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Free Cox Ingersoll Ross Process (Free CIR)

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Abstract: *The Cox Ingersoll Ross (CIR) process is consider recently one of the most important models in finance for the modeling the term structure of interest rate as constructed in 1985. In this paper, we mainly present the CIR process in the context of free probability theory which was introduced by D. Voicelescu in 1980.The important part in this paper is transformation from CIR process driven by Brownian motion to CIR process driven by free Brownian motion is called Free CIR process. Then, we estimate the free CIR process using Maximum Likelihood Estimation (MLE) method.it was concluded that the MLE method works well.*

Keywords - *Cox Ingersoll Ross(CIR) process, Free Brownian motion, Free probability, Free stochastic, Free Ito formula, Free CIR process.*

I. INTRODUCTION

In this paper , we present the CIR stochastics process with noncommutative random variable in a free probability theory framework which call Free CIR process. Free probability is a recent mathematical theory, studies noncommutative random variables. It was introduced at the beginning of the 1980s by Dan Virgil Voiculescu. The subject was relatively slow in its infancy , but it aroused the interest of many later. Its communicated with many topics as random matrices, will be transformation the CIR process driven by Brownian motion with Feller condition which guarantees a positive solution of CIR process to Free CIR process which is CIR process driven by Free Brownian motion.

Our problem is to find the best estimates for the parameters in the Free CIR process. To this process is considered one of the most complex random process at the current time. find the best estimates for these parameters using Maximum Likelihood Estimation (MLE) method. This method has a high estimation efficiency and effective in the cases of large samples.

Our aim is to use Bayesian estimators to find parameters estimates for CIR process using Markov Chain Monte Carlo (MCMC), where this process is considered a rather complex process because it contains the square root for a random variable whose values cannot be determined and can take negative values.

The rest of this article is arranged as follows: Section II introduces the CIR process. We present the Free CIR process in section III . In IV the simulation will be presented. Finally the conclusions are in section V.

II. COX INGERSOLL ROSS PROCESS (CIR)

Term structure of interest rates models are relevant and useful applications for grasp interest rate dynamics. Recently, it has been applied in a now widely using in finance field as modelling interest rates and stochastic volatility that are common applications. It was presented by John Carrington Cox, Jonathan Edwards Ingersoll and Stephen Alan Ross in 1989 [1] . In this process the short rate r_t is present by the stochastic differential equation (1) [2]:

$$dr_t = (\theta_1 - \theta_2 r_t)\, dt + \theta_3 \sqrt{r_t}\, dW_t \qquad ...(1)$$

where:-

θ_1 is the speed of mean reversion

θ_2 is the long term mean value

θ_3 is the volatility

r_t is a short rate

W_t is Brownian motion.

The CIR process remains non-negative, show that whenever $2\theta_1\theta_2 > \theta_3$, the interest rate is strictly larger than zero. Furthermore, there is empirical evidence that whenever interest rates are high, the volatility is likely to be high as well, which justifies the volatility term in the CIR process. The distribution function of r(t) is the non-central chi-square with 2q+2 degrees of freedom and parameter of non-centrality 2u proportional to the current spot rate. The probability density of the interest rate at time s, conditional on its value at the present time t as[1]:

$$f(r(s),s\,;\,r(t),t) = ce^{-u-v}\left(\frac{v}{u}\right)^{\frac{q}{2}} I_q\left(2(uv)^{1/2}\right) \quad \ldots (2)$$

where

$$c = \frac{2k}{\sigma^2(1-e^{-k(t-s)})} \quad , \ldots(3),$$

$$u = cr_t\, e^{-k(t-s)} \quad , \ldots(4),$$

$$v = cr_s \quad , \ldots(5),$$

$$q = \frac{2k\theta}{\sigma^2} - 1 \quad , \ldots(6)$$

The conditional expected and the conditional variance of the CIR process are [2] :-

$$E[r(s)|r(t)] = e^{-\theta_1(s-t)}\, r(t) + \theta_2\left(1 - e^{-\theta_1(s-t)}\right)$$

$$Var[r(s)|r(t)] = \theta_3\left(\frac{e^{\theta_1(s-t)} - e^{-2\theta_1(s-t)}}{\theta_1}\right) r(t) + \theta_3\frac{\theta_2}{2\theta_1}\left(1 - e^{-\theta_1(s-t)}\right)^2$$

