Simulation Analysis Based on Stochastic Delay Differential Equations

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Abstract: *This paper interested in studying the problem of the numerical solution for the stochastic delay differential equations (SDDE), we conducted few simulation scenarios based on the ordinary Black-Scholes process, and delay Black- Scholes process. It is well known that the finding of explicit solution of an SDDE is very hard to obtain, therefore, strong Euler-Maryama numerical method provided to find the approximation solution for the SDDE. The simulation examples results displayed by the pathways of the approximation solution for the s(t) process under different values of the process par for parameters, also we sketched the histogram of the s(t) that provides the log normal distribution of the solution values. Furthermore, we obtained some important statistics that described the approximation solution.*

Keywords *- Wiener process, SDDE, Euler-Maryama, approximation solution.*

I. Introduction:

Stochastic delay differential equation (SDDE) developed to study the behavior of some system. Generally, SDDE assumed as models to understand the behavior and the structure of some phenomena different area of sciences, such as, finance, economics, medicine, ecology, etc. In real life application modeling the phenomena to study the future behavior required the present information and then .We have what is called the stochastic differential equation in the presence of noise term. See Mao (٢٠٠٧) for more details. But, in many real data analysis, modeling the phenomena by modeling its functions and parameters do not appear their effects instantly of their occurrence, Mao (٢٠٠٧) and Aladagli (٢٠١٧). So based on the past information. The stochastic differential equation becomes realistic in presently the studied phenomenon. Consequently, by adding the time delay term to the SDE. We get what is called the stochastic delay to differential equation. Aladagli (٢٠١٧) referred financial assets. Scheinkman and Lebaron (١٩٨٩) stated that the stocks returns depends on the past information of stocks returns. Jassim (٢٠٠٦) proposed analytical solution for the SDE and SDDE with some examples. Stoica (٢٠٠٥) considered that the trader in the stock prices market follows the Black-Scholes diffusion process, but the insider in the market knows that the average rate of change and the dispersion of returns processes are affected by defined events that occur before the trading process period started. Zheng (٢٠١٥) studied the stochastic delay differential equation (SDDE) in real application of asset pricing through studying the mathematical properties of SDDE ,as well studying the estimation of its parameters from the Bayesian point of view and compared the results with the classical Geometric Brownian Motion model.

Explicit solutions for SDE is hardly to be obtained; which means ,there are no closed from for the analytical solution .Because of that ,various authors put in a lot of effort to find the analytical solution for SDE and SDDE ,see Smith (١٩٩٩), Muszta(٢٠٠٥). For further information .But some authors works of the numerical solution for SDE, Han (٢٠٠٥), and Mahony (٢٠٠٦). Arriojas et al (٢٠٠٧) developed an explicit formula for the solution of the SDDE with pricing European options considering that the underlying stock price follows nonlinear stochastic differential equation. Reiss (٢٠٠٧) studied the adaptive estimation for the SDDE functions. Non parametric inference developed to solve the estimation problem by using the wavelet estimator. Mohammed (١٩٩٨) introduced the stochastic differential equation with memory term (delay) which evolution based on the past in formation of the underlying state .The existence of the unique solution have been proved, as well as , the asymptotic stability for the SDDE have developed. Buckwar (٢٠٠٠) developed numerical solution for the stochastic delay differential equation based on Itô formula. Exploit solution has proposed for the SDDE by illustrating some numerical examples using the Euler maruyama method. Ferrante and Rovira (٢٠٠٦) studied the stochastic delay differential equation through the Fractional Brownian motion with Hurst parameter $> \frac{١}{٢}$. The developed the existence and uniqueness solution of SDDE considering that the coefficient of the SDDEare sufficiently regular.

