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Use power series for solve differential equations

A Research

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بسم الله الرحمن الرحيم

﴿ وما أوتيتم من العلم إلا قليلا ﴾

صدق الله العلي العظيم

Dedication

To.....

My dear father .

My compassionate mother .

My brother and sisters .

My husband.

الشكر والتقدير

بعد شكري لله عز وجل أن أعانني على انجاز هذا البحث المتواضع اتقدم بجزيل الشكر والامتنان الى الاستاذ الفاضل الدكتور خالد منديل الإبراهيمي على تفضله بقبول الاشراف على بحثي هذا ، وعلى ما اسداه لي من نصائح وارشادات كانت بمثابة النبراس المنير في كل ولا يفوتني بهذه المناسبة ان اوجه شكري واحترامي الى كل من ساعدني من قريب او خطواتي بعيد في انجاز هذا الجهد المتواضع، واخص بالذكر اختي العزيزة زهراء

Supervisor is Certification

Certify that the research entitled ***"Use power series for solve differential equations"*** ,was prepared by ***" Huda Kadhem Abied"*** under my supervision at Baghdad University ,College of Science , Department of Mathematics as a partial fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematics.

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Date : / / 2019

Abstract

In this paper we in traduce New method for solve ordinary differential equation by power series and compare this method withe some of method We have compared the proposed method with some traditional methods and found our method followed by a more accurate, faster and less error ratio

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Chapter 1

Background

1. 1- Background

Many ordinary differential equations encountered do not have easily obtain able closed from solutions , and we must seek other methods by which solutions can be constructed . Numerical methods provide an alternative way of constructing solutions to these some times difficult problems . In this chapter we present an introduction to some numerical methods which can be applied to awide variety of ordinary differential equations . These methods can be programmed into a digital computer or even programmed into some hand – held calculators . Many of the numerical techniques introduced in this chapter are readily available in the from of subroutine packages available from the internet .

An equation that consists of derivatives is called a differential equation .

Differential equations have applications in all areas of science and engineering . Mathematical formulation of most of the physical and engineering problems lead to differential equations .

So , it is important for engineers and scientists to know how to set up differential equations and solve them .

Differential equations are of two types ordinary differential equation " ODE " and partial differential equations "PDE" .

An ordinary differential equation is that in which all the derivatives are with respect to a single independent variable . Examples of ordinary differential equation include .

1- $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0 \quad , \quad \frac{dy}{dx}(0) = 2 \quad , \quad y(0) = 4$.

2- $\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + y = \sin x \qquad \frac{d^2y}{dx^2}(0) = 12 \quad , \quad y(0) = 4$

$$\frac{dy}{dx}(0) = 2 \quad , \quad y(0) = 4$$

Note : In this first part , we will see how to solve ODE of the form.

$$\frac{dy}{dx} = f(x, y) \quad , \quad y(0) = y_0$$

In another sextion , we will discuss how to solve higher ordinary differential equations or coupled "simultaneous" differential equations .

But first , How to write a first order differentia

Equation for example

$$\frac{dy}{dx} + 2y = 1.3e^{-x} \qquad , \qquad y(0) = 5$$

Is rewritten as :

$$\frac{dy}{dx} = 1.3e^{-x} - 2y \qquad , \qquad y(0) = 5$$

Is this case : $f(x,y) = 1.3e^{x} - 2y$

Example (2) : $e^{y}\frac{dy}{dx} + x^{2}y = 2\sin(3x) \qquad , \qquad y(0) = 5$

We consider the problem of developing numerical methods to solve a first order initial value problem of the form.

$$\frac{dy}{dx} = f(x,y) \qquad , \qquad y(x_0) = y_0$$

and then consider how to generalize these methods to solve systems of ordinary differential equations having the form .

$$\frac{dy_1}{dx} = f_1(x, y_1, y_2, \ldots\ldots, y_m) \quad , \quad y_1(x_0) = y_{10}$$

$$\frac{dy_2}{dx} = f_2(x, y_1, y_2, \ldots\ldots, y_m) \quad , \quad y_2(x_0) = y_{20}$$

.

.

