Sparsity in Bayesian Elastic Net in Tobit Regression with an Application

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Abstract

In this paper we developed one of the most well-known regularization methods that is called elastic net method in tobit regression from the Bayesian point of views. This regularization adding the ridge and lasso penalty functions to the residual sum of squares term. In this paper, we developed new Bayesian hierarchical model for the tobit regression based on the proposed scale mixture of Li and Lin, (2010) that mixing the normal distribution with truncated gamma distribution (1,∞) as double exponential prior distribution for the interested parameter (β). Furthermore, the MCMC Gibbs sampling algorithm has developed for the posterior distribution of interested parameter (β). Analysis of real data has conducted for the proposed model; also a comparative has made with some regression models. The proposed model is outperformed and gives promised results.

Keywords: Bayesian estimation, Tobit regression, elastic net method, Gibbs sampler algorithm.

1. Introduction

Statistical regression models are the most widely tools in many fields of sciences, like biological, chemical, ecology, physical, social sciences, and in economics. These models are very useful to form the function of the dependent (response) variable with one or more independent (explanatory) variable (s). Regression models are used for planning future strategies as predicted model. So, the aims of the regression analysis is to select the best regression model that interpreted the functional form between the variable and produced the more interpretable model that included the most relevant predictor variables on the response variable and then use this model for prediction . Model selection is key idea behind the regression analysis through applying the variable selection procedure.

The problem of the many predictor variables and the problem of the multicollinearity motivated the researcher to find the solution for the variable selection methods. In the case of many predictor variables (p > n) and multicollinearity, the OLS estimates are meaningful because the variability of the estimates that lead to biased and high variances estimators. To overcome these problems, ridge method has developed to deal with these circumanterc. Ridge method gives not sparse solution, therefore lasso method have developed to produce sparse solution with biased small variances. Because of some drawbacks on lasso, elastic net method that combined ridge and lasso developed to produced sparse solution that cope with the effect of pairwise correlation between predictor variable in the group based of different variable .The main goal of this paper is to present new Gibbs sampler for the tobit regression model based or the elastic net.

In (1970) Hoerl-Kennard proposed a ridge regression method that estimates the coefficients of a multiple regression model based on adding a small positive amount to the diameter of the coefficient of (x′x) Studies reached when this positive amount is added increases the possibility of making the data perpendicular and thus to obtain better estimates of the regression model coefficients. In (1996) Tibshirani proposed a new method for estimating parameters of linear models called Lasso, which are least absolute shrinkage and selection operators, that reduce the sum of squares of residuals subject to the sum of the absolute value of the coefficients, it tends to produce some coefficients that are equal to zero. Tibshirani concluded that Lasso has better properties than the sequential step method and the ridge regression method. In (2005) Zou and Hastie. Suggested the elastic net ,a new regularization and variable selection method. Reality world data and a simulation study show that the elastic net often surpass the lasso, whilst enjoy a similar sparsely of representation. Moreover, the elastic net encourages a grouping effect, where robustly correlated predictors tend to be in or out of the model together. The elastic net in particular usefully when the number of predictors (p) is much bigger than the number of observations (n).

By contrast, the lasso is not a very favorable variable selection method in the p $\gg$ n case. An algorithm called LARS-EN is suggestion for computing elastic net regularization paths efficiently, much like algorithm LARS does for the lasso. In (2010) Li and Lin, N, proposed Bayesian method to solve the elastic net model using a Gibbs sampler. The Bayesian elastic net has two major advantages. Firstly, as a Bayesian method, the distributional results on the estimates are straightforward, secondly, it chooses the two penalty parameters simultaneously, avoiding the double shrinkage problem" in the elastic net method. Real data examples and simulation studies show that the Bayesian elastic net it performs well compared to other methods. In (2013) Alhamzawi introduced the tobit quantile regression model using the adaptive lasso penalty function new hierarchical model and new Gibbs sample algorithm have developed through employing of the location – scale mixture of normal as formula for the skewed Laplace prior distribution. The proposed model performs well comparing with other regularization method .In (2014). Alhamzawi introduced the Bayesian Tobit quantile regression model by employing the g-prior density additionally to using the ridge parameter. In this paper adding ridge parameter was to deal with some challenges that comes with censored data, like collinearity between the covariates. This work also deal with variable selection procedure based on the g-prior .The results of simulation and real data analysis illustrated the outperformance of the proposed model.

