Republic of Iraq

Ministry of Higher Education

And

Scientific Research

AL - Qadisiya University

College of Education

A NOTE ON JORDAN DERIVATIONS OF TRIVAL GENERALIZED MATRIX ALGEBRA

A research

Submitted to the department of education AL- Qadisiya University in Partial fulfillment of the requirement for the degree of mathematical

BY

Hiba Kamel Obaid

Supervised By

Dr.Rajaa Chaffat Shaheen

1441 A.H

2019 A.D

بسم الله الرحمن الرحيم

﴿ ذلـــك الكتـــاب لا ريـــب فيـــه
هدى للمتقين ﴾

صدق الله العلي العظيم

سورة البقرة

الآية (٢)

الإهداء

إلى من جرع الكأس فارغاً ليسقيني قطرة حب

إلى من كلّت أنامله ليقدم لنا لحظة سعادة إلى من حصد

الأشواك عن دربي ليمهد لي طريق العلم

إلى القلب الكبير (والدي العزيز)

إلى من أرضعتني الحب والحنان

إلى رمز الحب وبلسم الشفاء

إلى القلب الناصع بالبياض (والدتي الحبيبة)

إلى القلوب الطاهرة الرقيقة والنفوس البريئة إلى

رياحين حياتي (إخوتي)

الآن تفتح الأشرعة وترفع المرساة لتنطلق السفينة في

عرض بحر واسع مظلم هو بحر الحياة وفي هذه

الظلمة لا يضيء إلا قنديل الذكريات ذكريات الأخوة البعيدة إلى الذين أحببتهم وأحبوني (أصدقائي)

شكر وتقدير

الحمد لله يقول الله في محكم كتابه { لئن شكرتم لأزيدنكم } والصلاة والسلام على اشرف خلق الله سيدنا محمد (صلى الله عليه واله وسلم) القائل: من لم يشكر المخلوق لم يشكر الخالق.

بداية اشكر الله عز وجل الذي ساعدني على اتمام بحثي وتفضل علينا بإتمام هذا العمل.. وبعد

شكرا وتقديرا لحضرة الاستاذة الفاضلة **رجاء جفات شاهين** على ما بذلته من سعة صدر وكرم طبعها ورحابة خاطرها وارشاد وتوجيه وتسديد لأفكاري

فجزاه الله خير جزاء المحسنين

الباحث

ABSTRACT

H. R. Ebrahimi Vishki et al, conjectured in [1], tha.t if every Jordan higher derivation on a trivial generalized matrix algebra $\mathcal{G} = (A, M, N, B)$ is a higher derivation, then either $M = 0$ or $N = 0$. In this note, we will give a class of counter examples. Let A be a unital R-algebra and M be a unital A-bimodule. It is shown that every Jordan derivation of the trivial extension of A by M, under some conditions, is the sum of a derivation and an antiderivation.

1. Introduction

Generalized matrix algebras were introduced by Morita in [12] to study Morita duality theory in 1958. They are natural generalizations of triangular algebras. All associative algebras with nontrivial idempotents are isomorphic to generalized matrix algebras. When both the pairings are zero, a generalized matrix algebra is called trivial. A class of trivial generalized matrix algebras is the so-called generalized one-point extension algebras introduced in [7] by Li and Wei. In [2]' Haghany studied hopfcity and co-hopfcity for trivial generalized matrix algebras.

Recent years, it has been an active research area to study various mappings on generalized matrix algebras, such as commuting mappings, Lie derivations, Jordan derivations, higher derivations and so on. We refer the reader to [5, 6, 7, 8, 9, 10, 11, 13, 15] for details. Note that Jordan derivations are related to the problem of the decomposition of Jordan homomorphisms [4]. The standard problem for Jordan derivations is to find out whether a Jordan derivation degenerate to a derivation. In [3]' Herstein showed that every Jordan derivation from a 2-torsion free prime ring into itself is a derivation. Zhang and Yu proved in [18] that all Jordan derivations of a triangular algebra with the faithful condition are derivations. More examples include incidence algebras [14] and dual extension algebras [8]. It is helpful to point out that the

problem has the following higher version: to find out whether a Jordan higher derivation degenerate to a higher derivation. We refer the reader to [16, 17] for some results on it.