.

$$\frac{dy_m}{dx} = f_m(x, y_1, y_2, \ldots\ldots, y_m) \quad , \quad y_m(x_0) = y_{m0}$$

Coupled systems of ordinary differential equations are sometimes written in the vector from

$$\frac{d\vec{y}}{dx} = \vec{f}(x, \vec{y}) \qquad \vec{y}(x_0) = \vec{y}_0$$

Where $\vec{y}$, $\vec{y}(x_0)$ and $\vec{f}(x, \vec{y})$ are column vectors given by

$$\vec{y} = \mathrm{col}(y_1, y_2, \ldots\ldots, y_m) \quad , \quad \vec{y}(x_0) = \mathrm{col}(y_{10}, y_{20}, \ldots\ldots y_{m0})$$

and $\vec{f}(x, \vec{y}) = \mathrm{col}(f_1, f_2, \ldots\ldots f_m)$

We start with developing numerical methods for obtaining solutions to the first order initial value problem over an interval $x_0 \le x \le x_n$ many of the techniques developed for this first order equation can with modifications , also be applied to solve a first order system of differential equations.

1.2 Higher order equations

By defining new variables , higher order differential equations can be reduced to a first order system of differential equations . As an example , consider the problem of converting order liner homogeneous differential equation .

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \dots\dots\dots + a_{n-1} \frac{dy}{dx} + any = 0$$

To a vector representation . To convert this equation to vector from we define new variables . Define the vector quantities .

$$\vec{y} = col(y_1, y_2, y_3, \dots\dots\dots, y_m) = col(y, \frac{dy}{dx}, \frac{d^2 y}{dx^2}, \dots\dots\dots, \frac{d^{n-1} y}{dx^{n-1}})$$

$$\vec{f}(x, \overrightarrow{y}) = A\overrightarrow{y},$$

Where A = $\begin{bmatrix} 0 & 1 & 0 & 0 & \cdots\cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots\cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots\cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots\cdots & 0 & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & \cdots\cdots & 0 & 1 \\ -an & -an-1 & -an-2 & -an-3 & \cdots\cdots & -a_2 & -a_1 \end{bmatrix}$

Observe that the linear the order differential equation can now represented in the from of equation .In this way higher order linear ordinary differential equations can be represented as a first order vector system of differential equations .

1.3 Numerical Solution :

In our study of the scalar initial value problem .it is assumed that f(x,y) and its partial derivative f_y both exist and are continuous in a rectangular region about a point (x_0 ,y_0) . If these conditions are satisfied , then theoretically there exists unique solution of the initial value problem .

Which is a continuous curve y=y(x) , which passes through the point (x_0 , y_0) and satisfies the differential equation. In contrast to the solution being represented by continuous function y=y(x) , the numerical solution to the initial value problem is represented by a set of data points (x_i,y_i) for i = 0,1,2,…..,n where y_i is an approximation to the true solution $y(x_i)$. we shall investigate various methods for constructing the data points (x_i,y_i) , for i=1,2,…,n which approximate the true solution to the given initial value problem . The given rule or technique used to obtain the numerical solution is called anumerical method or algorithm . There are many numerical methods for solving ordinary differential equations . In this chapter we will consider only a select few of the mor popular methods . The numerical methods considered can be classified as either single – step methods or multi-step methods . We begin our

introduction to numerical methods for ordinary differential

equations by considering single step methods .

1.4 Initial – Value Problems For Ordinary Differential Equations

Many problems in engineering and science can be formulated in terms of differential equations . A differential equation is an equation involving a relation between an un known function and one or more of its derivatives . Equations involving derivatives of only one independent variable are called ordinary differential equations and may be classified as either initial – value problems " IVP " or boundary value problems "BVP" . Examples of the two types are :

$$\text{IVP} : y'' = -yx$$

Y(0) = 2 , $y'(0) = 1$

$$\text{BVP} : y'' = -yx$$

Y(0) = 2 , y(1) = 1

Where the prime denotes differentiation with respect to x. The distinction between the two classifications lies in the location where the extra conditions are specified . For an IVP , the conditions are given at the same value of , where as in the case of the BVP , They are prescribed at two different values of x .

Since there are relatively few differential equations arising from practical problems for which anal arising from practical problems for which analytical solutions are known , one must resort to numerical methods . In this situation it turns out that the numerical methods for each type of problem , IVP or BVP , are quite different and require separate treatment . In this chapter we discuss .