In (2016) Alhamzawi introduced the Bayesian elastic net for the tobit quantile regression model. The new regularization method deals with the variable selection procedure and parameters estimation for the tobit quantile model by using the elastic net penalty function through employing the gamma priors. In this work Alhamzawi treated the hyper parameters of the proposed gamma priors. The results of simulation and real data analysis were comparable with some exists methods .In (2017) Alhusseini introduced new hierarchal model for the tobit regression by using lasso penalty function. In this work the scale mixture for uniforms mixing with special case of gamma distribution as representation of the Laplace prior distribution employed for develop. New Gibbs samples algorithm. Parameter estimation and variable selection were performs. Simulation example, and real data analysis have been showed that the proposed model performs well comparing with some other methods .In (2018). Alhamzawi and Haithem suggested a new Bayesian elastic net (EN) approach for variable selection and coefficient estimation in tobit regression. Mostly, we present a new hierarchical formularization of the Bayesian en by utilizing the scale mixture of truncated normal distribution (with exponential mixing distributions) of the laplace density part. The Proposal method is an alternate method to Bayesian method of the en problem. It is shown up that the model performs well Comparison with old elastic net representation. In (2018) Alhusseini introduced the composite tobit quantile regression model from the Bayesian point of view. In this work MCMC algorithm has developed by employing scale mixture for the skewed laplace prior distribution as formula of normal mixing with exponential distribution. The results of simulation scenarios and real data analysis illustrate that the proposed method that combine the information of covariates for the different quantiles is outperforms the other methods. In (2019) Hilali, introduced new regularization method by using transformation for the scale mixture of laplace prior distribution that proposed by Ma lick and yi (2014). Also, new Gibbs sampling algorithm proposed for the Bayesian adaptive lasso tobit regression. The results of simulation and practiced side were very promising.

2. Bayesian Elastic Net Regression Model

The elastic net overcomes Lasso drawbacks because it uses the two penalty functions and we can work with the elastic net when there are many correlated predictor variables, see (Li and Lin 2010). The elastic net estimator is defined by

$$\beta = \text{argmin}\, ||Y - X\beta||^2 + \lambda 2||\beta||^2 + \lambda 1||\beta||^1 \qquad (1)$$

Where $\lambda 1, \lambda 2 \geq 0$ are the shrinkage parameters. By motivation of Li and Lin (2010) and Alhamzawi (2014) works, we investigates the Bayesian elastic net tobit regression (BENTR) model through employing new hierarchical model for the Bayesian elastic net tobit model, and proposed new Gibbs sapling algorithm for BENTR. The classical elastic net estimator defined by

$$\hat{\beta}_{EN} = \underbrace{\text{argmin}}_{\beta} (y - x_i'\beta)^T (y - x_i'\beta) + g(\beta), \quad (2)$$

Where $g(\beta)$ is the penalty function defined by $g(\beta) = \lambda_1||\beta||_1 + \lambda_2||\beta||^2$, $\lambda_1 \geq 0$, and $\lambda_2 \geq 0$ are the shrinkage parameters guarantees the strictly convex. The ridge penalty can be obtained if $\lambda_1 = 0$ and the lasso penalty if $\lambda_2 = 0$. The parameters λ_1 and λ_2 controls the amount of shrinkage and selection and the amount grouping, respectively.

We proposed a BENTR analysis in this thesis for the parameters estimation and the variable selection procedure. We employed the prior distribution of $\boldsymbol{\pi}(\boldsymbol{\beta}|\boldsymbol{\sigma}^2, \boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2)$, which is defined by

$$\pi(\beta|\sigma^2, \lambda_1, \lambda_2) \propto \exp\{-\frac{1}{2\sigma^2}(\lambda_1 \sum_{j=1}^{p}|\beta_j| + \lambda_2 \sum_{j=1}^{p} {\beta_j}^2)\} \quad (3)$$

Then, in general the posterior marginal distribution of the parameter $\boldsymbol{\beta}$ of the Tobit regression model conditioning on latent variable $\boldsymbol{y}^*$ is

$$p(\beta|y^*) \propto \exp\{-\frac{1}{2\sigma^2}(y_i - \max(y^0, y^*) - \frac{1}{2\sigma^2}(\lambda_1 \sum_{j=1}^{p}|\beta_j| + \lambda_2 \sum_{j=1}^{p} {\beta_j}^2)\} \quad (4)$$

Where y^0 is a censoring point. We exploits the above formulas (4 and 3) to setup the Bayesian elastic net tobit regression through the following general posterior marginal density of β,

$$f\ (\beta/y) = \int_0^\infty h\ (2\pi\sigma^2)^{-n/2}\ \exp\left\{-\frac{1}{2\sigma^2}\left((y_i - \max(y^0, y^*)) - \frac{1}{2\sigma^2}(\lambda_1 \sum_{j=1}^{p}|\beta_j| + \lambda_2 \sum_{j=1}^{p} {\beta_j}^2\right)\right\} \boldsymbol{\pi}\ \sigma^2\ d\sigma^2 \quad (5)$$

Where h is the normalizing constant of $\boldsymbol{\lambda}_1$, $\boldsymbol{\lambda}_2$, and $\boldsymbol{\sigma}^2$.