II. Stochastic Delay Differential Equation :

Suppose that (Ω, F_t, P) be a complete probability space, which is defined to describe any random experiment, with a filtration$(F_t)_{t\geq ٠}$. In this paper we will study the convergence of the Euler numerical method assuming that the functions of stochastic delay differential equation such that,

$$X(t) \in L^{٢}(\Omega, F_t, P), \quad \text{if} \quad E[\sup\|X(t)\|^{٢}] < \infty.$$

See Arnold (١٩٧٤), Karatzas and Shreve (١٩٩١), and Williams (١٩٩١) for the background of probability fundamental the Brownian motion and stochastic differential equation properties. Now, let $T \in [t_{٠} = ٠, \infty]$ and $w(t)$ is the Brownian motion process .One dimensional, thus under the filtered probability space (Ω, F_t, P), let us consider we have the following SDDE,

$$\begin{cases} dx(t) = g(x(t), x(t-\lambda), t)dt + u(x(t), x(t-\lambda), t)dw(t) & ; ٠ \leq t \leq T \\ x(t) = \psi(t) \quad ; \quad -\lambda \leq t \leq ٠ & \ldots\ldots (١) \end{cases}$$

Where $\lambda \geq ٠$ is the time –delay parameter, g and u are functions defined as follows:

$$g: R \times R \times R_+ \to R$$

$$u: R \times R \times R_+ \to R$$

$$R_+ = [٠, \infty)$$

Also, $\psi(t)$ is the initial solution defined as measurable valued random variable defined on $[-\lambda, ٠] \to R^n$ that satisfy the condition$E\|\psi\|^{٢} < \infty$. In equation (١), the function g(.) can be considered as the drift coefficient, u(.) as volatility coefficient, and w(t) is the Wiener process. Moreover, the equation (١) can be rewritten in terms of integration from as follows:

$$\begin{cases} x(t) = x(t_{٠}) + \int_{t_{٠}}^{t} g(x(t), x(s-\lambda)ds + \int_{t_{٠}}^{t} u(x(s), x(s-\lambda)dw(s) \quad ; ٠ \leq t \leq T \\ x(t) = \psi(t)\ for\ -\lambda \leq t \leq ٠ \\ \qquad (٢) \end{cases}$$

The following definitions are necessary for understanding the behavior of SDDE, see Mohammed (١٩٨٤), Mao (١٩٩٧), Stoica (٢٠٠٥), and Zleny (٢٠١٥).

Definition (١): For the SDDE in (١), the strong solution is R-valued stochastic process,

$$x(t): [-\lambda, T] \times \Omega \to R$$

Which is measurable and continuous process? Such that $x|[٠, T]$ is $F_{٠\leq t\leq T}$ is adaptive process and $x(.)$ Satisfies (١) and (٢) almost surely and satisfies the condition of the initial solution $x(t) = \psi(t)$; $t \in [-\lambda, ٠]$. Hence, the solution of the process $x(t)$ is called path-wise unique if

$$P[x(t) = \hat{x}(t)] = ١ \quad ; \forall\, t \in [-\lambda, T]$$

Definition (٢): The functions g and u in (٢) satisfies the local Lipchitz condition if for every integer $i \geq ١$, there is positive constant K_i such that:

$$|g(x_١, y_١, t) - g(x_٢, y_٢, t)| \vee |u(x_١, y_١, t) - u(x_٢, y_٢, t| \leq K_i[|x_١ - x_٢| + |y_١ - y_٢|]$$

and

$|x_١| \vee |y_١| \vee |x_٢| \vee |y_٢| \leq i$; $x_١, y_١, x_٢, y_٢ \in R$,

Where $|x| \vee |y| = \max(|x|, |y|)$.

Definition (٣): The functions g and u in (٢) satisfies the linear growth condition, such that

$$|g(x, y, t)| \vee |u(x, y, t)| \leq K(١ + |x| + |y|)$$

Where K is appositive constant, and $(x, y, t) \in R \times R \times R_+$

Definitions (١) and (٢) provides path-wise unique strong solution to equation (١).

Numerical solution for SDDE

Euler–Maruyama (E-M) is one of most popular numerical method that provides the strong solution for SDDE. E-M employed the stochastic calculus to find the numerical solution for the SDDE in (١)

Theorem (٢): Let an SDDE is defined as follows,

$$dx(t) = g\big(t, x(t), x(t-\lambda)\big)dt + u\big(t, x(t), x(t-\lambda)\big)dw(t) \quad ; t \in [a, b]$$

$$x(t_٠) = \psi_٠(t) \qquad ; t_٠ - \lambda \leq t \leq t_٠$$

and suppose that the interval $[a, b]$ partitioned as follows

$$٠ = t_٠ < t_١ < \cdots < t_N = t$$

$$\Delta t_{n+١} = t_{n+١} - t_n = h$$

Where $\Delta t_{n+١}$ represent the time increments and *h* is uniform step size defined as $h = \frac{b-a}{N}$.

