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**Republic of Iraq**

**Ministry of Higher Education & Scientific Research**

**University of AI-Qadisiyah**

**College of Computer Science and Information Technology**

**Department of Mathematics**

**A Study of Differential Subordination and Coefficients Properties of Univalent and Multivalent Functions**

**A Thesis**

**Submitted to the Council of the College of Computer Science and Information Technology, University of AI-Qadisiyah as a partial fulfilment of the requirements for the degree of Master of Science in Mathematics**

**By**

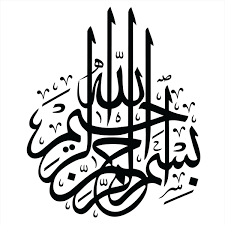
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 ﴿ يَرْفَعِ اللَّهُ الَّذِينَ آمَنُوا مِنْكُمْ وَالَّذِينَ أُوتُوا الْعِلْمَ دَرَجَاتٍ وَاللَّهُ بِمَا تَعْمَلُونَ خَبِير ﴾

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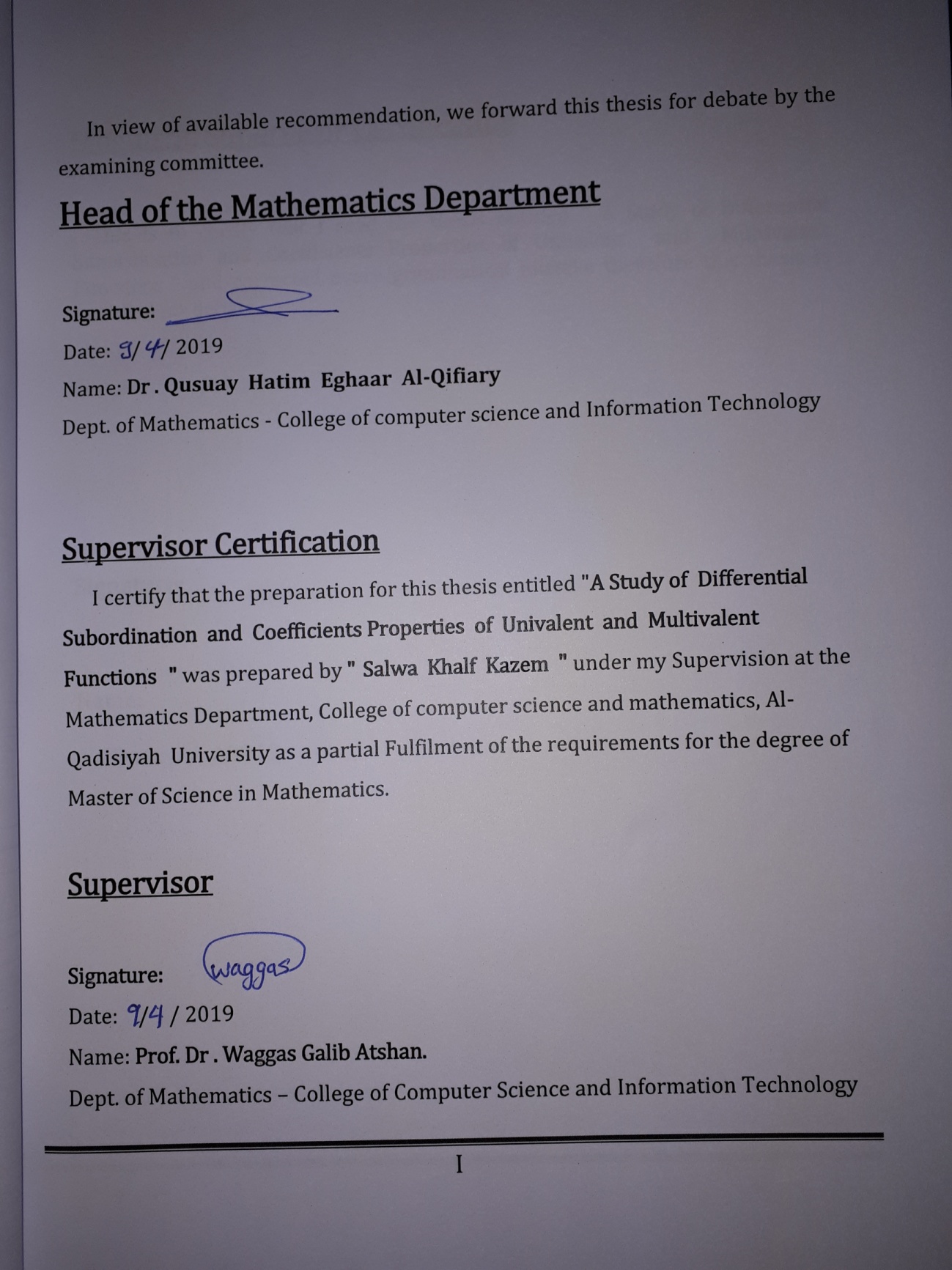
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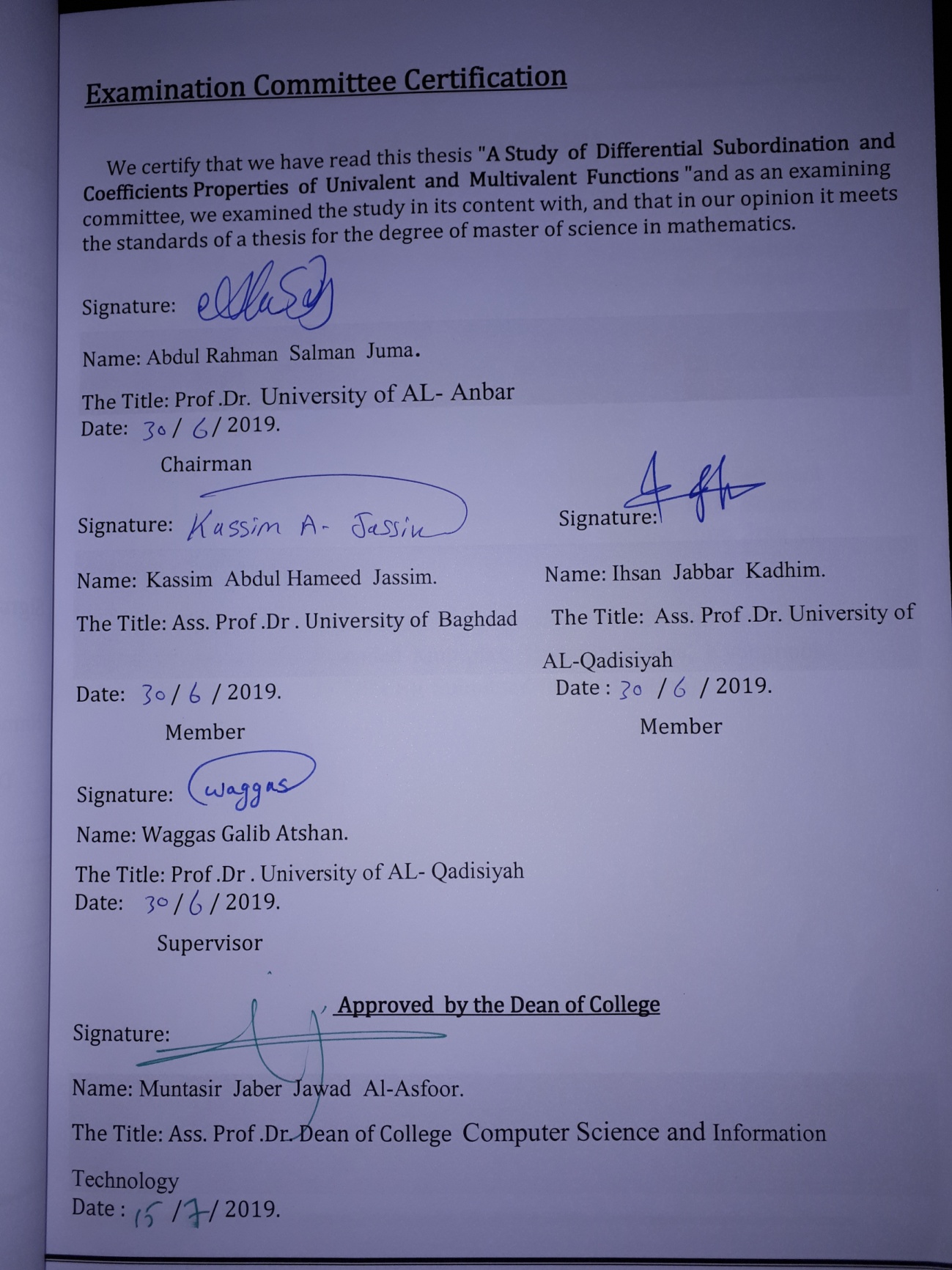
Special thanks go to my husband Wisam Hadi .