3. Hierarchical Model and Prior Distributions for BENTR

By using the structure equation of tobit regression (7) and the prior proposed by Li and Lin (2010),

$$\exp\{-\frac{1}{2\sigma^2}(\lambda_1 \sum_{j=1}^{p}|\beta_j| + \lambda_2 \sum_{j=1}^{p} {\beta_j}^2)\} \propto$$
$$K \prod_{j=1}^{p} \int_1^\infty \sqrt{c}\ \exp\ \{-\frac{\beta_j^2}{2}(\frac{\lambda_2}{\sigma^2}\ c)\ \} w^{\frac{-1}{2}}\ \exp(-\frac{1}{2\sigma^2}\frac{\lambda_1^2}{4\lambda_2}w)\ dw \quad (6)$$

$$y_i = \begin{cases} y^* \ \text{if} \ \ y^* > 0 \\ 0 \ \text{if} \ \ y^* \le 0 \end{cases} \quad (7)$$

where $\boldsymbol{K}$ is the normalizing constant and $\boldsymbol{c} = \frac{\boldsymbol{w}}{\boldsymbol{w-1}}$. The prior formula (6) represent the scale mixture of normal mixing with truncated gamma. Suppose that $y^0 = 0$, then we list the following proposed hierarchical model for the Bayesian elastic net regression model

$$y_i = \begin{cases} y_i^* & \text{if} \quad y_i^* > 0 \\ 0 & \text{if} \quad y_i^* \leq 0 \end{cases},$$

$$y_i^* = x_i^T \beta + \epsilon_i,$$

$$y_i^* | \beta, \sigma^2 \sim \text{N}\ (x_i^T \beta, \sigma^2 I_n),$$

$$\beta_j | w_j, \sigma^2 \sim \prod_{j=1}^{p} \text{N(mean=0, var} = (\frac{\lambda_2}{\sigma^2} \frac{w_j}{w_j - 1})^{-1}),$$

$$w_j | \sigma^2 \sim \prod_{j=1}^{p} \text{TG (mean} = \frac{1}{2}, \text{var} = \frac{8\lambda_2 \sigma^2}{\lambda_1^2}),$$

$$\sigma^2 \sim \frac{1}{\sigma^2}.$$

Where **TG** is the truncated gamma supported on **(1,0)**. Our contribution is employing the hierarchy model (5) to develop new Bayesian computation for the elastic net tobit regression.

4. Conditional Posterior Distributions.

Supposing that all priors for the different parameters are independent, then we can write down the full conditional distribution as follows,

$$y_i^* / \beta, \sigma^2 \sim \text{N}\ (x_i' \beta, \sigma^2 I_n),$$

Where $i = 1, 2, \ldots, n$.

Following Alhamzawi (2014) and Li and Lin (2010) and conditioning on y^*, β the posterior distribution of β is

$$\pi(\beta / y^*, \sigma^2, \gamma) \propto \pi(y^* / \beta, \sigma^2, \gamma)\ \pi(\beta / \sigma^2)$$

$$\propto \exp\{-\frac{1}{2\sigma^2} (y^* - x'\beta)'(y^* - x'\beta)\} \exp\{-\frac{1}{2\sigma^2} \beta' Q_\gamma \beta)$$

$$\text{Where } Q = \text{diag}\ (\frac{\gamma_1}{\gamma_2 - 1}, \ldots, \frac{\gamma_p}{\gamma_p - 1})$$

$$= -\frac{1}{2\sigma^2} [\ \beta'(x'x)\ \beta - 2y^* x\beta + y'^* y^* + \beta' Q_\gamma \beta\ ]$$

$$= -\frac{1}{2\sigma^2} [\ \beta'\ (x'x - Q_\gamma)\ \beta - 2y^* x\beta + y^{*'} y^*]$$

$$\text{Let } s = x'x + \lambda_2 Q_\gamma \text{, then}$$

$$= -\frac{1}{2\sigma^2}[\beta' s\beta - 2y^* x\beta + y^{*'} y^*]$$

$$= -\frac{1}{2\sigma^2}(\beta - s^{-1}x'y^*)' c\,(\beta - s^{-1}x'y^*) \qquad (8)$$