And $\Delta w_n = w(t_{n+١}) - w(t_n)$

Represents the standard Brownian process with $n = ٠, ١, \ldots, N-١$, and $t_n = a + nh$

Then, the sequence of numerical solutions given as follows

$$x_{n+١} = x_n + g(t_n, x_n, \psi_{n-\lambda})h + u(t_n, x_n, \psi_{n-\lambda})\Delta w_n \quad ; ٠ \leq n \leq N-١$$

See Jassim (٢٠٠٦) for further details

Definition (٤): Local error is define as the difference between the approximation exact solution, and represent the sequence of the following random variables

$$S_n = x(t_n) - \bar{x}(t_n) \qquad ; n = ١, \ldots, N$$

Where $\bar{x}(t_n)$ is the approximation variable.

III.Simulations Analysis:

In this section we will use the numerical Euler-Maruyama method to study the behavior of sample path by conducting four simulation scenarios (note that there is no closed form for the SDDE solution), as well as, to see the impact of the initial values on the following SDDE.

١- Simulation example one

Let us consider we have the following ordinary Black- Scholes process:

$$dS_t = \mu S_t d_t + \sigma S_t dW_t,$$

Where μ=٠.٠٣, σ=٠.٠٥, $y_. =$ ١٠. The following figure (١) illustrates the pathway of the ordinary Black-Scholes process also figure (٢) displays that the generated process follows the log normal distribution.

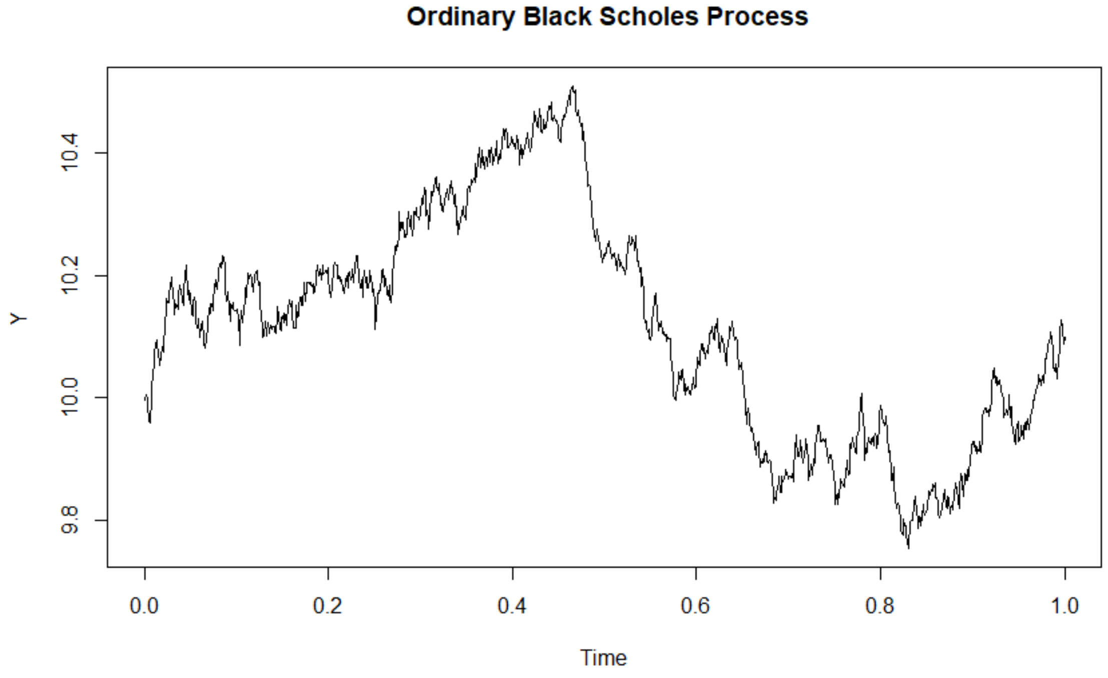

Figure ١: Sample path for the ordinary Black-Scholes values.