Salwa Khalf Kazem

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List of Publications

1. On Differential Sandwich Theorems of Multivalent Functions Defined by a Linear operator, Journal of Al-Qadisiyah for Computer Science and Mathematics, 11(1)(2019), 117-123.
2. Coefficient Estimates for Some Subclasses of Bi-Univalent Functions Related to m-fold Symmetry, Journal of Al-Qadisiyah for Computer Science and Mathematics, 11(2)(2019), 81-86.
3. Coefficient Estimates of New Subclasses of Bi-Univalent Functions, The 1st International Scientific Conference on Pure Science (ISCPS2019) ,University of Al-Qadisiyah, College of Education ,Iraq on the 23th – 24th January (2019).
4. On Third-Order Differential Subordination Results for Univalent Functions Associated Extended Multiplier Transformations, Kyungpook Mathematical Journal **(SCOPUS)(** Submitted for Publication).

**Listofsymbols**

The list should be rearranged Alphabetically

|  |  |
| --- | --- |
| Symbol | Description |
|  | Unit disk. |
|  | Punctured unit disk. |
|  |  |
|  | Domain |
|  | The boundary of . |
|  | The Complex plane . |
|  | . |
|  | The set of natural numbers. |
|  | The set of integer numbers. |
|  | The set of real numbers. |
|  |  |
|  | Class of analytic and p-valent functions in the unit disk . |
|  | Class of analytic functions in . |
|  | The subclasses of |
|  | The family of all functions analytic in and  . |
|  | Linear operator. |
|  | The set of all functions |
|  | Th Disk (or region) |
|  | Class of analytic functions in the unit disk . |
|  | Class of all functions in which are univalent and normalized in . |
|  | The class of all bi-univalent functions defined in |
|  | The class of m-fold symmetric bi-univalent functions in |
|  | Class of all starlike functions of order in |
|  | Class of all convex functions of order in |
|  | Hadamard product of and |
|  | subordinate to . |
|  | Class of all starlike functions of order 0 |
|  | Class of all convex functions of order 0 |
|  | Class of analytic functions in of the form:  . |
|  | Class of admissible functions of the form: |
|  | Class of admissible functions of the form: |
|  | New classes of analytic and bi-univalent functions in the unit disk |
|  | Classes of m-fold symmetric bi-univalent functions . |

**Abstract**

The purpose of this thesis is to study differential subordination and coefficients properties of univalent and multivalent functions .

