Heavy tailed estimation in Stochastic Differential Equations with an application

Abstract

Heavy-tailed distributions are very important branch in statistical analysis. In this paper, we will estimate the tail parameter (α) using three (the Direct, Bootstrap and Double Bootstrap) methods in two examples of Stochastic Differential Equations driven by (Brownian Motion and Levy Process). Our aim is to illustrate the best way to estimate the α-stable with $(0 < \alpha < 2)$ using simulation and real data for the daily Iraqi financial market dataset.

Keywords: Heavy-tailed; Stochastic Differential Equations; Geometric Brownian Motion; Levy process; Tail index; Hill estimator; Direct method; Bootstrap and Double Bootstrap.

1.Introduction

The tail index represents the shape parameter of heavy - tailed distributions. The term heavy-tailed can be used for distributions whose moments are not limited i.e., variance is infinite, in which case quintiles and order statistics are used. The Hill estimator **(Hill, 1975)** is one of the most widely used tools to infer the tail behavior of a distribution, but sometimes this estimator produces poor results. To get rid of the problem of the large bias of the Bootstrap method, **(Hall, 1990)** suggested using samples with a smaller size than the original sample size, provided that the sample size is very large and the second order parameter $(\boldsymbol{\rho})$ is known. To obtain a consistent estimator for the optimal number of order statistics that do not need restriction on the parameter $(\boldsymbol{\rho})$, **(Danielsson et al., 2001)** used a combination of subsample Bootstrap estimates of the difference between two estimators based on Bootstrap sample sizes of different order. Later, **(Drees & Kaufmann, 1998)** presented a sequential process, which has yielded a consistent estimator of $\boldsymbol{k_n^{opt}}$ in complete model with no need for prior information on the second order parameter. **(Gomes et al., 2002)** presented a class of semi-parametric estimators for the parameter $(\boldsymbol{\rho})$ with a regularly varying tail and showed that 2nd order parameter has a very significant impact when dealing with the problems of the optimization in the statistics of the extreme values. **(Ciuperca & Mercadier, 2010)** have generalized many studies on the extreme value theory to estimate 2nd order parameter $\boldsymbol{\rho}$ and extreme value index. By performing some numerical calculations and asymptotic normality and consistency are proven under classical

assumptions. **(Hashemifard et al., 2016)** focused on heavy-tailed stochastic signals generated through continuous time auto-regressive models evoked through the infinite-variance $\propto$-stable processes with $(0 < \alpha < 2)$. Their aim was estimating the continuous time model parameters. The consistency of the estimator of desired values is illustrated in the case where the sample size and sampling frequency approach infinity. The suggested method was applied to two real data types, and the experimental results showed good agreement between a model and this data. **(Nemeth, 2020)** presented new methods that combine the advantages of Kolmogrov-Smirnov and Bootstrap and showed that the estimators have the ability to estimate well the parameters of the large tail index and also the small sample sizes.

In this paper, we generate data using two famous examples of Stochastic Differential Equations **(S.D.E.)** which are the Geometric Brownian Motion **(G.B.M.)** model and Black-Scholes driven by Levy process. Our aim is to estimate the right- tail parameter using the Direct, Bootstrap and Double Bootstrap methods and then comparing the three methods using the mean square errors**.**

The rest of this article is arranged as follows: Section 2 introduces the heavy tail distributions and their types. Stochastic differential equations; Geometric Brownian motion and Black- Scholes driven by Levy processes are presented in section 3. In section 4, we have presented the tail index; Hill estimator and our methods (the Direct, the Hall ̓s Bootstrap and the Double Bootstrap). In section 5, the simulation and real data will be presented for our methods. Finally the conclusions are in section 6.

2: Heavy-tailed distributions

Heavy tail distributions tends to have many extreme values as there will be more density under the probability density function (p.d.f.) curve. It is a probability distribution that is not significantly restricted **(Asmussen, 2003)**. In numerous applications the distribution ̓s right tail is important, while the left tail could be heavy or both tails may be heavy. In this paper, we have focused on the heavy right- tailed distribution.
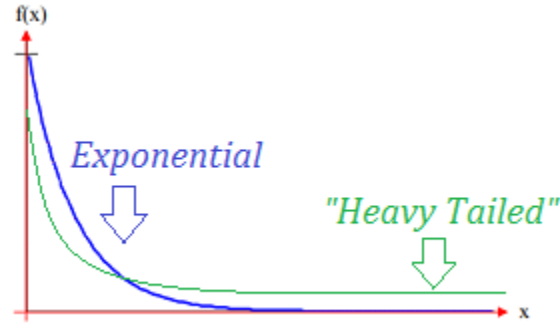
Figure1: the heavy -tailed distribution

Figure1: graph shows the right-heavy tail distribution. Obviously, the heavy tail goes to zero slower than the exponential distribution.