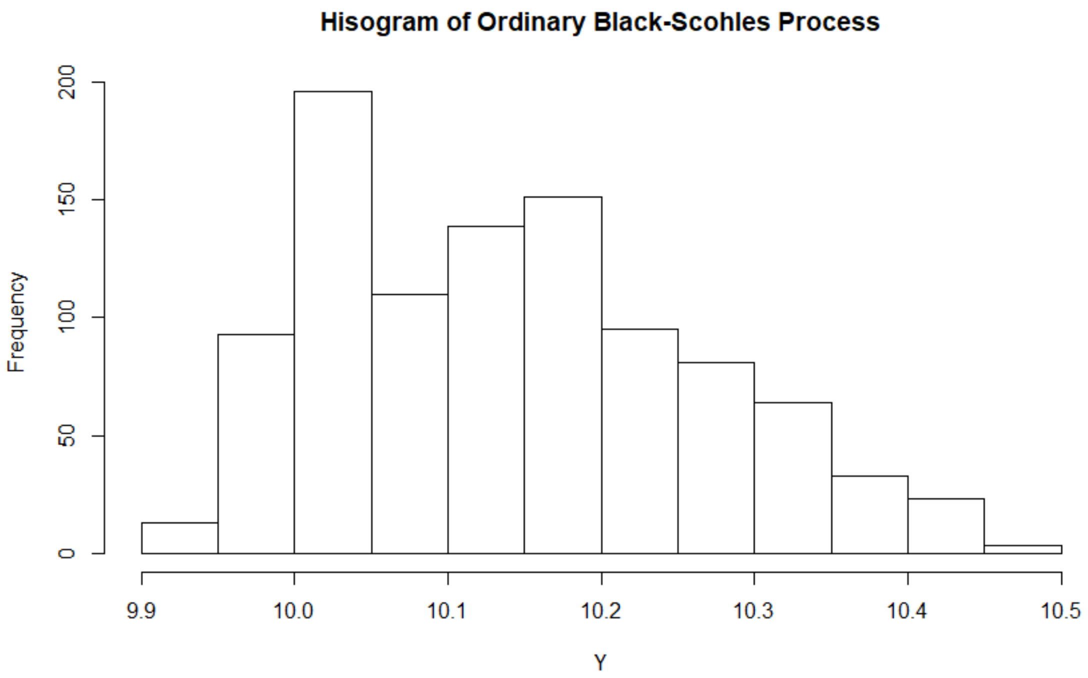

Figure (١) the Histogram of the stock price model in ordinary Black-Scholes

The following table explains some important values for the ordinary Black-Scholes process.

Table (١) some important statistics

Min.	١st. Qu.	Median	Mean	٣rd. Qu.	Max.
٩.٨٤١	١٠.٠١٨	١٠.١٩٥	١٠.١٩٥	١٠.٣٧٢	١٠.٥٥٠

٢- Simulation example two

Let us consider we have the following delay Black- Scholes process,

$$ds_t = \left(\mu s_t + ٢e^{-١}(s_t - \lambda)\right) + ٣.٢e^{-١})dt + \sigma dwt,$$

Where μ is a real number to see the impact of the delay time in drift term only, and σ is a constant (diffusion). In Figure (٣), we sketched one sample path with $\lambda = -٠.٦$, $\sigma = ٠.٥$, T=١.