H. R. Ebrahimi Vishki et al. conjectured in [1] that if every Jordan higher derivation on a trivial generalized matrix algebra $\mathcal{G} = (A, M, N, B)$ is a higher derivation, then either $M = 0$ or $N = 0$. In this note, we will give a class of counter examples.

Throughout is research R will denote a commutative ring with unity. Let A be an algebra over R. Recall that an R-linear map Δ from A into an A-bimodule M is said to be a Jordan derivation if $\Delta(ab + ba) = \Delta(a)b + a\Delta(b) + \Delta(b)a + b\Delta(a)$ for all $a, b \in A$. It is called a derivation if $\Delta(ab) = \Delta(a)b + a\Delta(b)$ for all $a, b \in A$. Each map of the form $a \longrightarrow am - ma$, where $m \in M$, is a derivation which will be called an inner derivation. Also Δ is called an antiderivation if $\Delta\ (ab) = \Delta(b)a + b\Delta(a)$ for all $a, b \in A$. We shall say that an antiderivation Δ is improper if it is a derivation; otherwise we shall say that Δ is proper. Clearly, each derivation or anti derivation is a Jordan derivation. The converse is, in general, not true (see [6]).

It is natural and very interesting to find some conditions under which a Jordan derivation is a derivation. Herstein [14] proved that every Jordan derivation from a 2-torsion free prime ring into itself is a derivation and that there are no nonzero antiderivations on a prime ring. Brcsar [8] showed that every additive Jordan derivation from a 2-torsion free semiprime ring into itself is a derivation. Sinclair [18] proved that every continuous linear Jordan derivation on semisimple Banach algebras is a derivation. Zhang in [21] proved that every linear Jordan derivation on nest algebras is an inner derivation. Lu [17] proved that every additive Jordan derivation on reflexive algebras is a derivation which generalized the result in [21]. Benkovic [6] determined Jordan derivations on triangular matrices over commutative rings and proved that every Jordan derivation from the algebra of all upper triangular matrices into its arbitrary bimodule is the sum of a derivation and an antiderivation. Zhang and Yu [23] showed that every Jordan derivation of triangular algebras is a derivation, so every Jordan derivation from the algebra of all upper triangular matrices into itself is a derivation.

In this note we study the Jordan derivations on trivial extensions and generalize the Zhang and Yu's result [23].

CHAPTER ONE

Preliminaries

Definition(1-1) : (Derivation)

A linear mapping θ from A is called a derivation if

$\theta\ (ab)\ =\ \theta\ (a)b\ +\ a\ \theta\ (b)$,for all $a, b \in\ A$

Definition(1-2) : (Anti-derivation)

A linear mapping θ from A is called anti-derivation if

$\theta\ (ab)\ =\ \theta\ (b)a\ + b\theta\ (a)$,for all $a, b \in\ A$

Definition(1-3) : (Jordan-derivation)

An -linear mapping θ from A is called a Jordan-derivation if

$\theta\ (a \circ b)\ =\ \theta\ (a) \circ b\ +\ a \circ\ \theta\ (b)$,for all $a, b \in\ A$

Definition(1-4) : (Higher derivation)

Let N be the set of all non-negative integers. Let $D = \{dn\}_{n \in N}$ be a sequence of R-linear mappings on A with $d_0\ =\ idA$ (the identity mapping of A) then $\{dn\}_{n \in N}$ is called a higher derivation if

$$dn(xy) = \sum_{i=0}^{n} d_i(x)dj(y) \quad ,\forall\ x, y\ \in\ A, \qquad \forall\ n\ \in\ N$$

Definition(1-5): (Jordan higher derivation):

A sequence D is called a Jordan higher derivation of A if

$$dn(x^2) = \sum_{i+j=n} d_i(x)d_j(x) \ \forall x \in A, \ \ n \in N.$$

Definition(1-6) : (Module)