Definition

F(X) is considered to be having a (right) heavy tailed with tail index if $(\alpha > 0)$ satisfies **(Peng & Qi, 2017) :-**

$$\lim_{t \to \infty} \frac{1-F(tx)}{F(t)} = x^{-\alpha} \quad \text{for all } x > 0 \quad \text{and } \alpha > 0 \tag{1}$$

Where:

t: is the time

x: is the random variable

F(t): the distribution function

Definition

A distribution of the (r.v.) x with the distribution function is considered to be having a heavy (right) tail in the case where $M_X(t)$ of x is infinite for each $t > 0$ **(Rolski & et.al., 2009), (Foss & et.al., 2011), i.e.,**

$$\int_{-\infty}^{\infty} e^{tx} dF(x) = \infty \quad \text{for all } t > 0$$

There are three types of heavy-tailed distributions as follows:-

2:1- Fat-tailed distribution

Fat – tailed distribution represents the heavy-tailed distribution with infinite variation. It is a probability distribution that shows a large skewness or kurtosis relative to the exponential or normal distribution. Several authors state that this type of distribution is a probability distribution with a tail that appears fatter than usual. The Log-normal distribution is one example of a fat – tailed distribution **(Bahat & et.al. , 2005).**

2:2- Sub - exponential distributions

It is a distribution in which the largest value in the sample makes a large contribtion to the overall total **(Mikosch, 1999).**

2:3- Long-tailed distribution

A distribution of the r.v. x with distribution function F is considered to be having a long right-tailed in the case where **(Asmussen, 2003)** :-

$$\lim_{x \to \infty} Pr\{X > x + t | X > x\} = 1 \qquad \forall\, t > 0$$

3: Stochastic Differential Equations (SDE$_s$)

SDEs are utilized for modeling many different phenomena like the physical systems and unstable stock prices. The direct application of $It\hat{o}$ lemma can be helped to find the solution of SDEs **(Imkeller & Schmalfuss, 2001) and (Iacus, 2009).**

The general formula of (SDE) is **(Franke & et.al., 2004):-**

$$dx_t = \theta_1 dt + \theta_2 dw_t \qquad\qquad \textbf{(2)}$$

Where:-

dx_t: represent change of x_t in a continuous time t.

θ_1: represent drift parameter.

θ_2: represent volatility parameter.

w_t: represent standard Brownian motion (continuous time and continuous space stochastic process) It is sometimes called the Wiener process **(Iacus, 2011) .**

We will present the Geometric Brownian Motion and Levy process as a popular examples of Stochastic Differential Equations.

3:1. Stochastic Differential Equations driven by Geometric Brownian Motion (GBM)

The **(GBM)** is a continuous- space and continuous-time stochastic process. As a simple model of market prices, many Economists prefer the **Geometric Brownian motion** because it is positive everywhere (with probability1)**(Dunbar, 2016).** It is an important example of SDE as it is used to model stock prices in mathematical finance which is called **Black & Scholes (Iacus, 2009) and (Mikosch, 2004).** It is modeled by **Fisher Black and Myron Scholes (Fisher & Scholes, 1973)**.

The general formula for G.B.M. is as follows **(Franke & et.al., 2004) :-**

$$ds_t = \theta_1 S_t d_t + \theta_2 S_t dw_t \qquad (3)$$

The analytical solution is:

$$S_{t+1} = S_t exp\left(\left(\theta_1 - \frac{1}{2}\theta_2^2\right)\Delta T + \theta_2 DW_t\right) \qquad (4)$$

It is easier way to work with returns:

$$y_t = \log\frac{S_t}{S_0}$$

Where:

S_0 is an initial value.

