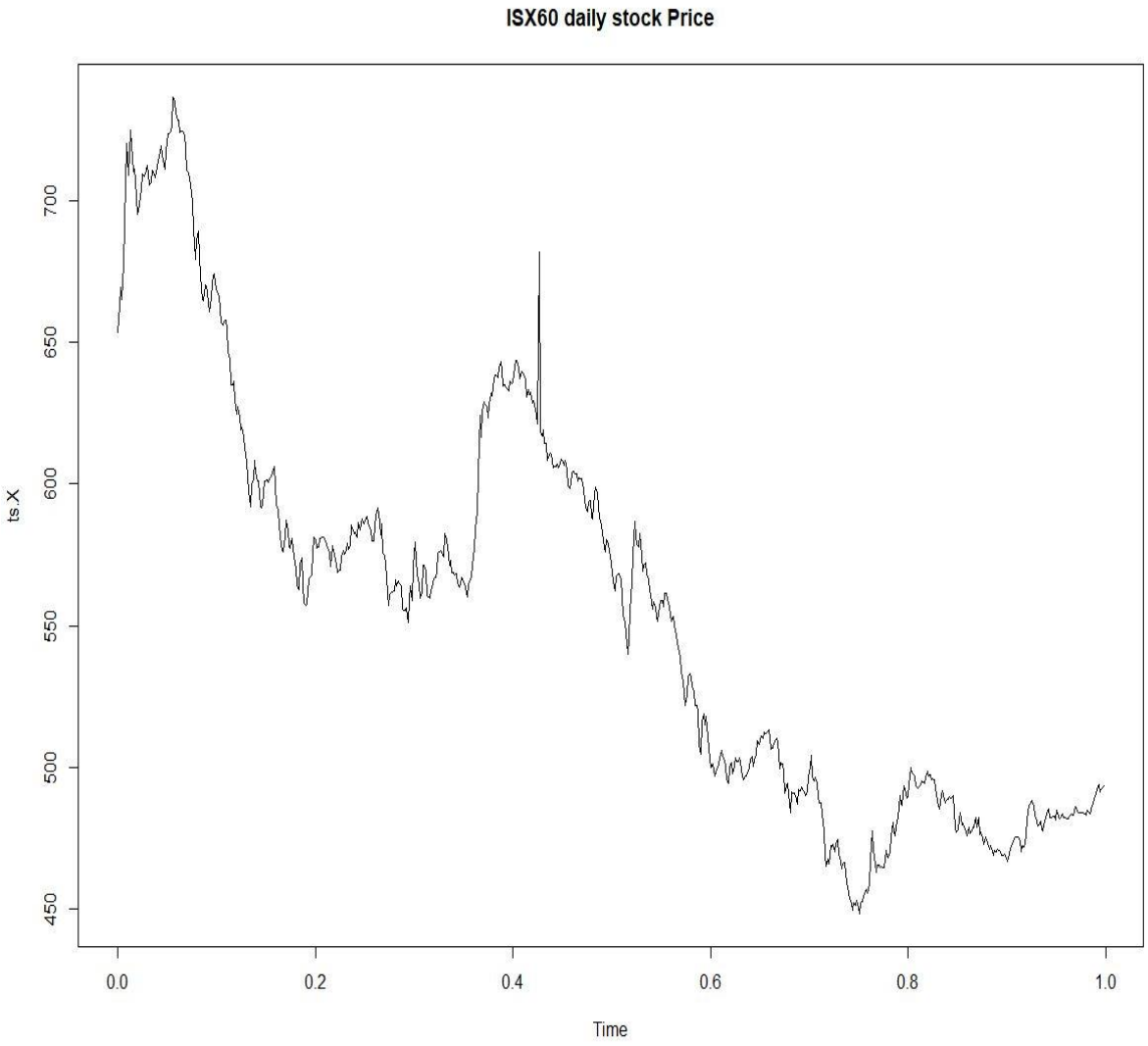

Figure 3.5: show the movement of index (ISX60), and present the path the CIR process for Iraq's stock market index for the period 2017 to 2019.

Table 3.4: shows estimation of parameters by using MLE for real data

Parameters	Coefficients Estimate	Standard Error
θ_1	8.726077	4.83431672
θ_2	5.465292	3.24816056
θ_3	0.500000	0.03590596

-2 Log .Likelihood = -5662.999

Table 3.4 shows estimation of parameters by using MLE estimation .Thus, the MLE estimation is so closer to the initial value. Therefore, it means that the MLE was able to estimate the parameters well, and the standard error was low especially since the θ_3 was a very good estimate because the S.D is its value 0.03590596.