Let $(R,+,\cdot)$ be a ring and let $(M,+)$ abelion group then $(M,+)$ is called Left-Module if there is a mapping $\cdot : R \times M \to M$ Such that

1) $r \cdot (M_1 + M_2) = rM_1 + rM_2 \ \ni r \in R, \ M \in M$
2) $(r_1 + r_2) \cdot M = r_1 M + r_2 M \in M \ni r_1, r_2 \in R, M \in M$
3) $(r_1 \cdot r_2) \cdot M = r_1 \cdot (r_2 \cdot M) \in M \ni r_1, r_2 \in R, M \in M.$

Definition(1-7) : (Bi Module)

an abelian group that is both a left and a right Module such that the left and right multiplication compliable.

Definition(1-8) : (Unital algebra)

An algebra that contains a multiplicative identity element.

Definition(1-9) : (Tensor Product of Matrices)

If $\delta: R^M \longrightarrow R^M$ and $T: R^N \longrightarrow R^N$ are Matrices , we define the linear extension of $\delta \otimes T \ (e_i \otimes e_j) = (\delta e_i) \otimes (Te_j)$. The linear mapping $\delta \otimes T$ is called tensor product of the matrices S and T.

Definition(1-10) : (Generalized Matrix algebras)

The definition of generalized matrix algebras is given by a Morita context . A Morita context consists of two R-algebras A and B, two bi modules $_{A}M_{B}$ and $_{B}N_{A}$, and two bi module homomorphism called the pairings $\Phi_{MN}: M \otimes_{B} N \rightarrow A$ and $\psi_{NM} : N \otimes_{A} M \rightarrow B$ satisfying the following commutative diagrams:

$$\begin{array}{ccc} M\otimes_B N\otimes_A M & \xrightarrow{\Phi_{MN}\otimes I_M} & A \otimes_A M \\ \Big\downarrow{\scriptstyle I_M\otimes\psi_{NM}} & & \Big\downarrow{\cong} \\ M\otimes_B B & \xrightarrow{\cong} & M \end{array} \quad \text{and} \quad \begin{array}{ccc} N \otimes_A M\otimes_B N & \xrightarrow{\psi_{NM}\otimes I_N} & B\otimes_B N \\ \Big\downarrow{\scriptstyle I_N\otimes\Phi_{MN}} & & \Big\downarrow{\cong} \\ N\otimes_A A & \xrightarrow{\cong} & N. \end{array}$$

Let us write this Morita context as $(A, B, {}_{A}M_{B}, {}_{B}N_{A}, \Phi_{MN}, \psi_{NM})$. If $(A, B, {}_{A}M_{B}, {}_{B}N_{A}, \Phi_{MN}, \psi_{NM})$ is a Morita context, then the set

$$\begin{bmatrix} A & M \\ N & B \end{bmatrix} = \left\{ \begin{bmatrix} a & m \\ n & b \end{bmatrix} \middle| a \in A, m \in M, n \in N, b \in B \right\}$$

forms an R-algebra under matrix-like addition and matrix-like multiplication. There are possibly equal to zeros. Such an R-algebra is called a generalized matrix algebra of order 2 and is, usually denoted by $\mathcal{G} = \begin{bmatrix} A & M \\ N & B \end{bmatrix}$ or $\mathcal{G} = (A, M, N, B)$ in brief. When the pairings are both zero, we call $\mathcal{G}$ a trivial generalized matrix algebra.

Definition(1-11) : (Triangular algebra)

Let $Tri(A, M, B)$ is an R-algebra of the form

$$Tri(A, M, B) = \left\{ \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \mid a \in A, m \in M, b \in B \right\}$$

under the usual matrix operations, where A and B are unital algebras over Rand M is a unital (A, B)-bimodule which is faithful as a left Amodule as well as a right B-module.

Definition(1-12) : (The trivial extension)

Let A be a unital algebra over R and M be a unital A-bimodule.