It studied differential sandwich theorems of multivalent functions defined by a linear operator . We derive some results for multivalent analytic functions by using differential subordination and superordination . our thesis also dealt with the third – order differential subordination results for univalent functions associated extended multiplier transformation . New results for third-order differential subordination in the open unit disk have been done . Also, the dealing has been done with coefficients estimates of new subclasses of bi-univalent functions . Here , new subclasses and of consisting of analytic and bi-univalent functions in the open unit disk *U* are investigated and studied , we obtain estimates on the coefficients . Lastly , coefficient estimates for some subclasses of bi-univalent functions related to m-fold symmetry are discussed . Here , we investigate and introduce new subclasses and of analytic and m-fold symmetric bi-univalent functions in , among other results belonging to these subclasses upper coefficients bounds and are obtained . Certain special cases are also indicated .

**Introduction**

The study of analytic univalent functions have been engaging the attention of researchers at least till as early as 1907 . This has been growing vigorously with added research . This field captioned as geometric function theory is found to be a hybrid or an interplay of geometric and analysis . Despite the classical nature of the subject , unlike contemporary areas , this field has been fascinating researchers , with stress on the interest based on investigations by function theorists. The main contents motivating this line of thought is based on the famous conjecture called the Bieberbach conjecture or coefficient problem offering vast scope for development from 1916 , till a positive settlement in 1985 be De Branges where innumerable results were obtained based on this problem . Since then, geometric function theory was a subject in its own right .

Geometric Function Theory is a classical subject. Yet, it continues to find new applications in an ever-growing variety of areas such as modern mathematical physics , more traditional fields of physics such as fluid dynamics , nonlinear integrable systems theory and the theory of partial differential equations . Detailed treatment of univalent functions are available in the standard books of Duren and Goodman .

A function analytic in a domain of the complex plane is said to be univalent or one-to-one in if it never takes the same value more than once in . That is , for any two distinct points and in , . The choice of the unit disk as a domain for the study of analytic univalent functions is a matter of convenience to make the computations simpler and leads to elegant formula . There is no loss of generality in this choice , since Riemann Mapping Theorem asserts that any simply connected proper subdomain of can be mapped onto the unit disk by univalent transformation

The class of all analytic functions in the open unit disk will be denoted by , consisting of functions of the form :

Also, denoted by the class of all functions in which are univalent and normalized by . Geometrically , the normalization amounts to only a translation of the image domain and corresponds to rotation and stretching or shrinking of the image domain .

The function called the Koebe function is defined by

which maps onto the complex plane except for a slit along the negative real axis from is a leading example of a function in . It plays a very important role in the study of the class . In fact, the Koebe function and its rotations are the only extremal functions for various extremal problems in . The study of univalent functions was initiated by Koebe . He discovered that the ranges of all functions in contain a common disk , later named as the Koebe domain for the class in honour of him .

For functions in the class , it is well known that the following growth and distortion estimates hold respectively as for

and

Further for functions in the class , it is well known that the following rotation property holds :

where . The bound is sharp .

In 1916, Bieberbach studied the second coefficient of function . He has shown that , with equality if and only if is a rotation of the Koebe function and he mentioned ‘’ is generally valid’’. This statement is known as the Bieberbach conjecture .

In 1923 Lwner proved the Bieberbach conjecture for , many others investigated the Bieberbach conjecture for certain values of . Finally, in 1985 De Branges proved the Bieberbach conjecture for all coefficients with the help of hypergeometric functions .

Since the Bieberbach conjecture was difficult to settle, several authors have considered classes defined by geometric conditions . Notable among them are the classes of starlike functions , convex functions and close-to-convex functions .

Subordination for analytic functions returns back to Littlewood and Lindelf , where Rogosinski introduced the term and established the basic results involving subordination . Quite recently, Srivastava and Owa investigated various interesting properties of the generalized hypergeometric function by applying the concept of subordination . Ma and Minda showed that many of these properties can be obtained by a unified method . For this purpose they introduced the classes of functions , for some analytic functions with positive real part on , with maps the open unit disk onto a region starlike with respect to 1 , symmetric with respect to the real axis , satisfying :

Geometric Function Theory has experienced resurgence in recently the methods of function theory on compact Riemann surfaces and algebraic geometry .

Early string theory models depend on elements of geometric function theory for the computation of so called Veneziano amplitudes was appeared .

This thesis is organized as follows . Chapter One , we present a brief introduction to some background in complex concepts and the basic ideas in geometric function theory .

Chapter two consists of two sections , in the first section , we deal with differential sandwich theorems of multivalent functions defined by a linear operator . Some results for multivalent analytic functions by using differential subordination and superordination are obtained. The second section deals with the third- order differential subordination results for univalent functions associated extended multiplier transformation . Here , we obtain new results for third –order differential subordination in the open unit disk .

Chapter three is divided into two sections : section one deals with coefficients estimates of new subclasses of bi- univalent functions . Here , we introduce and study new subclasses and of consisting of analytic and bi–univalent functions in the open unit disk *U* . We obtain estimates on the coefficients for functions of these classes . Section two deals with coefficient estimates for some subclasses of bi-univalent functions related to m-fold symmetry. Here , we investigate and introduce new subclasses and of analytic and m-fold symmetric bi-univalent functions in . Among other results belonging to these subclasses upper coefficients bounds and are obtained . Certain special cases are also indicated.

**Chapter One**

**Basic Definitions and Fundamental Results**

**Introduction :**

In this chapter , we have given all of the required definitions , some examples and standard results of analytic functions, univalent, multivalent and bi-univalent functions which are needed in subsequent chapters for research. The detailed proofs and further discussions can be found in standard texts such as Duren , Goodman Miller and Mocanu and other references .