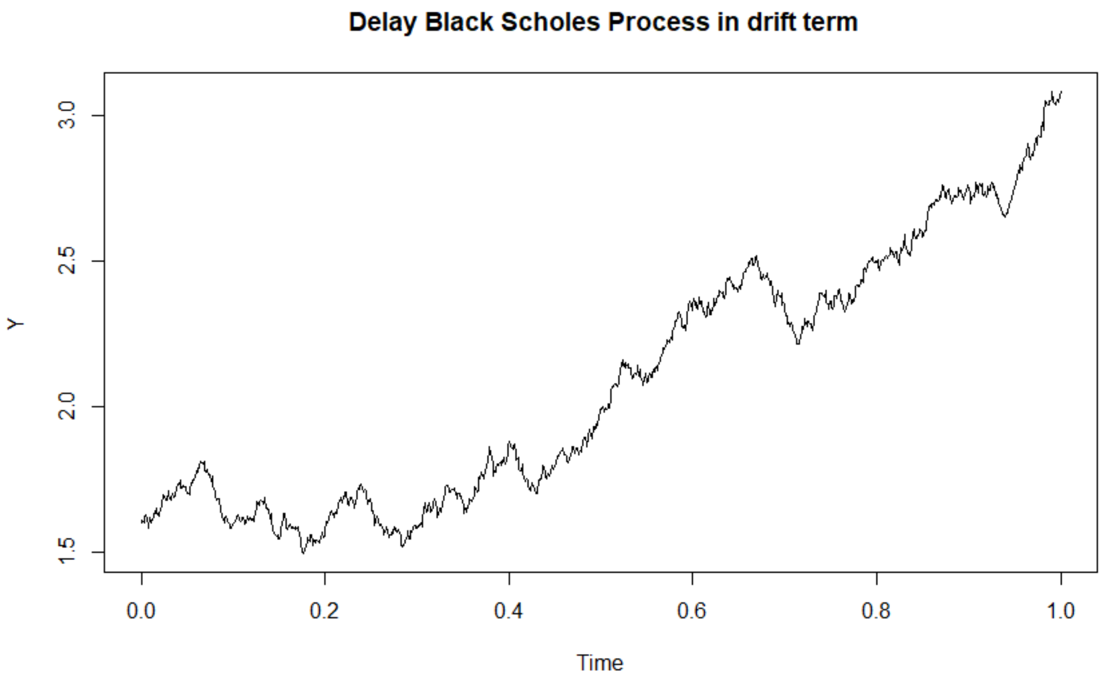

Figure ٣: Sample path with of delay-drift Black-Scholes process.

The Figure (٣) displays one sample path for the different initial values with $\lambda = -$٠.٦ ,$\mu =$ ٠.٠٠٣,$\sigma =$ ٠.٥ , T=١. With time t = ٠, the sample path decreases while path with time t =١ increases. At time t = ٠, it is seen that the graph take the value, ١.٧ and after time t = ٠, we observed that graph shows increases values for the sample paths between t = ٠ and t = ١. Also, figure (٤) displays that the generated process follows the log normal distribution.

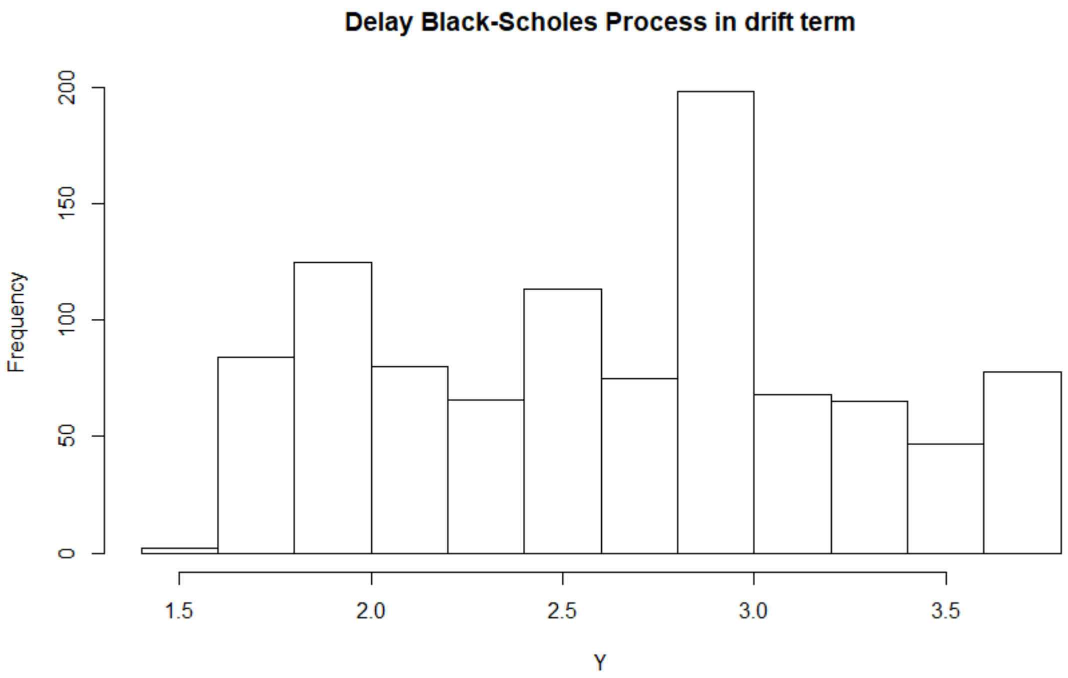

Figure (٤) the Histogram of the stock price model in delay-drift Black-Scholes

The following explains some important values for the delay-drift Black-Scholes process.