$A\, x\, M$ as an R-module together with the algebra product defined by:

$$(a, m).(b, n) = (ab, an + mb)\ (a, b \in A, \quad m, n \in M)$$

is an R-algebra with unity $(1,0)$, which is called the trivial extension of A by M and denoted by $T(A, M)$.

Definition(1-13) : (The direct sum)

Let $Tri(A, M, B)$ be a triangular algebra over R. Denote by $A \oplus B$ the direct sum of A and Bas R-algebra, and view M as an $A \oplus B$ bimodule with the module actions given by

$$(a, b).m = am, m.(a, b) = mb, \qquad a \in A, \quad b \in B, \quad m \in M.$$

Let $Tri(A, M, B)$ is isomorphic to $T(A \oplus B\ , M)$ as an R-algebra. So triangular algebras are examples of trivial extensions.

Definition(1-16) : (The left annihilator)

Let A be an R-algebra and M be an A-bimodule, define the left annihilator of M by $.ann_A M = \{a \in A : {}_aM = \{O\}\}$. Similarly, we define the right annihilator of M by $r.ann_A M = \{a \in A : M_a = \{O\}\}$. Also we denote the unity and zero of $T(A, M)$ by 1 and 0, respectively.

In order to give the main result of this note, we need the following lemmas about derivations and Jordan derivations of generalized matrix algebras.

CHAPTER TWO

Main results

<u>Lemma 2.1:</u> ([5, Proposition 4.2]).

An additive map θ from $\mathcal{G}$ into itself is a derivation. if and only if it has the form

$$\theta\left(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\right) = \begin{bmatrix} \delta_1(a) - mn_0 - m_0 n & am_0 - m_0 b + \tau_2(m) \\ n_0 a - bn_0 + v_3(n) & n_0 m + nm_0 + \mu_4(b) \end{bmatrix},$$
$$\forall \begin{bmatrix} a & m \\ n & b \end{bmatrix} \in \mathcal{G}$$

where$m_0 \in M$, $n_0 \ \in \ N$ and

$$\delta_1 : A \to A, \ \ \tau_2 : M \to M, \ \ v_3 : N \to N, \ \ \mu_4 : B \to B$$

are all R-linear mappings satisfying the following conditions:

(1) δ_1 is a derivation of A with $\delta_1\ (mn) \ = \tau_2(m)n \ + \ mv_3(n)$;

(2) μ_4 is a derivation of B with $\mu_4(nm) \ = \ n\tau_2(m) \ + \ v_3(n)m$;

(3) $\tau_2(am) \ = \ a\tau_2(m) \ + \delta_1(a)m \ and \ \tau_2(mb) = \tau_2(m)b \ + \ m\mu_4(b)$;

(4) $v_3(na) \ = \ v_3(n)a \ + \ n\delta_1(a) \ and \ v_3(bn) \ = \ bV3(n) \ + \mu_4(b)n$.

<u>Lemma 2.2 :</u> ([9, Proposition 4.2]).

An additive map θ from $\mathcal{G}$ into itself is a Jordan derivation if and only if it is of the form

$$\theta\left(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\right)$$

$$= \begin{bmatrix} \delta_1(a) - mn_0 - m_0 n & am_0 - m_0 b + \tau_2(m) + \tau_3(n) \\ n_0 a - bn_0 + v_2(m) + v_3(n) & n_0 m + nm_0 + \mu_4(b) \end{bmatrix},$$

$$\forall \begin{bmatrix} a & m \\ n & b \end{bmatrix} \in \mathcal{G}$$

where$m_0 \in M$, $n_0 \in N$ and

$$\delta_1 : A \to A, \;\; \tau_2 : M \to M, \;\; \tau_3 : N \to N,$$

$$v_2 : M \to N, \;\; v_3 : N \to N, \;\; \mu_4 : B \to B$$

are all R-linear mappings satisfying the following conditions:

(1) δ_1 is a Jordan derivation on A and $\delta_1(mn) = \tau_2(m)n + mv_3(n)$;