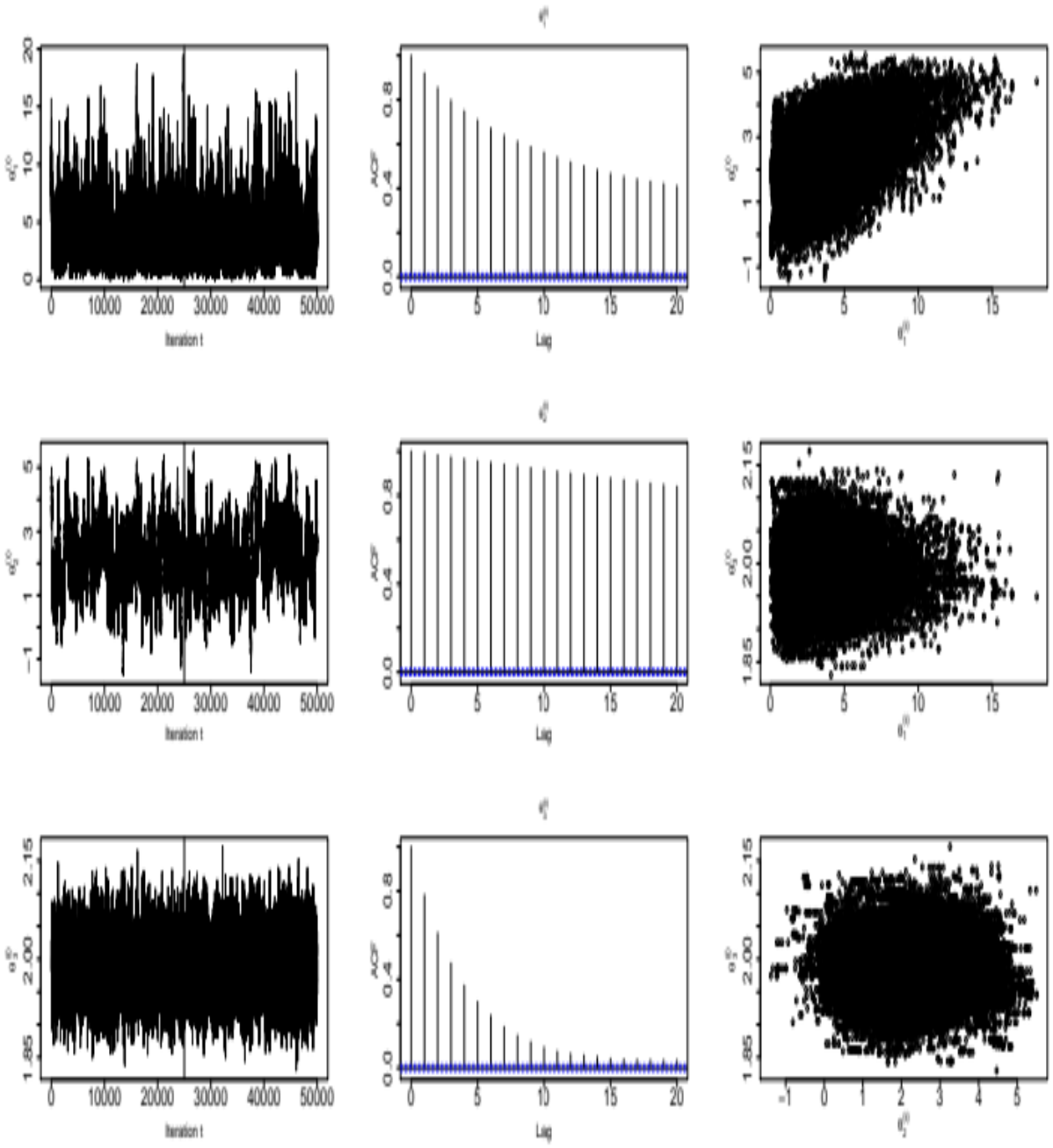

Figure 3.6: show the movement of index (ISX60), and present the Markov Chain Monte Carlo for Iraq's stock market index for the period 2017 to 2019.

Chapter Four
(Conclusion & Recommendations)

4.1 Introduction

In this chapter, we are going to mention the conclusion and the recommendations that the researcher reached through the theoretical part , the Simulation and Applied part.