Then $\boldsymbol{\beta}$ distribution is the multivariable normal with mean $\boldsymbol{s^{-1}x'y^*}$ and variance $\boldsymbol{\sigma^2 s^{-1}}$;

$$\beta / y^*, \sigma^2, \gamma \sim N\,(s^{-1}x'y^*, \sigma^2 s^{-1}) \qquad (9)$$

The second variable $\boldsymbol{\sigma^2}$, the terms that involves $\boldsymbol{\sigma^2}$ are

$$\pi(\sigma^2/y^*, \beta, \gamma) \propto \pi(y^*/\beta, \sigma^2, \gamma)\,\pi(\beta/\sigma^2)\,\pi(\sigma^2)$$

$$\propto (\sigma^2)^{\frac{-n}{2} - p - 1}\{\Gamma z\,(\tfrac{1}{2}, \tfrac{\lambda_1^2}{8\sigma^2\lambda_2})\}^{-p} \exp[-\tfrac{1}{2\sigma^2}\{(y^* - x'\beta)'(y^* - x'\beta) + \lambda_2 \sum_{j=1}^{p} \tfrac{\gamma_j}{\gamma_j - 1}\beta_j^2 + \tfrac{\lambda_1^2}{4\lambda_2}\sum_{j=1}^{p}\gamma_j\}], \qquad (10)$$

Where $\boldsymbol{\Gamma z\,(\alpha, x) = \int_x^{\infty} t^{a-1}\; e^{-t}\, dt}$ is the upper incomplete gamma function, see Armido and Alfred (1986) for more details, and $\boldsymbol{1_P}$ is the vector of **p**-dimensional of $1's$.

The third variable $(\boldsymbol{\gamma - 1_P})$, where the full conditional distribution is

$$(\gamma - 1_P)/y^*, \sigma^2, \beta \sim \prod_{j=1}^{p} \text{GIG}\,(\lambda = \tfrac{1}{2}, \varphi = \tfrac{\lambda_1}{4\lambda_2\sigma^2}, \chi = \tfrac{\lambda_2\beta_j^2}{\sigma^2}), \qquad (11)$$

Where **GIG (.)** is the generalized inverse Gaussian disruption, see Jorgensen (1982) for more details, **i.e.** we can say that **x ~ GIG** $(\boldsymbol{\lambda, \varphi, \chi})$ if its pdf as follows,

$$f\,(x/\lambda, \varphi, \chi) = \frac{(\varphi/\chi)^{\lambda/2}}{2k_\lambda(\sqrt{\varphi\chi})} x^{\lambda-1} \exp\{-\tfrac{1}{2}(\chi x^{-1} + \varphi\, \boldsymbol{x})\}, \qquad (12)$$

Where $\boldsymbol{x} > \mathbf{0}$, $\boldsymbol{k_\lambda}$ **(.)** is the based function of the third Kind with order $\boldsymbol{\lambda}$. So, we can easily say that

$$(\gamma_j - 1_p)^{-1}/y^*, \sigma^2, \beta \sim \text{IG}\,(\mu = \sqrt{\lambda_1}/(2\lambda_2\,|\beta_j|), \boldsymbol{\lambda} = \tfrac{\lambda_1}{4\lambda_2\sigma^2})$$

With the following pdf,

$$f(x/\mu,\lambda) = \sqrt{\frac{\lambda}{2\pi x^3}} \exp\{-\frac{\lambda(x-\mu)^2}{2\mu^2 x}\}.$$

See Chhikara and Folks,(1988) for more details.

Park and Casella (2008) and Casella (2001), Li suggested the empirical Bayes estimates for the shrinkage parameters $\boldsymbol{\lambda_1}$ and $\boldsymbol{\lambda_2}$ by using the marginal maximum likelihood of the data and use the Monte Carlo Expectation- maximization (**MCEM**) algorithm. Following Li and Lin (2010), we treated $\boldsymbol{\beta, \gamma, \sigma^2}$ as missing data and $(\boldsymbol{\lambda_1}, \boldsymbol{\lambda_2})$ as fixed parameters, the likelihood is