(2) μ_4 is a Jordan derivation on B and $\mu_4(nm) = n\tau_2(m) + v_3(n)m$;

(3) $\tau_2(am) = a\tau_2(m) + \delta_1(a)m \text{ and } \tau_2(mb) = \tau_2(m)b + m\mu_4(b)$;

(4) $v_3(bn) = bv_3(n) + \mu_4(b)n \text{ and } v_3(na) = v_3(n)a + n\delta_1(a)$;

(5) $\tau_3(na) = a\tau_3(n), \tau_3(bn) = \tau_3(n)b, n\tau_3(n) = 0, \tau_3(n)n = 0$;

(6)$v_2(am) = v_2(m)a$, $v_2(mb) = bv_2(m)$,
$mv_2(m) = 0$, $v_2(m)m = 0$.

Clearly, the mappings τ_3 and v_2 in **(2)** play an important role for a Jordan derivation. Let us 'study them now.

Lemma 2.3:

Let K be a field and A a finite dimensional K -algebra. Let M be a simple left A-module and N a simple right A-module with $dim_K M \neq dim_K N$.

Then we have:

(1) Let v be a K -linear map from M to N. If $v(am) = v(m)a \, for \, all \, a \in A, m \in M, then v = O$.

(2) Let τbe a K -linear map from N to M. If $\tau(na) = ar(n) for \, all \, a \in A, n \in N, then \, \tau = O$.

Proof.

Since $dim_K M \neq dim_K N$, without loss of generality, suppose that $dim_K M < dim_K N$

(1) If $v \neq 0$, then there exists some $m \in M$ such that $v(m) \neq O$. It follows from $v(am) = v(m)$a that the cyclic module $\langle v(m) \rangle$ generated by $v(m)$ is contained in Im v, the image of v. Note that N is a simple as a right A-module. This implies that $\langle v(m) \rangle = N$ and consequently, Im $v = N$. However, this is impossible for $dim_K M < dim_K N$.

(2) Since $dimM < dimN$, the kernel of τ is not zero. Let $0 \neq n \in Ker \, \tau$.

Then the condition $\tau(na) = a\tau(n)$ gives that $0 \neq \langle n \rangle \subseteq Ker \, \tau$ and hence $Ker \, \tau = N$ for N is simple as a right A-module, that is, $\tau = 0$.

Now we are in: a place to give the main result of this note.

<u>Theorem 2.4 :</u>

Let A and B be two finite dimensional K-alqebras and let $\mathcal{G} = (A, {}_AM_B, {}_BN_A, B)$ be a trivial generalized matrix algebra. Suppose that $\mathcal{G}$ satisfies the following conditions:

(1) All Jordan derivations of A and B are derivations.

(2) M is simple as a left A-module and N is simple as a right A-module; (3) $dim_K\ M \neq dim_K N$.

Then each Jordan derivation of $\mathcal{G}$ is a derivation.

Proof

Since all Jordan derivations of A and B are derivations, comparing the form **(1)** of **Lemma (2.1)** with that **(2)** of **Lemma (2.2)** yields that we only need to prove $v_2 = 0$ and $\tau_3 = 0$. By **(Lemma 2.3)** this is clear and we complete the proof now.

In order to give a counter example for [1, Conjecture 3.2]' let us recall a result of Xiao and Wei on Jordan higher derivations.

<u>**Lemma 2.5:**</u> **([17, Proposition 3.1]):**

Let A be an associative algebra over a field of characteristic zero. If every Jorddn derivation on A is a derivation, then every Jordan higher derivation on A is a higher derivation.

CHAPTER THREE

The main result of the paper is the following theorem

Theorem 3.1.