1. Free probability Theory

Then, we present the CIR process stochastics process with noncommutative random variable in a free probability theory framework which call Free CIR process. Free probability is a recent mathematical theory, studies noncommutative random variables. It was introduced at the beginning of the 1980s by Dan Virgil Voiculescu. The subject was relatively slow in its infancy , but it aroused the interest of many later. Its communicated with many topics as random matrices.

We will transform the CIR process driven by Brownian motion with Feller condition which guarantees a positive solution of CIR process to Free CIR process which is CIR process driven by Free Brownian motion ,We will present some main definitions related to Free CIR process as follows:-

Free probability theory was presented in 1980 by Dan Voiculescu and since than it has been the focus of much attention because of its application with large size random matrices, used to calculate the moments or distribution of noncommutative random variable. It based on noncommutative probability space and freeness, it includes huge mathematical concepts and symbols , and is difficult for the public reader to understand without the following concepts [3] :-

- Non-commutative probability space is included (A, φ) where A is unite algebra , φ is Linear function φ : A → C, (Assume that φ(1) =1). Noncommutative probability space which is the element of A , the numbers φ $(a_{i(1)}, \dots, a_{i(k)}$ for random variables $a_1, \dots, a_m \in$ A are named moments, the non-commutative distribution of $a_1, \dots, a_m$ which is the set of moments [4].

- let (A, φ) be a non-commutative probability space and let I be an index set [4]. The sub algebras $(A_i)_{i \in I}$ is called free if $\varphi(a_1 \cdots a_k) = 0$, i=1,…,k , where k is a non-negative integer, $a_j \in A_{i(j)}$, where i(j) ∈ I , and any two neighboring indices i_1 and i_{1+1} are not equal, i.e. $i_1 \neq i_2,\ i_2 \neq i_3, \dots, i_{k-1} \neq i_k$.

- The random variables $(x_i)_{i \in I}$ are free, if their generated unite sub algebras are free, it means if $(A_i)_{i \in I}$ are free, where, for each i ∈ I , A_i is the sub algebra of A which is generated by x_i.

2. Free Stochastic Integral

Free stochastic integral is a section of mathematics. It is used as a solution to free stochastic differential equations. Indeed, important free stochastic processes to which free stochastic calculus is applied is free Brownian motion , which is used to model free Brownian motion. The free stochastic integral as follows:

$$I = \int_0^1 a_t (dB_t) b_t$$

Where :

a_t and b_t are operator coefficients , B_t is free Brownian motion , suppose that $a_t$and $b_t$are functions of B_τ, τ ≤ t. That is, let $a_t$and b_t belong to the sub algebra B_τ. Suppose that $\max\{\|a_t\|, \|b_t\|\} \leq C$ for all t ∈ [0, 1] and that t → a_t, b_t are continuous in the operator norm. Let $t_0, \dots, t_n$ and $\tau_1, \dots, \tau_n$ be real numbers as [5]

$$0 = t_0 \leq t_1 \leq \cdots \leq t_n = 1$$

And

$$0 \leq t_k \leq t_{k-1}$$

Indicate the set of $t_0, \dots, t_n$ and $\tau_1, \dots, \tau_n$ as Δ , so

$$d(\Delta) = \max_{1 \leq k \leq n} (t_k - \tau_k)$$

Consider the sum,

$$I(\Delta) = \sum_{i=1}^{n} a_{\tau i} (B_{t_i} - B_{t_{i-1}}) b_{\tau i}$$

It turns out that as d(Δ) → 0, the math problems I(Δ) converge in operator norm and the limit does not consist on the choice of t_i , τ_i. The limit is named "the free stochastic integral" .