4.2 Conclusion

By employing the methods presented in the unite three index real data from 2017 to 2019 Iraq stock exchange, to estimate the parameters of the short term interest rate CIR and Free CIR process. After explaining the importance of these methods and their advantages. We used two methods to estimate the parameters of the CIR process and Free CIR process which are Maximum Likelihood estimation (MLE) method and Metropolis-Hasting MCMC algorithm. We conclude from our study the following :-

1- In table 3.1, when we estimated the CIR process driven by Brownian motion in the method maximum likelihood estimation (MLE). We note that the standard error of parameter θ_3 is 0.0444 so the parameter θ_3 is the best among the other parameters θ_1, θ_2.

2- In table 3.2, when we estimated the CIR process driven by Free Brownian motion in the method maximum likelihood estimation (MLE). We note that the standard error of parameter θ_3 is 0.135 so the parameter θ_3 is the best among the other parameters θ_1, θ_2.

3- The Bayesian estimation method was the best for CIR and Free CIR process, and there is a very clear convergence in the results of the other method.

4.3 Recommendations and Future works:

The work in this thesis focuses on some statistical methods previously mentioned. The main contributions of the thesis and possible future research are summarized as following :

1. Given the importance of the subject, we find it necessary to conduct more studies that study these models on other companies listed on the Iraq stock exchange.

2. We have recently found that it can be very useful to extend the Lévy process to CIR process.

3. In future studies, it is recommended to

a. use other short term interest rate models such as Hull white model, vasicek model in free setting.

b. use the two, multi- factor CIR process in free setting.

c. use Gibbs Sampling Markov Chain Monte Carlo for CIR and Free CIR process.

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Appendix A.

The Exact solution of the Cox Ingersoll Ross (CIR) process

$$dr_t = (\theta_1 - \theta_2 r_t)dt + \theta_3\sqrt{r_t}\, dW_t \qquad \ldots(1)$$

$f(r_t,t) = e^{\theta_2} r_t$ ...(2)

We will get the following derivatives :

$$\frac{\partial f(r(t),t)}{\partial t} = \theta_2 e^{\theta_2} r_t \qquad \ldots(3)$$

$$\frac{\partial f(r(t),t)}{\partial r_t} = e^{\theta_2 t} \qquad \ldots(4)$$

$$\frac{\partial^2 f(r(t),t)}{\partial {r_t}^2} = 0 \qquad \ldots(5)$$

$f(r_t,t) = f(0,r_0) + \int_0^t \theta_2 e^{\theta_2 u} r_u du + \int_0^t e^{\theta_2 u} dr_u$

$= r_0 + \int_0^t \theta_2 e^{\theta_2 u} r_u du + \int_0^t e^{\theta_2 u}[(\theta_1 - \theta_2 r_u)du + \theta_3\sqrt{r_u}\, dW_u\,]$

$= r_0 + \int_0^t \theta_2 e^{\theta_2 u} r_u du + \int_0^t \theta_1 e^{\theta_2 u} du - \int_0^t \theta_2\, e^{\theta_2 u} r_u du + \int_0^t \theta_3 e^{\theta_2 u}\sqrt{r_u}\, dW_u$

$= r_0 + \int_0^t \theta_1 e^{\theta_2 u} du + \int_0^t \theta_3 e^{\theta_2 u}\sqrt{r_u}\, dW_u$

$= r_0 + \theta_1 \frac{e^{\theta_2 t} - 1}{\theta_2} + \int_0^t \theta_3 e^{\theta_2 u}\sqrt{r_u}\, dW_u$

$= r_0 + \frac{\theta_1}{\theta_2}\left(e^{\theta_2 t} - 1\right) + \int_0^t \theta_3 e^{\theta_2 u}\sqrt{r_u}\, dW_u$

$= r_0 + \frac{\theta_1}{\theta_2} e^{\theta_2 t} - \frac{\theta_1}{\theta_2} + \int_0^t \theta_3 e^{\theta_2 u}\sqrt{r_u}\, dW_u$

from which we obtain:

$$\frac{\theta_1}{\theta_2} + (r_0 - \frac{\theta_1}{\theta_2})e^{-\theta_2 t} + \int_0^t \theta_3 e^{\theta_2 t}\sqrt{r_t}\, dW_t$$

الخلاصة

تعد نماذج مصطلح هيكل اسعار الفائدة اصعب جزء في الرياضيات المالية الحديثة بسبب التعقيد النسبي للنموذج . اصبحت عملية Cox Ingersoll Ross مهمة ومستخدمة على نطاق واسع , انها صيغة رياضية ذات عامل واحد تعتمد على المعادلة التفاضلية العشوائية التي يكون فيها احد العوامل هو عملية عشوائية مما يعطى حلا يعرف باسم العملية العشوائية كما قدمه John Carrington Cox, Jonathan Edwards Ingersoll and Stephen Alan Ross في عام 1985. تعتبر هذه العملية واحدة من اكثر العمليات شهرة في دراسة الصناعة المالية وتصف تطور مصطلح هيكل اسعار الفائدة , وتحدد كيفية تطور سعر الفائدة بسبب التقلبات الحالية ومتوسط معدلات الانتشار. فقد اصبح موضوعا للعديد من الدراسات والاضافات الحديثة.