Let A be a unital algebra over the 2-torsion free commutative ring Rand M be a unital A-bimodule. Suppose that E is a non-trivial idempotent element in A and $E' = 1 - E$ such that

$$EAE'AE = \{O\}, \qquad E'AEAE' = \{O\},$$

$$E(l.ann_A M)E = \{O\}, \qquad E'(r.ann_A M)E' = \{O\},$$

and $EME' = M$ for all $M \in M$. Let $U = T(A, M)$ and $\Delta : u \longrightarrow u$ be a Jordan derivation and let $P = (E, O)$ and $Q = (E', 0)$. Then there exists a derivation $\delta : u \longrightarrow u$ and an antiderivation $J : u \longrightarrow u$ such that $\Delta = \delta + J$, $J(PXP) = O$ and $J(QXQ) = O$ for any $X \in u$. Moreover, δ and J are uniquely determined.

To prove the theorem we need some lemmas. We consider the conditions of this theorem in the lemmas. Note that, P and Q are idempotents of u such that $P + Q = 1$ and $PQ = 0$.

We will show that the Jordan derivation Δ is a sum of an antiderivation J (see Lemma 3.3), an inner derivation I (see Lemma 3.5) and a derivation D (see Lemma 3.8).

Lemma 3.2:

For every $X, Y \in u$, we have

$$PXQYP = 0 \quad and \quad QXPYQ = 0.$$

Proof.

For all $M \in M$, since $EM\,E' = M$, we have

$$EME = 0, \quad E'ME = 0, \quad E'ME' = 0,$$

$$EM = M, \quad ME' = M, \quad ME = O, \quad E'M = O.$$

Let $X = (A, M)$ and $Y = (B, N)$. So $PXQYP = (EAE'BE, EAE'NE + EM\,E'\,BE) = 0$ as $ECE = 0$ for all $C \in M$ and $EAE'\,AE = \{O\}$. Similarly, $QXPYQ = O$.

Lemma 3.3:

The mapping $J : u \longrightarrow u$ defined by

$$J(X) = P\Delta(QXP)Q + Q\Delta(PXQ)P$$

is an antiderivation. Also $J(PXP) = O$ and $J(QXQ) = 0$ for all $X \in u$.

Proof.

Clearly, J is an R-linear map. Since A is a Jordan derivation, for all $X, Y \in u$ we have

$$\Delta(QXPYP) = \Delta(QXPPYP)$$

$$= \Delta(QXPPYP + PYPQXP)$$

(3.1) $$= \Delta(QXP)PYP + QXP\Delta(PYP)$$

$$+PYP\Delta(QXP) + \Delta(PYP)QXP.$$

Similarly

$$\Delta(QXQYP) = \Delta(QXQ)QYP + QXQ\Delta(QYP)$$

$$+ QYP\Delta(QXQ) + \Delta(QYP)QXQ. \tag{3.2}$$

$$\Delta(PXPYQ) = \Delta(PXP)PYQ + PXP\Delta(PYQ)$$

$$+ PYQ\Delta(PXP) + \Delta(PYQ)PXP. \tag{3.3}$$

$$\Delta(PXQYQ) = \Delta(PXQ)QYQ + PXQ\Delta(QYQ)$$

$$+ QYQ\Delta(PXQ) + \Delta(QYQ)PXQ. \tag{3.4}$$

Thus,

$$P\Delta(QXPYP)Q = PYP\Delta(QXP)Q;$$

$$P\Delta(QXQYP)Q = P\Delta(QYP)QXQ;$$

$$Q\Delta(PXPYQ)P = Q\Delta(PYQ)PXP;$$

$$Q\Delta(P\,XQYQ)P = QYQ\Delta(P\,XQ)P.$$

From these relations and Lemma 3.2 we arrive at

$$\begin{aligned}
J(XY) &= P\Delta(QXYP)Q + Q\Delta(PXYQ)P \\
&= P\Delta(QXPYP)Q + P\Delta(QXQYP)Q \\
&\quad + Q\Delta(PXPYQ)P + Q\Delta(PXQYQ)P \\
&= PYP\Delta(QXP)Q + P\Delta(QYP)QXQ \\
&\quad + Q\Delta(PYQ)PXP + QYQ\Delta(PXQ)P \\
&= YP\Delta(QXP)Q + P\Delta(QYP)QX \\
&\quad + Q\Delta(PYQ)PX + YQ\Delta(PXQ)P \\
&= Y\,J(X) + J(Y)X.
\end{aligned}$$

So J is an anti-derivation. By the definition of J it is clear that $J(PXP) = 0$ and $J(QXQ) = 0$ for all $X \in u$. The proof is now complete.