**1.1 Basic Definitions**

**Definition :** A function of the complex variable is analytic at a point if its derivative exists not only at but each point in some neighborhoods of . It is analytic in region if it is analytic at every point in .

**Definition** : A function is said to be univalent if it does not take the same value twice i.e. for all pairs of distinct points . In other words, is one-to-one (or injective) mapping of onto another domain .

If assumes the same value more than one, then is said to be multivalent (-valent) in . We also deal with the functions which are meromorphic univalent in the punctured unit disk . is said to be meromorphic if it is analytic at every point inexcept finite elements in .

**Example (1.1.1)[ 37]:**  The function is univalent in . But the function is not univalent in .

**Definition :** Let denotes the class of functions of the form :

which are analytic in the open unit disk .

**Definition :** A function is said to be locally univalent at a point if it is univalent in some neighborhood of . For analytic function the condition is equivalent to local univalence at

**Definition :** We say that is normalized if satisfies the conditions .

**Definition :** A set is said to be starlike with respect to if the linear segment joining to every other point lies entirely in .In a more picturesque language, the requirement is that every point of is visible from . The set is said to be convex if it is starlike with respect to each of its points, that is , if the linear segment joining any two points of lies entirely in .

**Definition :** A function  is said to be conformal at point if it preserves the angle between oriented curves passing through in magnitude as well as in sense. Geometrically, images of any two oriented curves taken with their corresponding orientations make the same angle of intersection as the curves at both in magnitude and direction. A function is said to be conformal in the domain , if it is conformal at each point of the domain .

**Definition** : A function is said to be starlike function of order if

Denotes the class of all starlike functions of order in by and the class of all starlike functions of order 0, . Geometrically, we can say that a starlike function is conformal mapping of the unit disk onto a domain starlike with respect to the origin.

**Example:** The function is starlike function of order .

**Definition** : A function is said to be convex function of order if

Denotes the class of all convex functions of order in by and for the convex function .

**Definition :** A Mbius transformation ,or a bilinear transformation, is a rational function of the form :

where are fixed and .

**Definition :** Let denote the class of analytic valently functions in of the form:

We say that is valently starlike of order and valently convex of order , respectively if

**Definition :** Iffunctions and belonging to the class given by

then the Hadamard product or (convolution) of functionsanddenoted by is defined by

**Definition** : Let be a function . We call a Schwarz function, if for all , then

where "capital " is defined as follows:

Let and be any two sequences and ≥ 0 for all . If there exists a fixednumber 𝜂 > 0 such that ≤ 𝜂 (for all ), then we write

**Definition :** Let and be analytic functions in . The function is said to be subordinate to , written , if there exists a Schwarz function , which is analytic in , with such that

In particular, if the function is univalent in then if and only if and .

**Definition** : Let and let be univalent in . If is analytic in and satisfies the second- order differential subordination:

then is called a solution of the differential subordination . The univalent function is called a dominant of the solutions of the differential subordination , moreover simply dominant, if for all satisfying .

A univalent dominant that satisfies for all dominant of is said to be the best dominant of . Note that the best dominant is unique up to a relation of .

**Definition** : Let and let be analytic in . If and are univalent in and if satisfies the second-order differential superordination:

then is called a solution of the differential superordination . It is worth mentioning to the analytic function which is called a subordinant of the solutions of the differential superordination or more simply a subordinant , if for all satisfying . A univalent subordinant that satisfies for all subordinants of is said to be the best subordinant .

**Definition [37]:** Denote by the set of all functions that are analytic and injective on , where and

and are such that for .

Further , let the subclass of for which be denoted by .

**Definition :** Let and be univalent in . If is analytic function in and satisfies the following third-order differential subordination.

then the function is called a solution of the differential subordination. A univalent function is called a dominant of the solutions of the differential subordination if for all satisfying (1.9). A dominant that satisfies for all dominants of (1.9) is said to be thebest dominant .

**Definition :** Let and be analytic in . If the function and are univalent in and satisfies the following third-order differential superordination .

then function is called a solution of the differential superordination . An analytic function is called subordinant of the solutions of the differential superordination, or simply a subordination if for all satisfying .

A univalent subordinant that satisfies for all subordinants of is called the best subordinant .

**Definition:** Let be a set in . Also let and , being the set of positive integers . The class of admissible functions consists of those functions , which satisfy the following admissibility conditions :, whenever

and

where

**Definition :** Let be a set in and . The class of admissible functions consists of those functions that satisfy the following admissibility condition : whenever

,

where

**Definition :** An analytic function is said to be bi-univalent in a domain , if and its inverse are both univalent in . The class of all bi-univalent analytic functions in is denoted by .

**Example**: The function are bi-univalent.

**Example** : The functions are not bi-univalent.

**On Differential Sandwich Theorems of Multivalent**

**Functions Defined by a Linear Operator**

Let denote the class of functions of the form:

which are analytic in the open unit disk .

For two functionsand are analytic in , we say that the function is subordinate to in , written , if there exists Schwarz function , analytic in with Ifis univalent and , then .

If is given by (2.1) and given by

Then Hadamard product (or convolution) is defined by

Using the results, Bulboacă [11] considered certain classes of first order differential superordinations as well as superordination preserving integral operator [12]. Ali et al. [1], have used the results of Bulboacă [11] to obtain sufficient conditions for normalized analytic functions to satisfy:

where are given univalent functions in . Also, Tuneski [64] obtained a sufficient conditions for starlikeness of in terms of the quantity

Recently, Shanmugam et al*.* [49,50] and Goyal et al*.* [23] also obtained sandwich  
results for certain classes of analytic functions.