Note that the G.B.M has a log-normal distribution with drift θ_1 and volatility θ_2 **(Dunbar, 2016).**

3:2. Stochastic Differential Equations driven by Levy process

Levy process (L) can be defined as a stochastic process with stationary and independent increments. It was introduced by French mathematician Paul Levy in 1950 **(Applebaum, 2009), (Kessler et.al., 2012) and (Klebaner, 2012).** The main idea of the Levy process is to work with

a jump in continuous time stochastic process. The jump of Levy process $[\Delta L_t = L_t - L_{t-}]$ is very important to understand the behavior of these process. Inverse Gaussian process is a famous example of the Levy process **(Kyprianou, 2014).**

The probability density function of Levy process is **(Applebaum, 2009):-**

$$f_X(X) = \left(\frac{\theta_2}{2\pi}\right)^{\frac{1}{2}} \frac{1}{(X-\delta)^{\frac{3}{2}}} \, exp\left\{-\frac{\theta_2}{2(X-\delta)}\right\} \quad \forall \, X > \delta \qquad (5)$$

Where:

X: location parameter

δ: scale parameter

4: Tail index estimation

There are many estimate for (α), but in this paper we will use the known Hill estimator to find the important solutions for estimating α such as the optimal selection of the sample fraction (k) and goodness-of-fit test **(Peng and Qi, 2017).**

4:1. Hil estimator

It is one of the most important estimators used to detect the presence of heavy tails. To define the Hill estimator, we assume that the observations $(X_1, X_2, \dots \dots, X_n)$ are nonnegative . For $1 \leq i \leq n$, write $X_{(i)}$ for the (i th) largest value of $(X_1, X_2, \dots \dots, X_n)$, so that

$$X_1 \leq X_2 \leq \cdots \leq X_n$$

Then Hill's estimator is defined as **(Hill, 1975):-**

$$\widehat{\alpha} = \left[\frac{1}{k} \sum_{i=1}^{k} \log \frac{x_{n,n-i+1}}{x_{n,n-k}}\right]^{-1} \qquad (6)$$

Where:-

K:- number of upper-order statistics.

$\hat{\alpha}$: tail index estimator. It is a consistent estimator for the tail index if the following are achieved **(Masson[a], 1982)**:-

$$k \to \infty \quad and \quad \frac{k}{n} \to 0 \quad as \, n \to \infty$$

Hill estimator is strongly dependent on optimal k selection. when k is small, the variance is large and the bias is small, while it is the opposite if k is large **(Gomes & et.al., 2009)** and **(Danielsson & et.al., 2016).**

Now we will use three methods to estimate the tail parameter, which are

4.2:Direct estimation method

A simple method for selecting optimal k in the **equation (7)** directly **(Peng, 2017)** :

$$K_{opt} = argmin_k \left[\frac{\alpha^4}{(1+\rho)^2} \beta^2 \left(\frac{k}{n}\right)^{2\rho} + \frac{\alpha^2}{k} \right] \qquad (7)$$

Where:

α**:** is the Hill estimator mentioned in **(6)**

K: represent the **equation (8)**

n**:** sample size

ρ and β : are the second order regular variation parameters, they can be calculated using the following steps:-

    a) steps to calculate ρ **(Gomes & Pestana, 2007)**:

      1- $\qquad\qquad\qquad k = \min \left[n - 1, \left(\frac{2n}{\ln \ln n}\right) \right]$ **(8)**

      2- $\qquad\qquad S_n^{(\theta)}(k) = \left[\frac{1}{k} \sum_{i=1}^{k} \ln \frac{X_{n,n-i+1}}{X_{n,n-k}} \right]^{\theta} \qquad \theta = 1,2,3,4$ **(9)**

      3- if $l = 0$

$$L_{(n)}^{(l)}(k) = \left[\frac{\ln\left(S_n^{(1)}(k)\right) - \frac{1}{2}\ln\left(S_n^{(2)}(k)/2\right)}{\frac{1}{2}\ln\left(S_n^{(2)}(k)/2\right) - \frac{1}{3}\ln\left(S_n^{(3)}(k)/6\right)} \right] \qquad (10)$$