Lemma 3.4:

If $J : u \longrightarrow u$ is an improper antiderivation, $J(P\,X\,P) = 0$ and $J(QXQ) = 0$ for all $X \in u$, then $J = 0$.

Proof.

First, observe that $J(P) = J(PPP) = 0$. Similarly, we have $J(\,Q) = 0$. Then, since J is a derivation and an antiderivation, we have

$$J(PXQ) = PJ(XQ) + J(P)XQ = PJ(XQ)$$

$$= P(\,QJ(X) + J(\,Q)X) = 0.$$

Similarly, $J(\,QX\,P) = 0$. So

$$J(X) = J(PXP) + J(PXQ) + J(QXP) + J(QXQ) = 0$$

for all $X \in u$.

Lemma 3.5:

Let $T = P\,\Delta(P)Q - Q\Delta(P)P$ and the mapping $I : u \longrightarrow u$ be defined by

$$I(X) = P\Delta(PXP + QXQ)Q + Q\Delta(PXP + QXQ)P.$$

Then for every $X \in u$ we have

$I(X) = XT - TX.$

Proof.

All $Y \in u$ satisfy

$$0 = \Delta((PYP)(QYQ) + (QYQ)(PYP))$$

$$(3.5) \qquad = PYP\Delta(QYQ) + \Delta(PYP)QYQ$$

$$+QYQ\Delta(PYP) + \Delta(QYQ)PYP.$$

From this, for every $Y \in u$, we obtain

$$(3.6) \qquad PYP\Delta(QYQ)Q + P\Delta(PYP)QYQ = 0$$

and

$$(3.7) \qquad QYQ\Delta(PYP)P + Q\Delta(QYQ)PYP = O.$$

For any $X \in u$ replace Y by $X + P$ in (3.6). This gives

$$PXP\Delta(QXQ)Q + P\Delta(QXQ)Q + P\Delta(PXP)QXQ + P\Delta(P)QXQ = O.$$

Hence, replacing X by QXQ in the previous equation, we get that $P\Delta(QXQ)Q + P\Delta(P)QXQ = 0 \, for \, any \, X \in u. If \; X = Q$

In this relation, then $P\Delta(Q)Q + P\Delta(P)Q = O$.

Now, for any $X \in u$ replace Y by $PXP + Q$ in (3.6) we obtain

$$PXP\Delta(Q)Q + P\Delta(PXP)Q = 0.$$

According to these relations we have $-PXP\Delta(P)Q + P\Delta(PXP)Q = O$. Similarly, we can obtain from relation (3.7) that

$$Q\Delta(QXQ)P + QXQ\Delta(P)P = 0 \; and - Q\Delta(P)PXP + Q\Delta(PXP)P = 0$$

$for\ all\ X \in u$. These relations and Lemma 3.2 imply

$$I(X) = P\Delta(PXP)Q + P\Delta(QXQ)Q + Q\Delta(PXP)P + Q\Delta(QXQ)P$$
$$= PXP\Delta(P)Q - P\Delta(P)QXQ + Q\Delta(P)PXP - QXQ\Delta(P)P$$
$$= XP\Delta(P)Q - P\Delta(P)QX + Q\Delta(P)PX - XQ\Delta(P)P$$
$$= XT - TX.$$

Lemma 3.6:

Let $X \in u$. Then

(a) $If\ PXPZQ = O\ for\ all\ Z \in u, then\ PXP = O$;

(b) $If\ PZQXQ = O\ for\ all\ Z \in u, then\ QXQ = O$.

Proof.