The linear operator defined by

where

and

For

Then linear operator

is defined by

Where is the function defined in terms of the Hadamard product by the following condition:

We can easily find from (2.3) - (2.5) that

It is easily verified from (2.6) that

Note that the linear operator unifies many other operators considered earlier. In particular

1. .
2. .
3. .
4. .

The main object of this idea is to find sufficient conditions for certain normalized analytic functions to satisfy:

and

where and are given univalent functions in with =

**Theorem 2.1.1.** Let be convex univalent in with . Suppose that

If is satisfies the subordination

where

then

and is the best dominant.

**Proof:** Define a function by

then the function is analytic in and therefore, differentiating logarithmically with respect to and using the identity in the resulting equation,

Thus the subordination (2.9) is equivalent to

An application of Lemma with and , we obtain .

Taking in Theorem , we obtain the following Corollary.

**Corollary 2.1.1.** Let . Suppose that  
.

If is satisfy the following subordination condition:

where given by , then

and is the best dominant .

Taking in Corollary (2.1.1) , we get following result.

**Corollary 2.1.2.** Let

.

If is satisfy the following subordination

where then

and is the best dominant.

**Theorem 2.1.2.** Let be convex univalent in unit disk with , let, and suppose that and satisfy the following conditions

and

If

where

then

is best dominant .

**Proof :** Define analytic function by

Then the function is analytic in and ,

differentiating logarithmically with respect to , we get

By setting and it can be easily observed that is analytic in , is analytic in and that . Also , if we let

, and

,

we find that is starlike univalent in , we have

and

,

hence that

By using , we obtain

Apply to obtain

and by using Lemma , we deduce that subordination implies that and the function is the best dominant .

Taking the function , in Theorem (2.1.2) , the condition becomes.

hence, we have the following Corollary.

**Corollary 2.1.3.** Let . Assume that holds. If and

where is defined in , then

is best dominant .

Taking the function in Theorem (2.1.2), the condition becomes

hence ,we have the following Corollary .

**Corollary 2.1.4.**  Let . Assume that holds. If and

where is defined in then

is the best dominant.

**Theorem 2.1.3.** Let be convex univalent in with

and

If the function defined by is univalent and the following superordination condition:

holds , then

and is the best subordinant.

**Proof:** Define a function by

Differentiating with respect to logarithmically , we get

A simple computation and using from , we get

now , by using Lemma , we get the desired result .

Taking we get the following Corollary.

**Corollary 2.1.5.** Let  and such that

.

If the function given by is univalent in and satisfies the following superordination condition:

then

and the function is the best subordinant .

**Theorem 2.1.4.** Let be convex univalent in unit disk , Let Suppose that

and

.

If the function is given by is univalent in ,

implies is the best subordinant.

**Proof:** Let the function defined on by .

Then a computation show that

by setting it can be easily observed thatis analytic inis analytic in Also , we get it observed that is starlike univalent in . Since is convex , it follows that

By making use of the hypothesis can be equivalently written as

thus , by applying Lemma , the proof is complete.

Combining Theorem with Theorem , we obtain the following sandwich Theorem.

**Theorem 2.1.5.**  Let and be convex univalent in with and satisfies . Suppose that . If

,

and the function defined by is univalent and satisfies

then

where and are respectively , the best subordinant and the best dominant of . Combining Theorem (2.1.2) with Theorem (2.1.4) , we obtain the following sandwich Theorem.

**Theorem 2.1.6.** Let be two convex univalent functions in , such that , (i=1,2). Suppose that and satisfies and , respectively.

If and suppose that satisfies the next conditions:

, and

,  
 and is univalent in , then

implies

and and are the best subordinant and the best dominant respectively and is given by .

**On Third-Order Differential Subordination Results for Univalent Functions Associated with Extended Multiplier Transformations**

Let be the class of analytic function in open unit disk and let denote the subclass of the functions of the form

Also, let be the subclass of the functions of the form

For , we say that the function is subordinate to , written symbolically as follows :

If there exists a Schwarz function , which (by definition) is analytic in with and , , such that

In particular, if the function is univalent in then we have the following equivalence (cf.,e.g.,[36] ; see also [37,p.4]):

The concept of differential subordination is a generalization of various inequalities involving complex variables . We recall some more definitions and terminologies

from the theory of differential subordination and differential superordination .

**Definition 2.2.1**  Let the function , the extended multiplier transformation on is defined by the following infinite series :

We can write as follows : where

It is easily verified from , that

We note that :

Also by specializing the parameters and m we obtain the following operators studied by various authors :

;

;

;

.

We first define the following class of admissible functions , which are required in proving the differential subordination theorem involving the operator defined by .

**Definition 2.2.2.** Let be a set in and . The class of admissible function consists of those functions that satisfy the following admissibility conditions :

whenever

,

and

*,*

where

**Theorem 2.2.1.** Let . If the function satisfy the following conditions :

and

then

**Proof.** Define the analytic function by

Then , differentiating with respect to and using , we have

Further computations show that

and

Define the transformation from by

,

,

.

Let The proof will make use of Theorem . Using the equations to , and from the equation , we have

Hence becomes

.

Note that

and

.

Thus , clearly , the admissibility condition for in Definition is equivalent to the admissibility condition for as given in Definition with . Therefore , by using and Theorem , we have

or , equivalently ,

.

This completes the proof of Theorem(2.2.1).

Our next result is a consequence of Theorem (2.2.1) for the case when the behavior of is not known .