4- if $\rho < 0$

$$\hat{\rho}_{(k)} = -\left|\frac{3\left(L_n^{(l)}(k)-1\right)}{L_n^{(l)}(k)-3}\right| \qquad \textbf{(11)}$$

b) we can calculate β as follows **(Caeiro & Gomes, 2006)**:

$$\hat{\beta}_\tau(k) = -\frac{2(2+\hat{\rho}_k)}{l\,\hat{\rho}_k\hat{\alpha}_k}\left(\frac{n}{k}\right)\hat{\rho}_k \frac{\left[\left(S_n^{(1)}(k)\right)^l-\left(S_n^{(2)}(k)/2\right)^{\frac{l}{2}}\right]^2}{\left(S_n^{(2)}(k)/2\right)^l-\left(S_n^{(4)}(k)/24\right)^{\frac{l}{2}}} \qquad \textbf{(12)}$$

Where $\hat{\alpha}_{(k)}$, $\hat{\rho}_{(k)}$ and $\hat{\beta}_\tau(k)$ are consistent estimators for α, ρ and β respectively

4.3:Hall's Bootstrap method

It is an important method used to estimate the parameter of the tail index by sampling data set with replacement. **(Hall, 1990)** suggested the Bootstrap method for the estimation of the Mean Squared Error (MSE) and selection of the smoothing parameter in non-parametric methods. Suppose $X_1, X_2, \ldots, X_n$ denote observations from the distribution function (F) and assume :

$$1 - F(X) \sim CX^{-\alpha} \qquad \textbf{C and } \boldsymbol{\alpha} > 0 \qquad \textbf{(13)}$$

Where:

c: is a constant value.

α: is a tail index

We will estimate $\boldsymbol{\alpha}$ using Hill estimator in equation**(6).** Let $[X_{n1} \geq X_{n2} \geq \cdots \geq X_{nn}]$ denote the order statistics of x_n and k is a smoothing parameter. Then, we will choose k to minimize mean square error (MSE) of $\hat{\alpha}$ **(Hall, 1990).** Put

$$\textbf{MSE(n, k)} = \textbf{E}[\hat{\alpha}(K) - \boldsymbol{\alpha}]^2 \qquad \textbf{(14)}$$

Where:-

$\hat{\alpha}(K)$: is the Hill estimator.

To select k we do the following steps:-

Drawing a resample $x_{n1}^* = \{x_1^*, x_2^*, \dots\dots x_{n1}^*\}$ from x_n; $n_1 \leq n$. Let $\{ x_{n1,1}^* \geq x_{n1,2}^* \geq \cdots \geq x_{n1,n1}^*\}$ denote the order statistics of $\{ x_1^*, x_2^*, \dots\dots x_{n1}^*\}$, and let **(Peng & Qi, 2017):**

$$\widehat{\alpha}^*(n1, k1) = \left[\frac{1}{k} \sum_{i=1}^{k1} \log x_{n1,n1-i+1}^* - \log x_{n1,n1-k1}^* \right]^{-1}$$

Then, the Bootstrap estimate of $MSE(n_1, k_1)$ is:-

$$\widehat{MSE}\,(n_1, k_1) = E[\{\widehat{\alpha}^*(n_1, k_1) - \widehat{\alpha}(n, k)\}^2 | x_1, x_2, \dots\dots, x_n] \qquad (15)$$

Then choosing $\widehat{k}_1$ to minimize $\widehat{MSE}\,(n_1, k_1)$. the optimal k is:

$$k_{opt} = cn^y$$

for a known y $\in$ (0,1) but an unknown $c > 0$, **(Hall, 1990)** proposed to estimate k_{opt} by:

$$\widehat{k}_H = \widehat{k}_1 (\frac{n}{n1})^y \qquad (16)$$

4.4: Double bootstrap method

 Double Bootstrap consider the most accurate estimates which was introduced by **(Danielsson et.al., 2001)** and improved by **(Qi, 2008)**. This method offers a solution for selecting the sample fraction by a two-step sub-sample Bootstrap approach**.** Then, we reduces the asymptotic mean square error Q(n,k) instead of MSE (n,k) **(Peng & Qi, 2017),** where:

$$Q(n, k) = E \left(\frac{1}{2} \widehat{\alpha}^2(k) - \widehat{M}(k) \right)^2 \qquad (17)$$

Where

$$\widehat{M}(k) = \left\{ \frac{1}{k} \sum_{i=1}^{k} \left(\log \frac{X_{n,n-i+1}}{X_{n,n-k}} \right)^2 \right\}^{-1} \qquad (18)$$

We will explain the steps of this method:

-   Drawing a resample $[X_1^*, \dots\dots, X_{n1}^*]$ from $[X_1, \dots\dots, X_n]$ with $n_1 = O(n^{1-\delta})$ for some $\delta \in \left(0, \frac{1}{2}\right)$.