Consider the problem of solving the mth-order differential equation .

$$y^{(m)} = f(x, y', y'', ..., y^{(m-1)}) \quad \text{with initial conditions}$$

$$y(x_0) = y_0$$
$$y''(x_0) = y'_0$$
$$\vdots$$
$$y^{(m-1)}(x_0) = y_0^{(m-1)}$$

Where f is known function and $y_0, y'_0, \ldots\ldots, y_0^{(m-1)}$ are constants . It is customary to rewrite as an equivalent system of m first – order equations . To do so , we define a new set of dependent variables y(x) , y_2(x) , ……. , y_m(x) by :

$$y_1 = y$$
$$y_2 = y'$$
$$y_3 = y''$$
$$\vdots$$
$$y_m = y^{(m-1)}$$

And transform into :

$$
\begin{aligned}
y_1' &= y_2 && = f_1(x, y_1, y_2, \dots\dots\dots, y_m) \\
y_2' &= y_3 && = f_2(x, y_1, y_2, \dots\dots\dots, y_m) \\
&. && \qquad\qquad (1.5) \\
&. \\
&. \\
y_m' &= f(x, y_1, y_2, \dots\dots\dots, y_m) = fm(x, y_1, y_2, \dots\dots\dots, y_m)
\end{aligned}
$$

$$
\begin{aligned}
y_1(x_0) &= y_0 \\
y_2(x_0) &= y_0' \\
&. \\
&. \\
&. \\
y_m(x_0) &= y_0^{m\text{-}1}
\end{aligned}
$$

In vector notation because :

$$
\begin{aligned}
y'(x) &= f(x, y) \\
y(x_0) &= y_0
\end{aligned}
$$

Where

$$y(x)=\begin{bmatrix} y_1(x) \\ y_2(x) \\ . \\ . \\ . \\ y_m(x) \end{bmatrix} \quad , \quad f(x,y)=\begin{bmatrix} f_1(x,y) \\ f_2(x,y) \\ . \\ . \\ . \\ f_m(x,y) \end{bmatrix} \quad , \quad y_0=\begin{bmatrix} y_0 \\ y_0' \\ . \\ . \\ . \\ y_0^{(m-1)} \end{bmatrix}$$

It is easy to see that can represent either an mth-order different equation . A system of equations of mixed order but with total order of m , or a system of , first order equations .

In general , subroutines for solving IVPS as same that the problem is in the form. In order to simplify the analysis , we begin by examining a single first – order IVP , after which we extend the discussion to include systems of the form .

Consider the initial – value problem .

$$y' = f(x,y) \qquad y(x_0) = y_0$$

$$x_0 \le x \le x_N$$

We assume that $\partial f / \partial y$ is continuous on the strip $x_0 \le x \le x_N$, thus guaranteeing that possesses unique solution If y(x) is the exact solution to its graph is a curve in the xy-plane passing through the point (x_0 , y_0) .

A discrete numerical solution of is defined to be a set point $(x_i , u_i)_{i=0}^{N}$, Where $u_0 = y_0$ and each point $(x_i , y(x_i))$ on the solution curve . Note that the numerical solution is only a set of points , and nothing is said about values between the points . In the remainder of this chapter we describe various methods for obtaining a numerical solution $[(x_i , u_i)]_{i=0}^{N}$.

Chapter 2

Using power Series to Solve Differential Equations

2.1 Using power Series to Solve Differential Equations

Many differential equations can't be solved explicitly in terms of finite combinations of simple familiar functions. This is true even for a simple-looking equation like

$$y'' - 2xy' + y = 0$$

But it is important to be able to solve equations such as Equation 1 because they arise from physical problems and, in particular, in connection with the Schrödinger equation in quan- tum mechanics. In such a case we use the method of power series; that is, we look for a solution of the form

$$y = f(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

The method is to substitute this expression into the differential equation and determine the values of the coefficients $c_0, c_1, c_2, \ldots$.

Before using power series to solve differential Equation , we illustrate the method on the simpler equation .

2.2 Examples

Example 1: Use power series to solve the equation $y'' + y = 0$.