**Corollary 2.2.1.** Let and let the function be univalent in with . Let for some , where . If the function satisfies the following conditions :

and

then

.

**Proof:** By applying Theorem (2.2.1) , we get

The result asserted by Corollary (2.2.1) is now deduced from the following subordination property

If is a simply connected domain , then for some conformal mapping . In this case , the class is written as The following two results are immediate consequence of Theorem (2.2.1) and Corollary (2.2.1).

**Theorem 2.2.2.** Let . If the function and satisfy condition and

Then

**Corollary 2.2.2.** Let and let the function be univalent in. Also let for some , where If the function satisfy the condition , and

Then

The following result yields the best dominant of the differential subordination .

**Theorem 2.2.3.** Let the function be univalent in . Also let be given by . Suppose that the differential equation

has a solution and satisfies the condition . If the function satisfies condition and

is analytic in , then

and is the best dominant .

**Proof .** From Theorem (2.2.1) , we deduce that is a dominant of . Since satisfies , it is also a solution of . Therefore , will be dominated by all dominated. Hence is the best dominant . This completes the proof of Theorem .

In view of Definition (2.2.2) , and in the special case when , the class of admissible functions , denoted by , is expressed as follows:

**Definition 2.2.3.** Let be a set in The class of admissible functions consists of those functions such that

whenever

**Corollary 2.2.3.** Let . If the function satisfies :

and

Then

.

In the case for simplification, we denote by to the class .

**Corollary 2.2.4.** Let If the function satisfies the following conditions :

and

then

**Corollary 2.2.5.** Let If the function satisfy the following conditions :

then

**Proof .** We define where

Use Corollary (2.2.3), we need to show that that is , that the admissibility condition is satisfied . This follows readily , since it is seen that

whenever The required results now follows from Corollary (2.2.3) . This completes the proof of Corollary .

**Definition 2.2.4.** Let be a set in and . The class of admissible functions consists of those functions that satisfy the following admissibility condition :

whenever

and

where

**Theorem 2.2.4.** Let . If the function , satisfy the following conditions :

and

Then

**Proof .** Define the analytic function by

By using and , we get

Further computations show that

and

Define the transformation from by

and

.

Let

The proof will make use of Theorem . Using the equations to . and from , we obtain

Hence becomes

.

Note that

and

.

Thus , the admissibility condition for in Definition (2.2.4) is equivalent to the admissibility condition for as given in Definition with n=2 . Therefore , by using and Theorem, we have

which completes the proof of Theorem (2.2.4) .

If is simply connected domain and for some conformal mapping , then the class written as .

The following result is immediate consequence of Theorem (2.2.4).

**Theorem 2.2.5.**  Let . If the functions satisfy the condition and

then

.

The next result is an extension of Theorem (2.2.4) to the case where the behavior of is not known .

**Corollary 2.2.6.** Let and let be univalent function in with Let for some where . If satisfy the following conditions:

and

then

, .

**Proof .** As a consequence of Theorem (2.2.4) , that

.

Now, the proof of Corollary (2.2.6) can be deduce from the following subordination property :

.

The proof of Corollary (2.2.6) is complete .

**Corollary 2.2.7.** Let and let be univalent in U , . Let for some where . If satisfy the following conditions :

and

then

, .

The next Theorem yields best dominant of the differential subordination

**Theorem 2.2.6.** Let be univalent function in and and be given by. Suppose that the differential equation

has a solution . If the function satisfy the condition and

is analytic in , then implies that

and is the best dominant .

**Proof .** By applying Theorem (2.2.4) , we deduce that is a dominant since is satisfies , it is also a solution of and therefore will be dominated by all dominants . Hence is best dominant .

**Definition 2.2.5.** Let be a set in , with . The class of admissible functions consists of those functions which satisfy the following admissibility condition :

whenever

and

where .

**Theorem 2.2.7.** Let . If the functions

satisfy the following conditions:

and

univalent in, then

implies

**Proof .** Let the function be defined by and be defined by . Since . Therefore and imply

From , we deduce that the admissible condition for in Definition . Hence by using the conditions in and using Theorem , we have

, or , equivalently ,

Therefore we completes the proof of Theorem(2.2.7) .

If is simply connected domain and for some conformal mapping of .In this case the class is written as .

**Theorem 2.2.8.** Let .Also , let the function be analytic in . If the functions satisfy the condition

and

is univalent in , then

implies that

The following Theorem proves the existence of best subordinant of for a suitable chosen .

**Theorem 2.2.9.** Let the function be analytic in given by Suppose that the differential equation

has a solution . If satisfy the condition

and

is univalent in , then implies that

and is the best subordinant .

**Proof .**The proof is similar to the proof of Theorem (2.2.3) and is therefore omitted . By combining Theorem (2.2.2) and Theorem (2.2.8) , we obtain the following sandwich type result .

**Theorem 2.2.10.**  Let be analytic in . Also , let be univalent in , with and . If the functions and

is univalent in and the conditions and are satisfied , then

implies that

Next , we introduce a new admissible class , as follows .

**Definition 2.2.6.** Let be a set in with and

The class of admissible functions consists of those functions which satisfy the following admissibility condition :

whenever

and

where .

**Theorem 2.2.11.**  Let  **.** If the functions and and with satisfy the following conditions:

and

,

univalent in, then

implies

.

**Proof .** Let the function be defined by and be defined by . Since . Therefore and imply

From , we deduce that the admissible condition for in Definition . Hence by using the condition and using Theorem , we have

Therefore , we complete the proof of Theorem (2.2.11) . If is simply connected domain and for some conformal mapping of . In this case the class is written as .