- Determining the estimators of $\hat{\alpha}(k)$ and $\widehat{M}(k)$ based on the bootstrap sample as $\hat{\alpha}^*(k)$ and $\widehat{M}^*(k)$, and choose:

$$\widehat{k}_1 = arg\ min_{k1}E\left\{\left(\frac{1}{2}\left(\hat{\alpha}^*(k_1)\right)^2 - \widehat{M}^*(k_1)\right)^2 \Big| x_1, \dots, x_n\right\} \qquad (19)$$

- Repeating the **equation (17)** with($n_2 = \frac{n_1^2}{n}$) and we get $\hat{k}_2$.
- The optimal k_{opt} is:

$$\widehat{k}_{DHPV} = \frac{\widehat{k}_1^2}{\widehat{k}_2}\left\{\frac{\left(log\ \widehat{k}_1\right)^2}{\left(2logn_1 - log\widehat{k}_1\right)^2}\right\}^{\frac{logn_1 - log\widehat{k}_1}{logn_1}} \qquad (20)$$

5: Simulation study

In order to compare the Hill estimator with other nonparametric estimators represented by the Direct method , Bootstrap and Double Bootstrap method, The simulation technique was adopted. We generate the data using two models of SDE with $\theta_1 = 0.5$ and $\theta_2 = 0.05$. the following sizes of the samples (N= 50, 100, 150, 200, 250, 500, 800) and MSE criterion are used to compare these methods. The method with less value of MSE is the best. Then we have get tail parameter (α); K_{opt} and MSE for each sample based on 100 replications . The results of the simulation were obtained based on a program written by R. The simulation study was carried out using two models.

First model

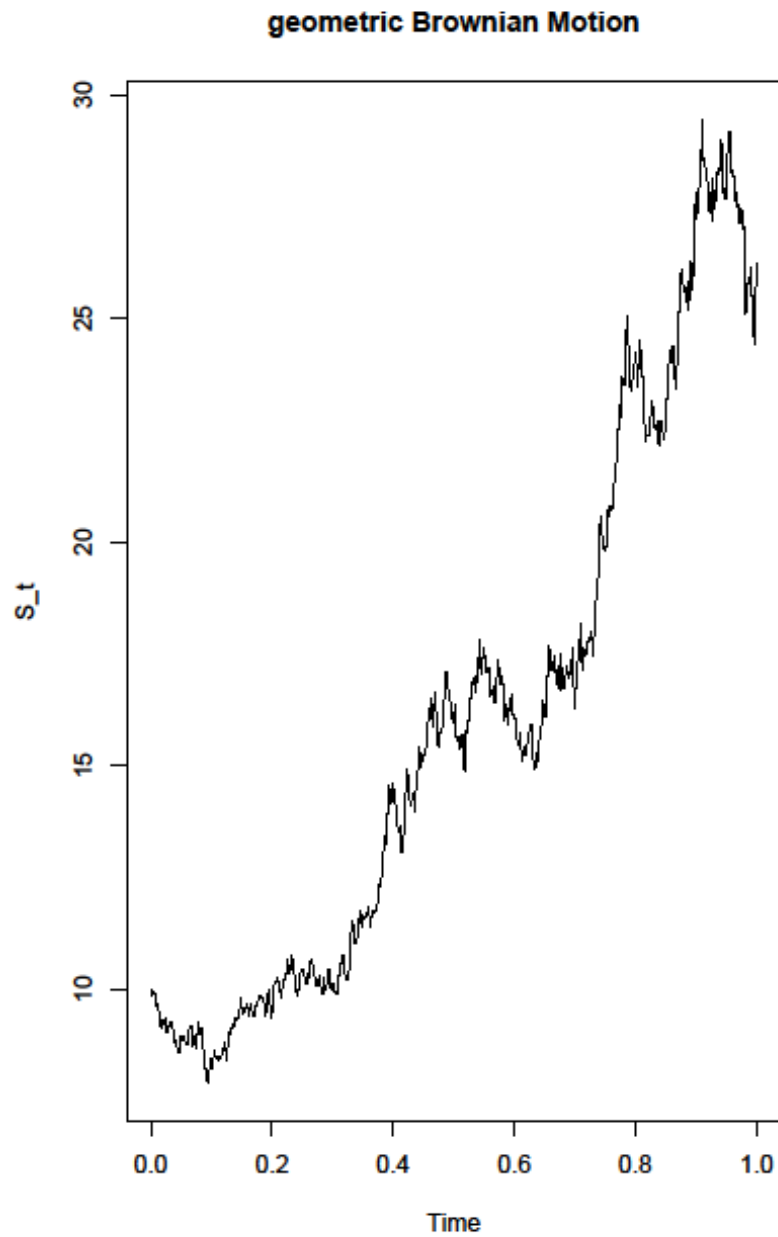The data is simulated using equation (4).