3. Free Ito Formula

Let a_t,b_t,c_t, d_t be operator-valued functions and $(W(t))_{t \geq 0}$ is a Brownian motion . Then [6][7][8]:

$a_t dt \cdot b_t dt = a_t dt \cdot b_t dW(t)\, c_t = a_t dW(t)\, b_t \cdot c_t\, dt = 0$

$a_t dW(t) bt \cdot c_t dW(t) dt = E(b_t c_t)\, a_t d_t dt$

4. Free Brownian motion

A free Brownian motion is a stochastic process given by a group $(B_t)_{t\geq 0} \subset (A,\varphi)$, where A is von Neumann algebra, φ is faithful trace. It has the following properties [9] :-

1. $B_t = 0$, initial value is zero.

2. increment B_t- B_s (s < t) follows semicircular distribution with expectation = 0 and variance = t-s .

$$d\mu B_t - B_s(x) = \frac{1}{2\pi(t-s)}\sqrt{4(t-s) - X^2}dx$$

3. disjoint increments are free for $0 < t_1 < t_2 < \cdot\cdot\cdot < t_n$,

$B_{t_1}, B_{t_2} - B_{t_1}, \dots, B_{t_n} - B_{t_{n-1}}$ are free .

Perhaps it is appropriate to present the Semicircle distribution with some properties.

- The Semicircle Distribution

The semicircle distribution is one of the important distributions that have a leading role in the stochastic matrices. The semicircle distribution plays an important role in many areas of mathematics, also appears in physics. It is consider the limiting distribution of Markov Chain of young diagrams and is the limiting distribution in the free version of central limit theorem. It is the free analogue of the normal distribution and also called the " Wigner distribution " in recognition of the physicist Eugene who did a great job on stochastic matrices [10] [11].

The semicircle distribution given by its probability density function (Pdf) : [7]

$$f(X) = \frac{2}{\pi R^2}\sqrt{R^2 - X^2} \qquad X \in [-R, R]$$

The pdf of the semicircle distribution has the properties :

be symmetric when X = a ,

at mode X = a is increases and then decreases with mode at X=a .

f is down concave .

The cumulative distribution function (CDF) for a distribution is known by the following :

$$F(x) = \frac{1}{2} + \frac{X\sqrt{R^2 - X^2}}{\pi R^2} + \frac{\arcsin\left(\frac{X}{R}\right)}{\pi}$$

The expectation of a distribution given by :

$$E(X) = 0$$

The variance of semicircle distribution as the following :

$$\mathrm{Var}\,(X) = \frac{R^2}{4}$$

The median of semicircle distribution given by the following :

$$F(M) = 0$$

The mode of a distribution is :

$$\mathrm{Mode} = 0$$

III. FREE CIR PROCESS (FREE CIR)

As we know the CIR equation is : [2]

$$dr_t = (\theta_1 - \theta_2 r_t)\, dt + \theta_3 \sqrt{r_t}\, dW_t$$

The main challenge here is to prove that the feller condition ensures the positive solution for free SDE in Free CIR.

$$dr_t = (\theta_1 - \theta_2 r_t)\, dt + \frac{\theta_3}{2} \sqrt{r_t}\, dB(t) + \frac{\theta_3}{2}\, dB(t)\sqrt{r_t}$$

condition, we will follow some step as:-

Firstly **:** [9][10][11]:

We start showing existence and uniqueness of these free SDEs by the following Stochastic differential equation with additive Brownian motion term , also called " square root process".

$$dV(t) = \left(\theta_1(t) - \frac{{\theta_3}^2(t)}{2}\right) \frac{1}{2}\, V^{-1}(t)) - \frac{1}{2}\theta_2 V(t) dt + \frac{\theta_3(t)}{2}\, dW(t)$$

where :

$\theta_1, \theta_2, \theta_3 > 0$, $t \in [0,\infty[$

$(W(t))_{t\geq 0}$ is a Brownian motion, $(V(t))_{t\geq 0}$ is a Von Neumann algebra-valued or vector-valued .