$$\lambda_1^P (\frac{1}{\sigma^2})^{\frac{n}{2}+p+1} \{\Gamma_u\left(\frac{1}{2}, \frac{\lambda_1^2}{8\sigma^2\lambda_2}\right)\}^{-p} \prod_{j=1}^{p} (\frac{1}{\gamma_j - 1})^{1/2} exp[-\frac{1}{2\sigma^2}\{(y^* - x'\beta)(y^* - x'\beta) + \lambda_2 \sum_{j=1}^{p} \frac{\gamma_j}{\gamma_j - 1} \beta_j^2 \frac{\lambda_1^2}{4\lambda_2} \sum_{j=1}^{p} \gamma_j\}], \quad (13)$$

and the we can take the log for the function (12) and the solve the maximization problem by MCMC algorithm. One can see Li and Lin (2010) for more details.

5. Real Data Description and Analysis

The following data have information that records for mother visits to the Salam Health Center in waist health department. Furthermore, I used (50) personal forms of mother that available in the above center, that is mean, I took simple random sample. Women was drawn to study the factors affecting the number of children born (response variable) y, while the independent variables were as follows:

X_1: Age of the mother

X_2: Mother's age at marriage

X_3: Academic achievement of mother

X_4: Academic level of the husband

X_5: Weight of mother

X_6: The length of the mother

X_7: Mother smoking status

X_8: Age of the husband

X_9: The occupation of the husband

$X10$: Number of dead children

$X11$: Use status of contraceptives

$X12$: Mother with thyroid disease

$X13$: The number of hours a mother sleeps a day

$X14$: Breastfeeding duration

$X15$: Mother's occupation

X_{16}: Viruses status

$X17$: Mother's food system

$X18$: Matching blood status

$X19$: Gestational diabetes status

$X20$: Psychological status

Parameters	**Tobit**	**BANET**	**Crq**	**Our method**
β_1	0.1117	0.0000	0.0167	**-9.8778**
β_2	-0.1188	0.0000	-0.0358	**0.0000**
β_3	0.1030	0.2408	-0.1288	**-0.3445**
β_4	0.0015	-0.2216	0.1258	**0.0000**
β_5	0.0453	0.0279	0.0686	**1.1694**

β_6	-0.1802	-0.1775	-0.2526	**0.9188**
β_7	0.6665	1.7751	0.9264	**-18.4727**
β_8	0.0572	0.0437	0.0633	**1.1728**
β_9	0.7540	-0.1622	0.4789	**-0.2375**
β_{10}	0.0320	0.8403	-0.0198	**0.8641**
β_{11}	-0.6230	-0.2304	-1.2832	**0.1860**
β_{12}	-1.8328	-0.6509	-2.5601	**0.2194**
β_{13}	0.5692	-0.0236	0.6496	**0.2061**
β_{14}	-0.1765	0.0515	-0.1420	**1.5064**
β_{15}	3.0677	1.1744	4.3379	**-7.1955**
β_{16}	-0.6657	0.1197	-0.0061	**0.4179**
β_{17}	-0.3281	1.1367	-0.4272	**-0.3860**
β_{18}	-2.2937	-0.5384	-2.7590	**0.6754**
β_{19}	-3.1945	0.2881	-3.1232	**5.8932**
β_{20}	0.6224	-1.3341	0.6829	**8.1628**

Table 1. parameters estimates of Q_1….., Q_{20} under four different models.

Table (1). Summarized the parameter estimates that captured from the posterior distributions for proposed model and the other three exists models. Gibbs sample algorithm estimates the mean of (B) for the posterior distribution estimates. We observed variable selection procedure in the proposed model in second and fourth variable (Mother age at marriage and academic level of the husband), where the parameters estimates were ($\beta 2$= 0, $\beta 4$= 0). The results of the proposed model were very meaning full estimates the proposed mode results are comparable to the other exists models. Furthermore, $\beta 1$= -9.8778,which is means that the age of the mother is very important variable and effect the response variable (weight of newborn child). Also variables (smoking status of mother, gestational diabetes statues, and psychological status of mother) are very important variables which are very effected on the response variable. The following figure illustrates the trace plot of the posterior densities for different (20) parameters. Trace plots are displayed the stability of the Gibbs sampler algorithm, which is mean that the appropriate prior distribution that formulated the posterior distribution.