Solution We assume there is a solution of the form

$$y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots = \sum_{n=0}^{\infty} c_n x^n \quad \ldots\ldots.(1)$$

We can differentiate power series term by term, so

$$y' = c_1 + 2c_2x + 3c_3x^2 + \cdots = \sum_{n=1}^{\infty} nc_nx^{n-1}$$

$$y'' = 2c_2 + 2 \cdot 3c_3x + \ldots = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} \quad \ldots\ldots\ldots(2)$$

In order to compare the e s for *y* and *y*" more easily, we rewrite *y*" as follows

$$4\text{-}\ y'' = \sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n$$

Substituting the expressions in Equations 2 and 4 into the differential equation, we obtain

$$\sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n + \sum_{n=0}^{\infty}c_n x^n = 0$$

or

$$\sum_{n=0}^{\infty}[(n+2)(n+1)c_{n+2} + c_n]\, x^n = 0 \quad \ldots\ldots\ldots(3)$$

If two power series are equal, then the corresponding coefficients must be equal. Therefore, the coefficients of x^n in Equation 5 must be 0:

$$(n+2)(n+1)c_{n+2} + c_n = 0$$

$$C_{N+2} = -\frac{c_n}{(n+1)(n+2)} \qquad n=0,1,2,3,\ \ldots. \qquad (4)$$

Equation 6 is called a recursion relation. If c0 and c1 are known, this equation allows us to determine the remaining coefficients recursively by putting n = 0, 1, 2, 3, . . . in succession.

Put $n = 0$: $\quad c_2 = -\dfrac{c_0}{1 \cdot 2}$

Put $n = 1$: $\quad c_3 = -\dfrac{c_1}{2 \cdot 3}$

Put $n = 2$: $\quad c_4 = -\dfrac{c_2}{3 \cdot 4} = \dfrac{c_0}{1 \cdot 2 \cdot 3 \cdot 4} = \dfrac{c_0}{4!}$

Put $n = 3$: $\quad c_5 = -\dfrac{c_3}{4 \cdot 5} = \dfrac{c_1}{2 \cdot 3 \cdot 4 \cdot 5} = \dfrac{c_1}{5!}$

Put $n = 4$: $\quad c_6 = -\dfrac{c_4}{5 \cdot 6} = -\dfrac{c_0}{4!\, 5 \cdot 6} = -\dfrac{c_0}{6!}$

Put $n = 5$: $\quad c_7 = -\dfrac{c_5}{6 \cdot 7} = -\dfrac{c_1}{5!\, 6 \cdot 7} = -\dfrac{c_1}{7!}$

By now we see pattern:

For the even coefficients, $c_{2n} = (-1)^2 \dfrac{c_0}{(2)!}$

For the add coefficients , $C_{2n} +1 = (-1)^n \dfrac{c_1}{(2n+1)!}$

Putting these values back into Equation 2, we write the solution as

$y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + \cdots$

$= c_0\left(1- \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \ldots + (-1)^n \dfrac{x^{2n}}{(2)!} + \ldots \right)$

$+ c_1 \left(x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \ldots + (-1)^n \dfrac{x^{2n+1}}{(2n+1)!} + \ldots \right)$

$= c_0 \displaystyle\sum_{n=0}^{\infty} (-1)^2 \frac{x^{2n}}{(2n)!} + c_1 \sum_{n=0}^{\infty} (-1)^2 \frac{x^{2n+1}}{(2n+1)!}$

Notice that there are two arbitrary constants, c_0 **and** c_1**.**

NOTE We recognize the series obtained in Example 1 as being the Maclaurin series for cos *x* and sin *x*. Therefore, we could write the solu- tion as

$$y(x) = c_0 \cos x + c_1 \sin x$$

But we are not usually able to express power series solutions of differential equations in terms of known functions

Example 2 Solve $y'' - 2xy' + y = 0$.

Solution We assume there is a solution of the form

$$y = \sum_{n=0}^{\infty} c_n x^n$$

Then $$y' = \sum_{n=0}^{\infty} n c_n x^{n-1}$$

and

$$y'' = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2}x^n$$

as in Example 1. Substituting in the differential equation, we get

$$\sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n - 2x\sum_{n=1}^{\infty} n c_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n - \sum_{n=1}^{\infty} 2n c_n x^n + \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=0}[(n+2)(n+1)c_{n+2} - (2n-1)c_n]\, x^n = 0$$