Secondly : [9][10][11]:

Then we will transform this process with two methods :

- first case transform the square root process into the setting of commutative function spaces by the following theorem (1).

Theorem (1) Let $\theta_1 \subset [0,\infty[$ be compact. Let $\theta_1, \theta_2, \theta_3 : [0,\infty[\to C(K)_+$ be continuous, feller condition ($2\theta_1 \geq {\theta_3}^2$) holds and let $\hat{V}_0 \in C(K)_+$. Then the SDE [7][8][9] :

$$d\hat{V}(t) = ((K(t) - \frac{{\theta_3}^2(t)}{2})\,\frac{1}{2}\,\hat{V}^{-1}(t) - \frac{\theta_2(t)}{2}\hat{V}(t))dt + \frac{\theta_3(t)}{2}\;1_k dW(t)\;\;, \hat{V}(0) = \hat{v}_0$$

for $t \in [0,\infty[$, has a global solution $V \in C([0,\infty[\,, L_2(p_\varphi, C(K)_+))$.

Proof : Consider a Brownian motion $(W(t))_{t\geq 0}$ on the probability space (Ω, F, p_φ). Then using point mass measures δ_k for $k \in K$ and the classical Feller condition prove the global existence of

$$d\hat{V}_k(t) = ((\theta_{1_k}(t) - \frac{\theta_{3_k}(t)^2}{2})\,\frac{1}{2}\,\hat{V}_k^{-1}(t) - \frac{1}{2}\theta_{2_k}(t)\hat{V}_k(t)dt + \frac{\theta_{3_k}(t)}{2}\;dW(t)\;\;, \hat{V}_k(0) = \hat{v}_{0,k}\;, \text{for } t \in [0,\infty[.$$

Using a countable dense subset $\tilde{K} \subset K$ and the point mass functional we show that the paths keep positive except at a P-zero set $N := \cup \cup_{k\in\tilde{K}} N_k$. So for all $\omega \in \Omega \setminus N$ the paths stay positive on $[0,\infty[$ for all k.

- Second case transform the square root process into the setting of the general non-commutative von Neumann algebra –valued case in the theorem (2). By employing the functional calculus and The spectral theorem we have an isometric homomorphism

$T : B(\sigma(V_0)) \to \langle V_0, id\rangle$
where :

$\langle V0, id\rangle$: is the von Neumann algebra generated by V0 and the identity

$B(\theta_3\,(V_0))$: is the function space of bounded, measurable functions on $\theta_3\,(V_0)$, the spectrum of V_0. If φ is a until, faithful trace, then consider

$$Ep_\varphi = T*(\varphi) \text{ with the identity}$$

$$Ep_\varphi(g) = \int g\,d\,p_\varphi = \varphi(T(g)) \quad , \text{for all } g \in B(\sigma(V_0))$$

Theorem (2) Let $V_0 \in A_+$ and let $\theta_1, \theta_2, \theta_3 : [0,\infty[\to\langle V0, id\rangle_+$ Feller condition ($2\theta_1\theta_2 \geq {\theta_3}^2$) holds for all $t \in [0,\infty[$. Then the SDE [9][10][11]:

$$d\overline{V}(t) = ((\theta_1(t) - \frac{{\theta_3}^2(t)}{2})\,\frac{1}{2}\,\overline{V}^{-1}(t) - \frac{\theta_2(t)}{2}\overline{V}(t))dt + \frac{\theta_3(t)}{2}\;dW(t)\;\;, \overline{V}(0) = v_0$$

for $t \in [0,\infty[$, has a global solution in $V \in C([0,\infty[, L_2(p_\varphi, A_+))$. Note that id is the unit in the corresponding von Neumann algebra.