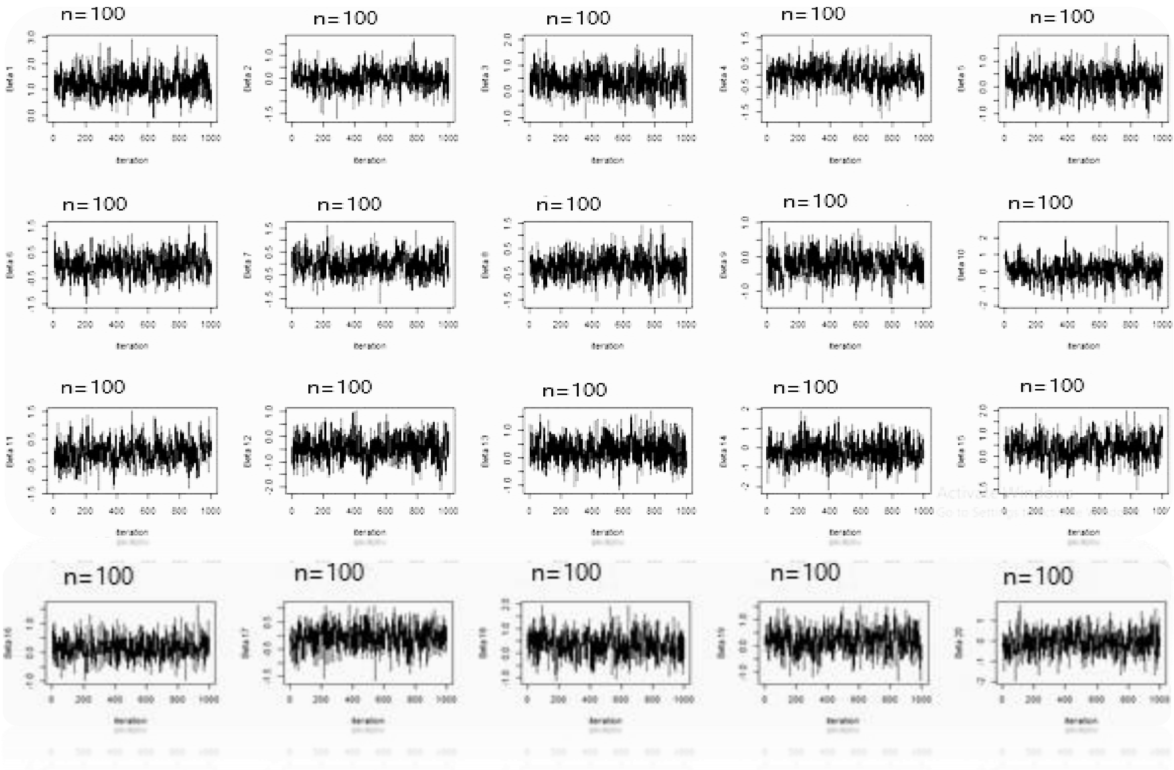

Figure 1. Trace plots for $\beta_1, \dots, \beta_{20}$ parameters.

6. Conclusions

This paper introduced new Bayesian elastic net tobit regression models by employing the double exponential density of parameter (β) which is the scale mixture of normal distribution mixing with truncated gamma distribution which is proposed by Li and Lin (2010). We introduced new Bayesian hierarchical model, also we provided MCMC Gibbs sampler algorithm for the introduced posterior distribution. The proposed model outperforms in real data analysis comparing with some regularization methods. The result shows that the proposed tobit regression model is comparable in terms of the estimation accuracy and in terms of the variable selection procedure.

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اختيار المتغيرات في انحدار توبت للشبكة المرنة مع تطبيق عملي

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المستخلص

في هذا البحث قمنا بتطوير إحدى طرق التنظيم الأكثر شهرة والتي تسمى طريقة الشبكة المرنة في انحدار توبت من وجهة نظر بيز يضيف هذا التنظيم وظائف جزاءات رج والخط إلى المجموع المتبقي لمصطلح المربعات. في هذا البحث ، قمنا بتطوير نموذج هرمي بايزي جديد لانحدار توبت استنادًا إلى مزيج المقياس المقترح ل لي و لين ، (2010) الذي يخلط التوزيع الطبيعي مع توزيع غاما المقطوع (1 ، ∞) كتوزيع أسي مزدوج مسبق للمهتمين. المعلمة (β). علاوة على ذلك ، تم تطوير خوارزمية أخذ عينات كبس MCMC للتوزيع اللاحق للمعلمة المهتمة (β). تم إجراء تحليل للبيانات الحقيقية للنموذج المقترح ؛ كما تم إجراء مقارنة مع بعض نماذج الانحدار. لقد تفوق النموذج المقترح في الأداء ويعطي نتائج واعدة.