بالإضافة الى ذلك, نقدم عملية Cox Ingersoll Ross في اطار نظرية الاحتمالات الحرة التي قدمها Voicelescu Dan في عام 1985 وتسمى Free Cox Ingersoll Ross (Free CIR).

جاءت هذه الرسالة بهدف تقدير معلمات CIR و Free CIR باستخدام طريقة تقدير الامكان الاعظم Maximum Likelihood Estimation (MLE) وطريقة Metropolis- Hasting التي تعتبر احد الانواع الشائعة من طريقة Markov Chain Monte Carlo (MCMC). نستخدم البيانات الحقيقة في (ISX60) , وقد تم تحليل بيانات الدراسة باستخدام لغة البرمجةR.

جمهورية العراق

وزارة التعليم العالي والبحث العلمي

جامعة القادسية

كلية الادارة والاقتصاد

التقدير البايزي لعملية CIR مع التطبيق المالي

رسالة مقدمة الى

مجلس كلية الادارة والاقتصاد/ جامعة القادسية

وهي جزء من متطلبات نيل درجة الماجستير في علوم الاحصاء

من الطالبة

ريام ابو الهيل عكلة

إشراف

ا.م.د. مهند فائز السعدون

١٤٤٢هـ القادسية ٢٠٢٢م

البيانات اليومية للأسواق العراقية المالية للعام 2017

653.570	695.890	715.160	724.010	673.240	656.780	619.800	592.330	576.110	558.160	580.150	576.710	588.360	561.250	551.240	560.320
662.850	701.580	718.940	724.430	665.010	655.820	616.730	600.870	581.070	557.300	578.550	578.900	586.760	557.110	558.780	559.800
669.360	703.810	715.970	724.220	664.460	657.410	612.430	601.120	587.070	557.540	577.770	576.890	585.120	561.210	564.160	562.390
665.200	709.240	712.190	722.950	670.020	657.780	607.890	601.300	583.760	566.070	575.840	578.240	583.680	562.030	559.060	564.970
676.940	708.250	710.930	718.150	669.570	655.180	602.880	600.390	578.300	567.200	570.980	585.470	580.040	561.880	570.980	566.700
692.190	709.390	717.420	710.740	667.420	646.500	597.490	601.890	577.520	568.090	578.400	583.610	580.040	562.390	579.460	567.200
720.180	712.270	723.420	709.860	660.800	643.850	591.880	602.860	580.640	575.610	576.660	582.720	584.910	566.140	573.280	569.070
714.790	709.580	723.570	708.000	662.390	634.980	599.920	603.850	577.010	581.040	575.200	582.950	589.040	564.080	568.430	575.510
709.050	705.510	724.260	702.320	670.170	634.890	601.500	606.200	572.040	580.230	571.930	581.330	591.540	565.720	564.800	576.040
724.920	706.320	726.410	699.330	673.680	636.020	608.090	600.140	570.270	577.160	568.670	586.270	589.750	565.070	559.860	576.600
719.560	710.740	736.330	690.080	674.090	629.470	603.800	592.190	564.190	577.980	569.470	583.960	582.130	564.130	561.960	575.210
709.940	709.380	734.840	679.100	668.640	624.800	600.920	590.900	562.910	580.640	569.730	585.840	585.830	556.260	571.320	574.330
710.820	707.820	730.830	686.020	667.910	627.210	600.850	584.980	571.590	580.840	574.510	587.600	575.570	555.620	571.550	582.620
707.110	711.800	728.310	689.070	666.430	623.630	591.870	578.660	574.080	581.110	576.580	586.030	574.890	555.600	569.060	580.540
695.140	713.840	728.310	680.480	661.850	618.850	591.440	577.020	564.980	581.250	575.120	587.220	572.050	556.300	560.490	