Table (٢) some important statistics

Min.	١st. Qu.	Median	Mean	٣rd. Qu.	Max.
١.٥٩٨	٢.٠٣٧	٢.٧٤٢	٢.٦٣٠	٣.٠٩١	٣.٧٣٠

٣- Simulation example three

Let us consider we have the following delay Black- Scholes process,

$$ds_t = (\mu s_t)dt + \left(\sigma s_t + b(s_t - \lambda)\right)dwt,$$

Where σ is a real number to see the impact of the delay time in diffusion term only, and μ is a constant (drift). In Figure (٥), we sketched one sample path with $\lambda = -٠.٥$, $\sigma = ٠.٥$, T=١.

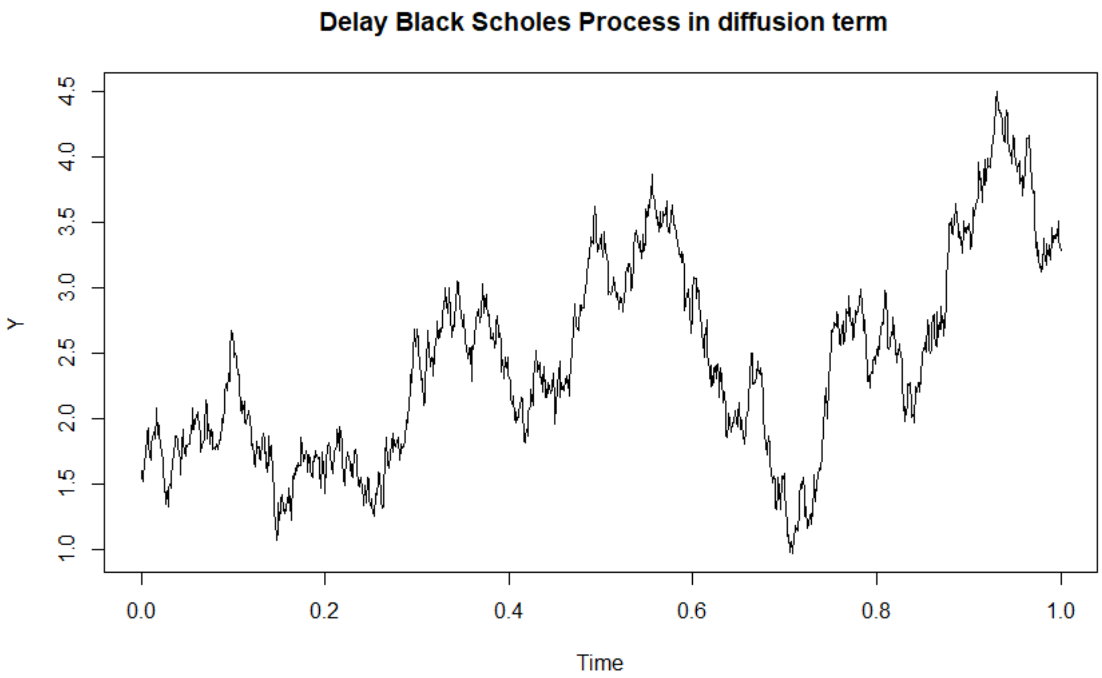

Figure ٥: Sample path with different drift and diffusion values.

The Figure (٥) displays one sample path for the different initial values with λ = −٠.٦ ,μ = ٠.٠٠٣,σ = ٠.٥ , T=١. With time t = ٠, the sample path decreases while path with time t =١ increases. At time t = ٠, it is seen that the graph take the value, ١.٧ and after time t = ٠, we observed that graph shows increases values for the sample paths between t = ٠ and t = ١.