Proof : Let $\hat{V}(t)$ be a global solution in $C(K)_+$ by Theorem 1. By the functional calculus we see that $T(\hat{V}(t))$ is a positive solution to the above equation under the Feller condition, where $T : C(K) \to \langle V0, id\rangle$ is the functional calculus mapping for the self-ad joint element V_0. In particular we have that $\overline{V}(t) \in \langle V0, id\rangle_+$ for all $t \in [0, \infty[$. After these two cases the process still with a classical Brownian motion as driving process.

Proposition. Let T > 0 and let $V \in C([0,T[,A) \cap C([0,T], L_2(\varphi))$ be a free Itô process and let $\overline{V} \in C([0,T],L_2(P_\varphi,A))$ be a vector-valued Itô processes. Suppose for all $t \in [0,T[$ and all orthogonal projections p free of V (t) that we have

$\| pV(t)^2p \|_1 = \| p\overline{V}(t)^2p \|_1$

[1] If $V(t) \in A$ and $V(T) \in L_2(\varphi)$ then $V(T) \in A$.

[2] If $\overline{V}(t)$ is invertible for all $t \in [0,T]$, V (t) is invertible for all $t \in [0,T[$ and $V(T) \in A$, then $V(T) \in A$ is also invertible.

Theorem (3) Let $V_0 \in A_+$ be given. Let $K : [0,\infty] \to A_+$, $\sigma : [0,\infty] \to \langle V_0, id\rangle_+$ and $\theta > 0$ a constant , Feller condition ($2k\,\theta \geq \sigma^2$)holds. Then the free SDE [9][10][11]:

$$dV(t) = \left(\left(\theta_1(t) - \frac{\theta_3{}^2(t)}{2}\right)\frac{1}{2}\,V^{-1}(t) - \frac{\theta_2}{2}V(t)\right)dt + \frac{\theta_3(t)}{2}\,dB(t) \quad , V(0) = v_0$$

for $t \in [0,\infty[$, has a global solution $V \in C([0,\infty[, A_+)$.

Proof:

1- In the first step we choose a maximal interval [0,T [, where a solution of the equation

$$dV(t) = \left(\left(\theta_1(t) - \frac{\theta_3{}^2(t)}{2}\right)\frac{1}{2}\,V^{-1}(t) - \frac{\theta_2}{2}V(t)\right)dt + \frac{\theta_3(t)}{2}\,dB(t) \quad , V(0) = v_0$$

for $t \in [0,T[$, By Theorem (2) we know that the solution for

$$d\overline{V}(t) = \left(\left(\theta_1(t) - \frac{\theta_3{}^2(t)}{2}\right)\frac{1}{2}\,\overline{V}^{-1}(t) - \frac{\theta_3}{2}\overline{V}(t)\right)dt + \frac{\theta_3(t)}{2}\,dB(t) \quad , \overline{V}(0) = v_0$$

for $t \in [0,\infty[$, exists globally.

2- we know that the process V (t) exists for t < T . The reference process V (t) exists globally with values in A. Because of the isometry we can define

$$V(T) = L_2\text{–}\lim_{t\to T} V_0 + \int_0^T (\tfrac{1}{2}(\theta_1 - \frac{\theta_3{}^2}{2})V^{-1}(t) - \theta_2 V(t)dt + \theta_3{}^2 B(T)$$

Using Proposition 1(1.), we can deduce that $V(T) \in A$

3- Step 2 told us $V(T) \in A$. We want to extend the (unique) solution V (t) beyond T . we need the inevitability of V (T). This would allow us an extension and the solution is global. Again by Proposition 1 (2.), we see that V (T) is invertible. Therefore the solution of square root process is global.