**Theorem 2.2.12.** Let . Also , let the function be analytic in . If the functions satisfy condition and

is univalent in , then

implies

The following Theorem proves the existence of best subordinant of for an appropriate .

**Theorem 2.2.13.**  Let the function be analytic functions in and and be given by .Suppose that the differential equation

has a solution If and satisfy the condition and

is univalent ,then implies that

and is the best subordinant .

**Proof :** The proof is similar to that of Theorem (2.2.6) and it is being omitted here .

Combining Theorems (2.2.5) and (2.2.12) , we obtain the following Sandwich type Theorem.

**Theorem 2.2.14.** Let and be analytic in . Also , let be univalent in , with and if and

and

is univalent in and the conditions and are satisfied , then

implies that

**Coefficient Estimates of New Subclasses of bi-univalent**

**functions**

Let be the class of functions of the form:

,  (3.1)

which are analytic in the open unit disk .

Also, let denoted the subclass of all function in which are univalent and normalized by the conditions It is well known that every univalent function has inverse

satisfying :   
,

and   
,

where

A function is said to be bi-univalent in if both and are univalent in .

Let denote the class of bi-univalent functions in given by (3.1). For a brief history and interesting example in the class .

For we consider the differential

operator was introduced by Amourah and Darus

.

Brannan and Taha introduced certain subclasses of the bi-univalent function class similar to the familiar subclasses. of starlike and convex functions of order respectively Thus, following Brannan and Taha a function in the class of strongly bi-starlike functions of order if each of the following conditions are satisfied :

,

and

where is the extension of of The classes and of bi-starlike functions of order and bi-convex functions of order , corresponding to the functions classes , were also introduced analogously. For each of the function classes , they found non-shap estimates on the first two Taylor-Maclaurin coefficients (cf. ) .

The object of this work is to introduce a new subclass of the function class and find estimates on the coefficients for

functions in these new subclass of the function class employing the techniques used earlier by Srivastava et al .

**Definition 3.1.1:** A function given by is said to be in the class if the following conditions are satisfied :

f and

and

,,

where the function is given by

**Theorem 3.1.1:** Let the function given by be in the class  
. Then

and

**Proof**: Let . Then

and

where and are in and have the forms

and

Now, equating the coefficients in (3.8) and (3.9) , we obtain

From (3.12) and (3.14) , we obtain

and

Now from (3.13) , (3.15) and (3.17) , we obtain

*=*

Therefore, we have

Applying Lemma (1.2.5) for the coefficients and . We immediately have

Next , in order to find the bound on by subtracting (3.15) from (3.13) , we obtain

It follows from (3.16) , (3.18) and (3.17) ,that

.

Or , equivalently ,

.

Applying Lemma (1.2.5) for the coefficients and , we readily get

.

This completes the proof of Theorem (3.1.1).

**Definition 3.1.2:** A function given by (3.1) is said to be in the class if the following conditions are satisfied :

*and*

and

where the function is given by (3.2)

**Theorem 3.1.2:** Let given by (3.1) be in the class and  **.** Then

and

**Proof :** Let .Then

and

1+

where and have the forms and , respectively . Equating coefficients in and yields

and

From (3.25) and (3.27) , we get

Also , from (3.26) and (3.28) , we find that

Thus , we have

Applying Lemma (1.2.5) for the coefficients and . We obtain

,

which is the bound on as given in . Next , in order to find the bound on , by subtracting and , we get

Or , equivalently ,

Upon substituting the value of from (3.30) , we obtain

Applying Lemma (1.2.5) for the coefficients we readily get

,

which is the bound on as asserted in (3.22) .

If we set in Theorems (3.1.1) and (3.1.2) .

**Definition :** A function given by (3.1) is said to be in the class if the following conditions are satisfied :

*and*

and

where the function is given by (3.2) .

**Definition 3.1.4:** A function given by (3.1) is said to be in the class if the following conditions are satisfied :

and

where the function is given by (3.2)

**Corollary :** Let given by (3.1) be in the class

. Then

and

**Corollary :**  Let given by (3.1 ) be in the class

. Then

and

**Coefficient Estimates for Some Subclasses of m-Fold Symmetric Bi-univalent Functions**

Let denote the family of functions analytic in the open unit disk

and normalized by the conditions and having the form:

Also let denote the subclass of functions in which are univalent in .

The koebe One Quarter Theorem ensures that the image of under every univalent function contains the disk of radius . Thus every univalent function has an inverse satisfying :

and

where

A function is said to be bi-univalent in if both and are univalent in

Let denotes the class of analytic and bi-univalent functions in .

Some examples of functions in class where

For each function , the function , is univalent and maps the unit disk into a region with m-fold symmetry . A function is said to be m-fold symmetric if it has the following normalized form :

(3.34)

We denote the class of m-fold symmetric univalent functions in , which are normalized by the series expansion . In fact , the functions in the class are one-fold symmetric . Analogous to the concept of m-fold symmetric univalent functions , we here introduced the concept of m-fold symmetric univalent functions , we here introduced the concept of m-fold symmetric bi-univalent functions . Each function generates an m-fold symmetric bi-univalent function for each integer Furthermore, for the normalized form of is given by (3.34) , they obtained the series expansion for as follows :

where We denote by the class of m-fold symmetric bi-univalent functions in . It is easily seen that for , the formula (3.35) coincides with the formula (3.33) of the class . Some examples of m-fold symmetric bi-univalent functions are given as follows :

with the corresponding inverse functions

respectively .

Recently , many authors investigated bounds for various subclass of m-fold bi-univalent functions .The aim of this work is to introduce the new subclass and find estimates on the coefficients for functions in each of these new subclass .