Figure 5.1: the Geometric Brownian Motion model through the time.

Figure (5.1) shows the movement of Geometric Brownian motion through time. It is clearly that the process affect by Brownian motion and always is positive.

Table 5.1: α, MSE and K_{opt} for five samples based on 100 replications driven by Geometric Brownian motion for simulation data.

N=50	Direct	Bootstrap	Double Bootstrap
α	0.07107605	0.0102245247	0.033000000
MSE	0.01004159	0.0001292511	0.001716803
K_{opt}	7.11581631	3.0000000000	2.000000000
N=100			
α	0.041829245	0.0105181566	0.0223157895
MSE	0.002247387	0.0001373524	0.0004991821
K_{opt}	0.490268148	3.0000000000	2.0000000000
N=150			
α	0.04216307	0.0162245871	0.045960000
MSE	0.00261230	0.0003937763	0.002791125
K_{opt}	0.25461379	3.4100000000	2.610000000
N=200			
α	0.050240318	0.0164169738	0.054640000
MSE	0.002988345	0.0003870336	0.003323239
K_{opt}	0.168603382	3.9500000000	2.970000000
N=250			
α	0.056774807	0.0224340013	0.066000000
MSE	0.003709766	0.0008237568	0.005643042
K_{opt}	0.041839528	4.1200000000	3.350000000
N=500			
α	0.051385442	0.0229479856	0.13840000
MSE	0.002843486	0.0007349886	0.02520428
K_{opt}	0.028233823	6.0200000000	5.24000000
N=800			
α	0.045265601	0.0234060623	0.24053333
MSE	0.002127585	0.0007046003	0.07481925
K_{opt}	0.021606282	8.2000000000	7.44000000

Table (5.1) represent the value of α, MSE and K_{opt} for our model using Direct, Bootstrap and Double Bootstrap methods. When 100 replications, it is obvious that the Bootstrap method is much better than the others. We also note that the Direct method is better than the Double Bootstrap method when the sample size is greater than or equal to 150.

Second model

The data is simulated using the following model:

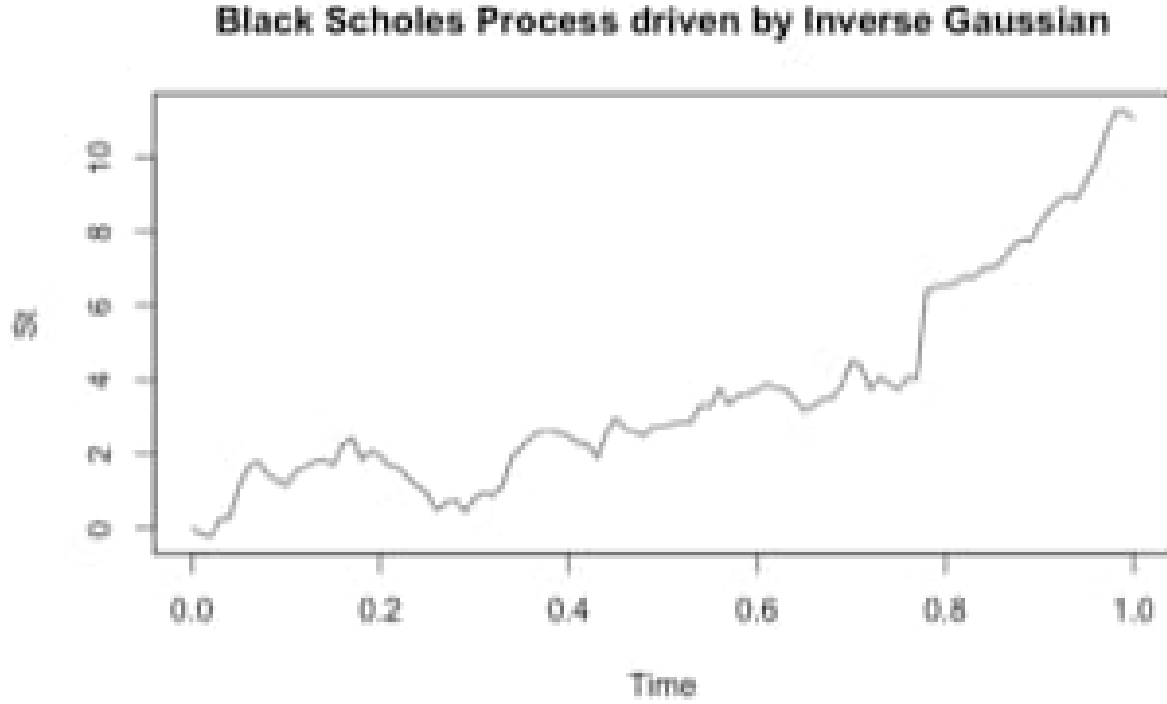$$S_{t+1} = S_t exp\left(\left(\theta_1 - \frac{1}{2}\theta_2^2\right)\Delta T + \theta_2 DL_t\right) \qquad \text{(21)}$$

Figure 5.2: the Levy process through time.

Figure (5.2) shows the movement of Levy process through time. It is clear that the process affects by Inverse Gaussian and always is positive.

Table 5.2: α, MSE and K_{opt} for five samples based on 100 replications.

N=50	Direct	Bootstrap	Double Bootstrap
α	0.035319927	0.0111697464	0.0210000000
MSE	0.002396321	0.0001245574	0.0004164086
K_{opt}	5.465085974	3.0000000000	2.0000000000
N=100			
α	0.031221234	0.0172252821	0.045560000
MSE	0.001191061	0.0004293483	0.002385551
K_{opt}	0.215970520	3.4400000000	3.000000000
N=150			
α	0.038914153	0.0205659604	0.053600000

MSE	0.001836429	0.0007201316	0.003359408
K_{opt}	0.146922287	4.0000000000	3.000000000
N=200			
α	0.038106131	0.022992670	0.077800000
MSE	0.001644134	0.000778499	0.007320747
K_{opt}	0.100875938	4.870000000	4.000000000
N=250			
α	0.034531429	0.030054351	0.081680000
MSE	0.001272489	0.001684765	0.007830792
K_{opt}	0.087406402	5.000000000	4.110000000
N=500			
α	0.031824993	0.026969259	0.19573333
MSE	0.000177517	0.000893379	0.04707668
K_{opt}	0.059326323	7.980000000	7.00000000
N=800			
α	0.0293744686	0.028605673	0.348000
MSE	0.0009357992	0.001203129	0.147316
K_{opt}	0.0492388616	10.900000000	10.860000

Table (5.2) represent the value of α, MSE and K_{opt} for our model using Direct, Bootstrap and Double Bootstrap methods. When 100 replications, it is obvious that the Bootstrap method is better than other methods when the sample size is 50,100,150,200 while the direct method is better when the sample size is greater than or equal to 250.

6: Real data

In this section we will apply our methods to a set of real data represented by the daily Iraqi financial market dataset (ISX) for the dinar for the period 1/1/2017 - 1/1/2020.The data was obtained from (Homepage www.isx.iq.net). We used the daily returns for the mentioned period as follows:

$$rS = \log \frac{S_t}{S_{t-1}} \qquad \textbf{(4-1)}$$

where

rS: represent daily returns at time t.

S_t: exchange rate at time t.