Let $r_t = V(t)^2$. Then according to the free Itô formula:

$$dr_t = (V(t) + dv(t))^2 - (V(t))^2$$

$$= (V(t))^2 + 2\,(V(t))\,dV(t) + (dV(t))^2 - (V(t))^2$$

$$= (dV(t))^2 + 2\,(V(t))\,dV(t)$$

$$= [2\,(\frac{\theta_1 - \frac{\theta_3{}^2}{2}}{2V(t)} - \frac{\theta_2}{2}V(t))V(t) + \frac{\theta_3{}^2}{2}]\,dt + \frac{\theta_3}{2}V(t)\,dB(t) + \frac{\theta_3}{2}\,dB(t)V(t)$$

$$= [2(\frac{\frac{2\theta_1 - \theta_3{}^2}{2}}{2V(t)} - \frac{\theta_2}{2}V(t))V(t) + \frac{\theta_3{}^2}{2}]\,dt + \frac{\theta_3}{2}V(t)\,dB(t) + \frac{\theta_3}{2}\,dB(t)V(t)$$

$$= [\,\frac{4\theta_1 - 2\theta_3{}^2}{4}\;4\,V(t)V(t) - \frac{\theta_2}{2}\;2\,V(t)V(t) + 2\frac{\theta_3{}^2}{2}]dt + \frac{\theta_3}{2}V(t)\,dB(t) + \frac{\theta_3}{2}\,dB(t)V(t)$$

$$= [\theta_1 - \frac{{\theta_3}^2}{2} 4\,V(t)^2 - \theta_2\,V(t)^2 + 2\frac{{\theta_3}^2}{2}]dt + \frac{\theta_3}{2} V\,(t)\,dB(t) + \frac{\theta_3}{2}\,dB(t)V(t)$$

$$= [\theta_1 - \theta_2 V(t)^2]\,dt + \frac{\theta_3}{2} V\,(t)\,dB\,(t) + \frac{\theta_3}{2}\,dB(t)V(t)$$

According to the above hypothesis replace every $V(t)^2 = r_t$ to get the final form of free CIR process as following [7][8][9]:

$$dr_t = (\theta_1 - \theta_2 r_t)dt + \frac{\theta_3}{2}\sqrt{r_t}\,dB\,(t) + \frac{\theta_3}{2}\,dB(t)\sqrt{r_t}$$

We can present the Expected and the variance of the free CIR process as following :-

$$E[r_t] = r_0 e^{-\theta_1 t} + \theta_2\left(1 - e^{-\theta_1 t}\right)$$

$$Var[r_t] = \frac{{\theta_3}^2}{4}\left(\frac{e^{\theta_1 t} - e^{-2\theta_1 t}}{\theta_1}\right)r_0 + \frac{{\theta_3}^2}{4}\frac{\theta_2}{2\theta_1}(1 - e^{-\theta_1 t})^2$$

We conclude from the above expectation and variance the distribution function of Free CIR process is non-central chi-square distribution with and parameters of non- centrality 2u and 2q+2 degree of freedom.

IV. SIMULATION STUDY

To study the behavior of any estimates distributions stochastic statistical models, we use a simulation. Analysis using simulation is one of the important tools that can be employed in formulating and solving mathematical and statistical models , especially during the last decades. The basic idea of simulation is to build a model that simulates the studied phenomenon and then, conducting experiments. A simulation experiment will be described for the purpose of achieving the research. Analytically, the free CIR process ensures an analytical non-negative solution, which makes the free CIR process interest rate model more in line with the actual behavior of interest rates. This makes the free CIR model popular among existing interest rate models. We appreciate the parameters of the mean-reverting model for free CIR process driven by Free Brownian motion.

We use R languages to simulate the CIR process driven by free Brownian motion. In the main model the initial values of the parameters are $\theta 1 = 1$, $\theta 2 = 0.5$ and $\theta 3 = 1$ also that the initial value of the interest rate $r_t = 1.2$.