البيانات اليومية للأسواق العراقية المالية للعام 2018

577.84	564.26	628.20	636.09	632.57	610.49	603.84	587.86	563.53	587.01	556.70	546.33	521.79	499.99	502.25	507.98
571.22	566.37	631.99	635.17	628.57	605.65	604.46	596.76	562.28	581.80	553.38	544.28	520.39	501.35	503.66	509.68
572.89	567.52	631.05	635.94	629.22	606.31	603.42	598.66	567.51	578.49	551.44	542.66	507.74	503.41	502.38	511.34
568.95	574.57	637.18	640.84	626.32	606.35	603.68	597.18	568.41	577.63	556.54	539.11	504.72	506.15	496.90	510.50
568.93	576.85	638.45	643.61	624.33	606.85	601.00	592.17	568.49	582.44	559.05	533.55	516.24	504.01	495.66	512.39
568.04	584.32	638.46	643.39	621.01	605.61	602.43	588.10	566.42	574.68	558.86	530.60	519.11	502.50	496.83	511.96
568.49	590.43	637.66	640.94	681.63	607.01	601.90	585.48	560.78	569.21	556.79	526.33	515.24	500.55	497.22	512.66
564.85	603.48	641.05	637.14	617.98	608.66	601.98	582.40	553.61	571.17	561.44	522.05	517.99	495.87	498.02	513.26
563.61	624.10	643.11	639.56	616.95	607.91	597.94	577.90	551.33	572.02	561.38	525.96	511.37	494.44	499.48	510.12
564.39	616.46	640.48	639.26	618.84	606.52	594.27	575.94	547.30	568.52	559.35	532.33	507.07	499.92	503.19	
566.87	625.90	634.60	638.77	614.16	608.17	591.85	580.42	540.07	565.99	556.53	533.21	499.94	501.76	503.86	
565.65	629.11	635.07	636.80	614.20	605.39	590.37	579.06	545.12	562.61	553.79	532.11	500.24	497.68	500.52	
565.08	628.13	633.99	630.58	608.36	599.36	593.78	576.67	555.29	560.90	551.77	528.81	501.21	498.87	502.79	
561.98	627.26	633.11	633.17	610.51	598.40	593.98	572.01	565.28	555.70	553.22	526.75	497.08	503.29	504.85	
560.38	623.20	632.64	631.50	610.76	599.95	588.73	568.50	575.42	558.45	550.37	522.08	498.21	502.22	509.37	

البيانات اليومية للأسواق العراقية المالية للعام 2019

506.32	491.49	497.04	472.75	452.46	468.47	476.06	493.46	488.29	478.01	482.24	470.67	474.14	481.14	482.75	484.27
506.92	490.99	495.16	470.28	451.45	463.20	480.98	492.88	486.57	481.70	476.19	470.16	470.29	478.32	483.77	484.26
508.62	491.16	496.48	472.94	453.03	465.64	482.21	493.29	485.36	484.23	477.12	468.94	472.37	477.72	482.43	483.80
509.97	488.80	494.25	474.73	450.58	465.55	490.32	494.20	490.00	479.87	475.53	468.97	471.90	481.32	482.25	483.26
510.44	487.24	490.28	469.59	448.55	464.61	487.14	495.17	491.71	480.09	473.05	469.31	472.96	482.42	482.06	484.89
508.97	492.25	487.74	468.66	452.20	465.11	486.56	495.02	489.04	478.59	475.58	468.15	481.07	484.79	482.09	483.91
499.65	491.77	487.73	464.16	452.97	464.58	493.57	494.48	487.68	477.04	474.83	467.09	484.58	485.20	482.74	483.77
501.82	492.97	483.65	466.29	455.11	464.63	492.34	496.45	488.43	475.86	473.21	468.00	486.72	482.22	483.69	486.80
501.15	491.69	477.50	466.61	456.30	470.62	489.08	498.55	488.68	478.92	471.13	471.34	487.54	482.22	483.41	487.22
498.27	491.42	469.00	463.02	457.20	468.63	489.74	496.89	489.52	476.73	472.42	472.47	488.42	482.67	483.19	488.93
491.07	490.28	465.13	459.82	455.81	468.25	494.05	497.58	489.10	477.66	470.62	473.80	486.62	482.86	486.47	490.89
493.39	491.40	467.92	455.56	458.90	470.13	499.95	495.64	489.58	479.05	469.20	474.81	483.34	481.74	485.80	492.31
494.33	496.87	466.07	453.89	467.61	475.81	498.19	496.31	489.95	478.88	470.91	475.51	482.47	484.95	484.26	493.93
487.40	500.06	472.43	451.81	477.51	480.66	497.30	495.55	479.03	482.24	470.47	475.48	479.48	482.72	484.30	491.50
484.12	504.15	471.95	449.67	470.65	477.36	496.81	492.74	477.10	478.88	471.08	475.33	479.94	481.80	484.31	492.25
493.20															
493.76															