**Definition 3.2.1:** A function given by (3.34) is said to be in the class if the following condition are satisfied :

(3.36)

and

where the function is given by (3.35) .

**Definition 3.2.2:** A function given by (3.34) is said to be in the class if the following conditions are satisfied :

and

(3.39)

where the function is given by (3.35) .

We begin this section by finding the estimates on the coefficients for functions in the class

**Theorem 3.2.1:** Let be of the form (3.34) . Then

(3.40)

and

**Proof.**  It follows from (3.36) and (3.37) that

and

where the functions and are in and have the following series representations:

and

Now, equating the coefficients in (3.42) and (3.43) , we obtain

and

From (3.46) and (3.48) , we find

and

(3.51)

From (3.47), (3.49) and (3.51), we get

(3.52)

Therefore , we have

(3.53)

Applying Lemma (1.2.5) for the coefficients and , we have

(3.54)

This gives the desired bound for as asserted in(3.40). In order to find the bound on, by subtracting (3.49)from (3.47), we get

(3.55)

It follows from(3.50) and (3.55) that

(3.56)

Applying Lemma (1.2.5) once again for the coefficients and , we readily obtain 1

(3.57)

The following theorem is devoted to find the estimates on the coefficients for functions in the class .

**Theorem 3.2.2:** Let be of the form (3.34).

Then

(3.58)

and

(3.59)

**Proof.** It follows from (3.38) and (3.39) that there exist , such that

(3.60)

and

(3.61)

where have the forms (3.44) and (3.45) , respectively . By suitably comparing coefficients in (3.60) and (3.61), we get

, (3.62)

(3.63)

(3.64)

(3.65)

From (3.62) and (3.64) , we find

(3.66)

and

(3.67)

Adding (3.62) and (3.65) , we have

(3.68)

Applying Lemma (1.2.5) , we obtain

This is the bound on asserted in (3.58) .

In order to find the bound on by subtracting (3.65) form (3.63) , we get

.

Or , equivalently ,

(3.69)

It follows from (3.66) and (3.67) that

(3.70)

Applying Lemma (1.2.5) once again for the coefficients we easily obtain

(3.71)

For one-fold symmetric bi-univalent functions and Theorem (3.2.1) and Theorem (3.2.2) reduce to Corollary (3.2.1) and Corollary (3.2.2), respectively , which were proven very recently by Frasin

**Definition 3.2.3 :** A function given by (3.32) is said to be in the class if the following conditions are satisfied :

, (3.72)

and

(3.73)

and where the function is given by (3.33) .

**Definition 3.2.4**: A function given by (3.32) is said to be in the class if the following conditions are satisfied :

(3.74)

and

,

where the function is given by (3.33) .

**Corollary 3.2.1:** Let be of the form (3.32) . Then

and (3.76)

(3.77)

**Corollary 3.2.2:** Let be of the form (3.32) . Then

and (3.78)

(3.79)

If we set and in Theorem (3.2.1) and Theorem (3.2.2) , then the classes and reduce to the classes and investigated

recently by Srivastava et al. [56].

**Definition 3.2.5:** A function given by (3.34) is said to be in the class if the following conditions are satisfied :

(3.80)

and

(3.81)

and where the function g is given by (3.35) .

**Definition 3.2.6:** A function given by (3.34) is said to be in the class if the following

conditions are satisfied :

(3.82)

and

(3.83)

,

and where the function g is given by (3.35) .

**Corollary 3.2.3:** Let be of the form (3.34) . Then

(3.84)

and

(3.85)

**Corollary 3.2.4:** Let be of the form (3.35) . Then

(3.86)

and

(3.87)

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**المستخلص**

الغرض من هذه الرسالة هو دراسة التابعية التفاضلية وخواص المعاملات للدوال احادية التكافؤ والمتعددة التكافؤ . درسنا مبرهنات الساندوج التفاضلية للدوال متعددة التكافؤ و المعرفة بواسطة المؤثر الخطي . ثم اشتقينا بعض النتائج للدوال متعددة التكافؤ التحليلية باستخدام التابعية التفاضلية و التابعية التفاضلية العليا . تعاملت رسالتنا ايضا مع نتائج التابعية من الرتبة الثالثة للدوال احادية التكافؤ حيث حصلنا على نتائج جديدة للتابعية التفاضلية من الرتبة الثالثة في قرص الوحدة المفتوح . تعاملنا ايضا مع مخمنات المعاملات لأصناف جزئية جديدة من الدوال ثنائية التكافؤ .وهنا اصناف جزئية جديدة و من الصنف و المتكونة من الدوال ثنائية التكافؤ وتحليلية في قرص الوحدة المفتوح ودراستها حيث حصلنا مخمنات على المعاملات و . اخيرا مخمنات المعامل لبعض الاصناف الجزئية من الدوال ثنائية التكافؤ المرتبطة الى الطية من النمط m المتناظرة نوقشت وهنا قدمنا اصناف جزئية و من الدوال ثنائية التكافؤ المتناظرة ذات الطية من النمط m والتحليلية في *U*. ثم الحصول على حدود المعاملات الاعلى و . حالات خاصة اكيدة تم الاشارة اليها .

**الاهداء ...**

**الى من بلغ الرسالة وادى الامانة .. ونصح الامة..**

****

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**والمتعددة التكافؤ**

## رسـالة

**مقدمة إلى مجلس كلية علوم الحاسوب وتكنولوجيا المعلومات في جامعة القادسية كجزء من متطلبات نيل درجة ماجستير علوم في الرياضيات**

**من قبل**

**سلوى كلف كاظم**

# بأشراف

## أ. د. وقاص غالب عطشان

**نيسان 2019 م**

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