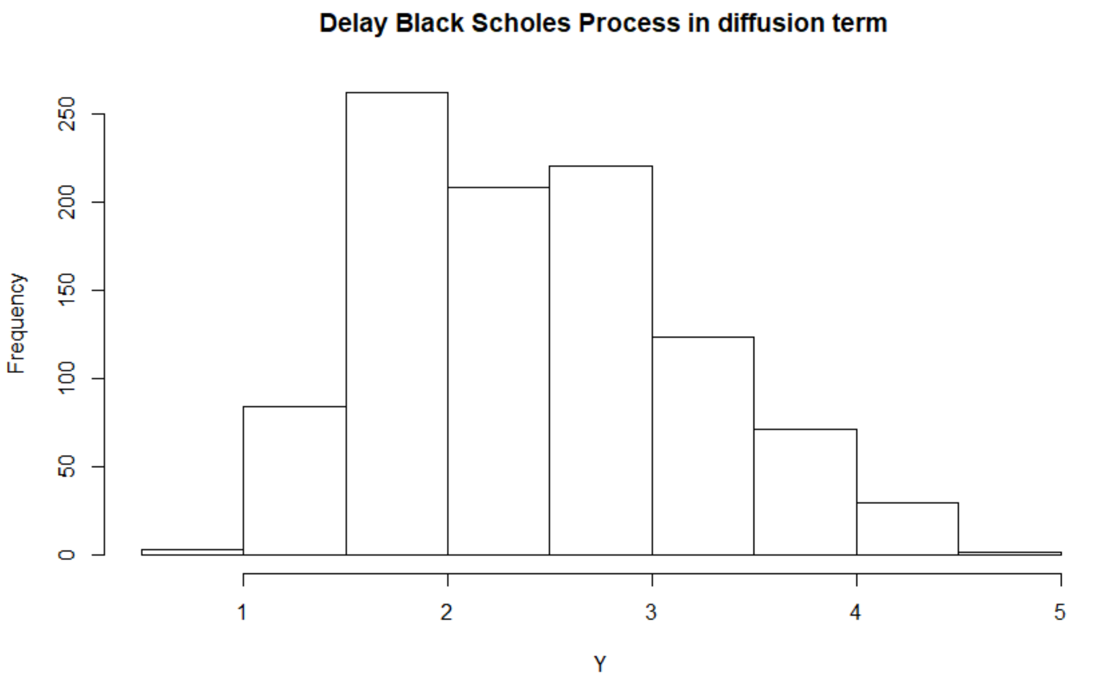

Figure (٦) the Histogram of the stock price model in delay - diffusion Black-Scholes

The following explains some important values for the ordinary Black-Scholes process.

Table (٣) some important statistics

Min.	١ st. Qu.	Median	Mean	٣rd. Qu.	Max.
٠.٩٧٤	١.٨٠٩	٢.٣٥٤	٢.٤٣٩	٢.٩٣٠	٤.٥٠٢

٤- Simulation example four

Let us consider we have the following delay Black- Scholes process,

$$ds_t = (\mu s_t + ٢e^{-١}(s_t - \tau) + ٣.٢e^{-١})dt + \sigma^{٢}(s_t + ٣(s_t - \tau))dwt,$$

Where μ and σ are real numbers to see the impact of the delay time in drift and diffusion terms. In Figure (٧), we sketched one sample path with $\lambda = -٠.٦$, $\sigma =$, T=١ and the initial values are …