4.2. Kolmgorov-Smirnov test

We use this test to see if the data follows a normal distribution or not. The null hypothesis of the test states that the data have a normal distribution. The p-value of the test is ($2.2e^{-16}$) at the significant level of (5%). Therefore, the null hypothesis was rejected, which means that the data do not follow a normal distribution.

4.3. Barndorff-Nielsen and Shephard jump test

In order to check for jumps in the data, we use the Barndorff-Nielsen and Shephard jump test. The null hypothesis of this test states that there are no jumps. At the significant level (5%), the test value is (1.3239) and the p-value is (0.09276)for the data. As for the returns, it was the test value is (-0.34325) and the p-value is (0.6343). Therefore, we will accept the null hypothesis, which means that there are no jumps, whether the test is for data or returns.

The comparison between the studied methods was done by calculating the MSE of the data, where the tail index of the studied methods was compared with the tail index of the Hill estimator. We used N=897, $\theta_1 = 0.5$ and $\theta_2 = 0.05$.
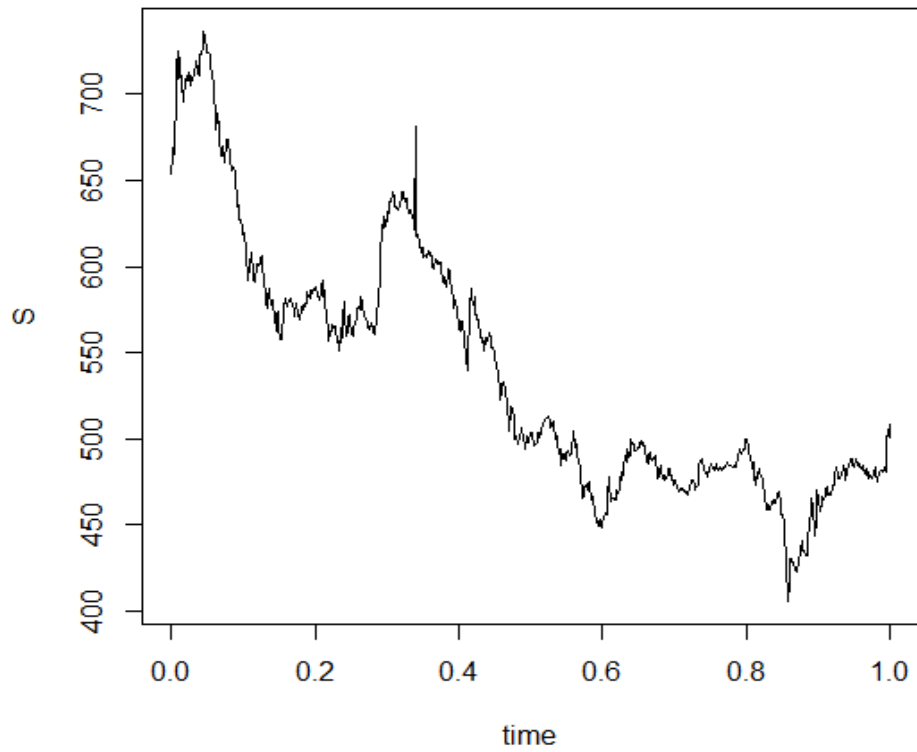
Figure 6.1: the real data through time.

Figure (6.1) represent the real data (ISX) during 2017-2020. It is clear that the behavior of our index follows the Stochastic Differential Equation.

After analyzing the data using R- program, we obtained some results presented in the following table.

Table 6.1: α, MSE and K_{opt} for five samples.

N=898	Direct	Bootstrap	Double Bootstrap
α	0.006878175	0.0017532978	0.02000000
MSE	0.000111708	0.0002463041	0.000006515888
K_{opt}	5.832650928	4.0000000000	2.00000000

Table 6.1: represent the value of α, MSE and K_{opt} for real data using Direct, Bootstrap and Double Bootstrap methods. it is obvious that the Double Bootstrap method is better than other methods when the sample size is N= 898.

7: Conclusion

In the simulation in first model, represented by SDE driven by G.B.M. the Bootstrap method was the best for all sample sizes. for second model, represented by SDE driven by Levy process the Bootstrap method was also best when the sample size was less than 250. while for large values, the Direct method outperformed the others.

In the real data, the Double Bootstrap method was the best, and there is a very clear convergence in the results of the other methods.

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