This equation is true if the coefficient of x^n is 0:

$$(n+2)(n+1)c_{n+2} - (2n-1)c_n = 0$$

$$c_{n+2} = \frac{2n-1}{(n+1)(n+2)} c_n \qquad n = 0, 1, 2, 3, \ldots$$

We solve this recursion relation by putting $n = 0, 1, 2, 3, \ldots$ successively in Equation :

$$c_{n+2} = \frac{2n-1}{(n+1)(n+2)} c_n \qquad n = 0, 1, 2, 3, \ldots$$

Put $n = 0$: $C_2 = \frac{-1}{1.2} C_0$

Put $n = 1$: $C_3 = \frac{1}{2.3} C_1$

Put $n = 2$: $C_4 = \frac{3}{3.4} C_2 = -\frac{3}{1.2.3.4} C_0 = \frac{3}{4!} C_0$

Put $n = 3$: $C_5 = \frac{5}{4.5} C_3 = \frac{1.5}{2.3.4.5} C_1 = \frac{1.5}{5!} C_1$

Put $n = 4$: $C_6 = \frac{7}{5.6} C_4 = -\frac{3.7}{4!5.6} C_0 = -\frac{3.7}{6!} C_0$

Put $n = 5$: $C_7 = -\frac{9}{6.7} C_5 = \frac{1.5.9}{5!6.7} C_1 = -\frac{1.5.9}{7!} C_1$

Put $n = 6$: $C_8 = \frac{11}{7.8} C_6 = -\frac{3.7.11}{8!} C_0$

Put $n = 7$: $C_9 = \frac{13}{8.9} C_7 = \frac{1.5.9.13}{9!} C_1$

In general, the even coefficients are given by

$$C_{2n} = -\frac{3.7.11 \ldots (4n-5)}{(2n)!} C_0$$

and the odd coefficients are given by

$$C_{2n+1} = \frac{1.5.9 \ldots (4n-3)}{(2n+1)!} C_1$$

The solution is

$$y = C_0 + C_1 X + C_2 X^2 + C_3 X^3 + C_4 X^4 + \dots$$

$$= C_0 \left(1 - \frac{1}{2!} X^2 - \frac{3}{4!} X^4 - \frac{3.7}{6!} X^6 - \frac{3.7.11}{8!} X^8 - \dots \right)$$

$$+ C_1 \left(X + \frac{1}{3!} X^3 + \frac{1.5}{3!} X^5 + \frac{1.5.9}{7!} X^7 + \frac{1.5.9.13}{9!} X^9 + \dots \right)$$

Or

$$y = C_0 \left(1 - \frac{1}{2!} X^2 - \sum_{n=2}^{\infty} \frac{3.7.\dots(4n-5)}{(2n)!} X^{2n} \right)$$

$$+ C_1 \left(X + \sum_{n=1}^{\infty} \frac{1.5.9\dots\dots(4n-3)}{(2n+1)!} X^{2n+1} \right)$$

Note : In Example 2 we had to assume that the differential equation had a series solution . But now we could verify directly that the function given by Equation 8 is indeed a solution .

Note : Unlike the situation of Example 1 , the power series that arise in the solution of Example 2 do not define elementary functions . The functions .

$$y_1(X) = 1 - \frac{1}{2!} X^2 - \sum_{n=2}^{\infty} \frac{3.7\dots\dots(4n-5)}{(2n)!} X^{2n}$$

and

$$y_2(X) = X + \sum_{n=1}^{\infty} \frac{1.5.9\dots\dots(4n-3)}{(2n+1)!} X^{2n+1}$$

are perfectly good functions but they can't be expressed in terms of familiar functions. We can use these power series expressions for y_1 and y_2 to compute approximate values of the functions and even to graph them. Figure 1 shows the first few partial sums T_0 ,T_2 , T_4 , . . .(Taylor polynomials) for $y_1(X)$, and we see how they converge to y_1 . In this way we can graph both and in Figure 2.

NOTE : If we were asked to solve the initial-value problem .

$y'' - 2xy' + y = 0$ $\qquad$ $y(0) = 0$ $\qquad$ $y'(0) = 1$

We would observe that

$C_0 = y(0) = 0$ $\qquad$ $C_1 = y'(0) = 1$

This would simplify the calculations in Example 2, since all of the even coefficients would be 0. The solution to the initial-value problem is :

$$y(X) = X + \sum_{n=1}^{\infty} \frac{1 . 5 . 9 \ldots\ldots (4n-3)}{(2n+1)!} X^{2n+1}$$

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