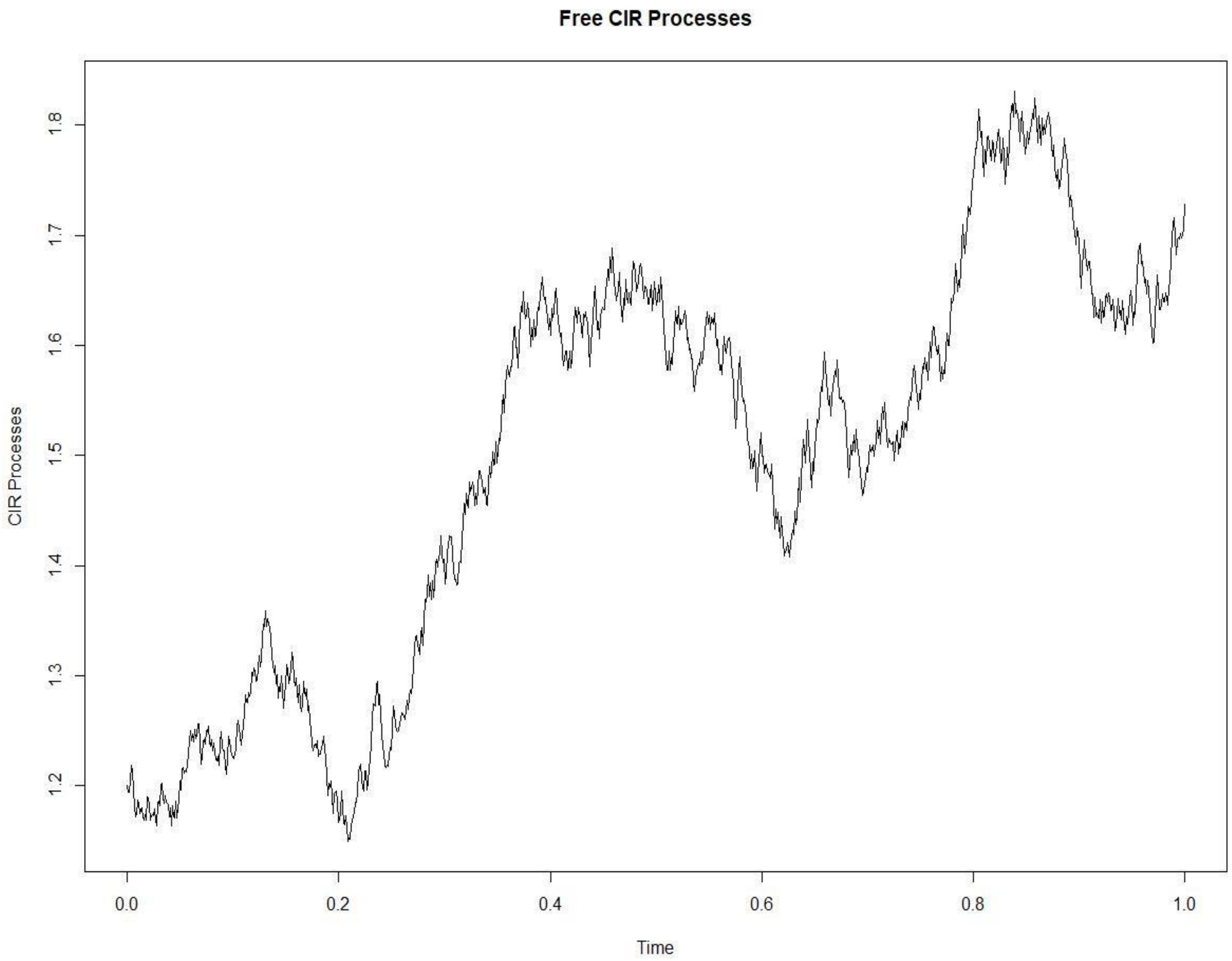

Figure 1**:** shows the Free CIR process in a simulation state when the initial value of parameters are $\theta_1 = 1, \theta_2 = 0.5$ and $\theta_3 = 1$, initial value of r_t =1.2

V. CONCLUSION

We used Maximum Likelihood estimation (MLE) method to estimate the parameters of the Free CIR process, we conclude from the simulation and based on the results of the MSE, we found that the MLE method works well.

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