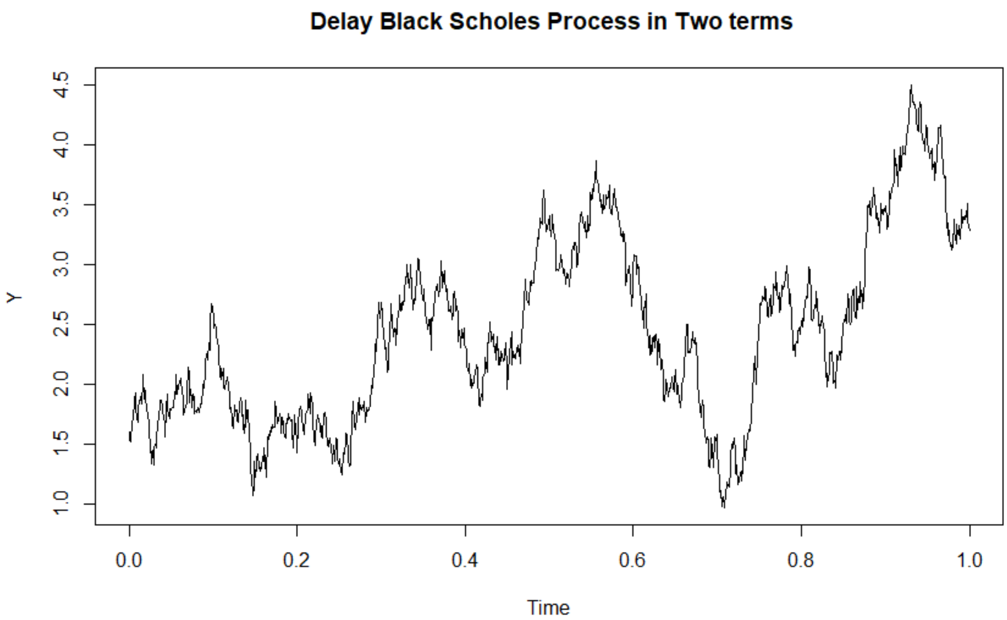

Figure ٧: Sample path with different drift and diffusion values.

The Figure (٧) displays one sample path for the different initial values with $\lambda = -٠.٦$,$\mu =, \sigma =$, T=١. With time t = ٠, the sample path decreases while path with time t =١ increases. At time t = ٠, it is seen that the graph take the value, ١.٧ and after time t = ٠, we observed that graph shows increases values for the sample paths between t = ٠ and t = ١.

The following explains some important values for the ordinary Black-Scholes process.

Table (٤) some important statistics

Min.	١ st. Qu.	Median	Mean	٣rd. Qu.	Max.
٠.٩٧٥	١.٨٠٩	٢.٣٥٤	٢.٤٤٠	٢.٩٣١	٤.٥٠٤

References:

X. Mao, Stochastic differential equations and applications, Elsevier, ٢٠٠٧.

E. Ezgi Aladagli, Stochastic delay differential equations, Master thesis, Middle East technical University, ٢٠١٧.

H.A Jassim, Solution of stochastic linear ordinary delay differential equations, Master thesis in Mathematics, Al-Nahrain University, ٢٠٠٦.

G. Stoica, A stochastic delay financialmodel, Proceedings of the American Mathematical Society, ١٣٣(٦), pp. ١٨٣٧–١٨٤١, ٢٠٠٥.

Y. Zheng, Asset pricing based on stochastic delay differential equations, ٢٠١٥.

L. Smith, Solving Stochastic Differential Equations by the Reduction Method, Internet Reference (http://www.personal.umich.edu/~lones/ftp/b١٠/sde٩٩), ١٩٩٩.

A.Muszta , Solving Stochastic Differential Equations, Internet Reference (http://www.math.chalmers.se/~eojan/SDE١/), ٢٠٠٥.

S. Han, Numerical Solution of Stochastic Differential Equations, M.Sc. Thesis, University of Edinburgh and Heriot-Watt, ٢٠٠٥.

C. O. Mahony, The Numerical Analysis of Stochastic Differential Equations, Transactions on Circuits and Systems II, ٢٠٠٦.

M. Arriojas, Y. Hu, S.-E. Mohammed, and G. Pap, A delayed Black and Scholes formula, Stochastic Analysis and Applications, ٢٥(٢), pp. ٤٧١–٤٩٢, ٢٠٠٧.

M. Reiss, Adaptive estimation for affine stochastic delay differential equations. Bernoulli ١١(١), ٦٧–١٠٢, ٢٠٠٥.

S.-E. A. Mohammed, Stochastic Differential Systems with Memory: Theory, Examples and Applications, Southern Illinois University Carbondale, ١٩٩٨.

E. Buckwar, Introduction to the numerical analysis of stochastic delay deferential equations, Journal of Computational and Applied Mathematics ١٢٥, ٢٩٧-٣٠٧, ٢٠٠٠.

M. Ferrante and C. Rovira, Stochastic delay differential equations driven by fractional Brownian motion with Hurst parameter H>١/٢, Bernoulli ١٢(١), ٨٥–١٠٠, ٢٠٠٦.

L. Arnold, Stochastic Differential Equations: Theory and Applications, Wiley-Interscience, New York, ١٩٧٤.

Karatzas, S.E. Shreve, Brownian motion and Stochastic Calculus, Springer, New York, ١٩٩١.

D. Williams, Probability with Martingales, Cambridge University Press, Cambridge, ١٩٩١.

S.E.A. Mohammed, Stochastic Functional Differential Equations, Pitman (Advanced Publishing Program), Boston, MA, ١٩٨٤.

X. Mao, Stochastic Differential Equations and their Applications, Horwood Publishing Limited, Chichester, ١٩٩٧.