(a) Write $X = (A, N)$. Let $M \in M$, and set $Z = (0, M)$. We have $EM\,E' = M$ by assumption and $EN = N$ for all $N \in M$ from the proof of Lemma 3.2. Hence,

$$ENE = 0\ and \quad 0 = PXPZQ = (O, EAEME') = (O, AM),$$

so $A \in l.ann_A M$. Hence, by assumptions we obtain $EAE = 0$, therefore $PXP = (EAE, ENE) = O$.

Similarly, we can show that (b) holds.

Lemma 3.7:

For every $X \in u$ we have

$P\Delta(QXQ)P = 0, \quad Q\Delta(PXP)Q = 0, \quad P\Delta(PXQ)P = 0,$

$Q\Delta(PXQ)Q = 0, \quad P\Delta(QXP)P = 0, \quad Q\Delta(QXP)Q = 0.$

Proof. Using (3.5) we see that for all $Y \in u$, we have

$$PYP\Delta(QYQ)P + P\Delta(QYQ)PYP = O.$$

For any $X \in u$ replace Y by $QXQ + P$, so $P\Delta(QXQ)P = O$. Similarly, replacing Y by $P X P + Q$ in (3.5), and multiplying the resulting equation by Q both on the left and on the right, yields $Q\Delta(P X P)Q = 0$, for all $X \in u$.

If we multiply (3.1) by P and replace Y by P, we obtain $P\Delta(QXP)P = 0$ for all $X \in u$, since Lemma 3.2 holds. Similarly, multiplying (3.1) by Q and replacing Y by P, we get $Q\Delta(QXP)Q = 0$ for all $X \in u$.

As above, from (3.4) and Lemma 3.2, we have $P\Delta(P XQ)P = 0$ and

$Q\Delta(PXQ)Q = 0, \quad for\ all\ X \in u.$

Lemma 3.8:

The mapping $D : u \longrightarrow u$ defined by $D(X) = P\Delta(P X P)P + P\Delta(PXQ)Q + Q\Delta(QXP)P + Q\Delta(QXQ)Q$ is a derivation.

Proof.

D is an R-linear map. From (3.3) and Lemma 3.7 it follows immediately that

$$P\Delta(PXPYQ)Q = PXP\Delta(PYQ)Q + P\Delta(PXP)PYQ$$

for all $X, Y \in u$. So for every $X, Y, Z \in u$ we have $P\Delta(PXPYPZQ)Q = PXPYP\Delta(PZQ)Q + P\Delta(PXPYP)PZQ$.

On the other hand,

$$P\Delta(PXPYPZQ)Q = PXPYP\Delta(PZQ)Q$$
$$+PXP\Delta(PYP)PZQ + P\Delta(PXP)PYPZQ.$$

By comparing the two expressions for $P\Delta(PXPYPZQ)Q$, we arrive at $P\big(\Delta(PXPYP) - \Delta(PXP)PY - XP\Delta(PYP)\big)PZQ = 0$.

for any $Z \in u$. Therefore, by Lemma 3.6, we have

$$P\Delta(PXPYP)P = P\Delta(PXP)PYP + PXP\Delta(PYP))P.$$

Similarly, from (3.4) we get

$$P\Delta(PXQYQ)Q = P\Delta(PXQ)QYQ + PXQ\Delta(QYQ)Q$$

and

$$Q\Delta(QXQYQ)Q = Q\Delta(QXQ)QYQ + QXQ\Delta(QYQ)Q$$

for all $X, Y \in u$.

Similarly, we can obtain from (3.1), (3.2) and Lemma 3.6 that

$$Q\Delta(QXPYP)P = Q\Delta(QXP)PYP + QXP\Delta(PYP)P$$

and

$$Q\Delta(QXQYP)P = QXQ\Delta(QYP)P + Q\Delta(QXQ)QYP$$

for all $X, Y \in u$.

These relations with Lemma 3.2 gives us that $D(XY) = X\,D(Y) + D(X)Y$ for all $X, Y \in u$. That is, D is a derivation from u into itself.

Proof of Theorem 3.1. For any $X \in u$we have

$$X = PXP + PXQ + QXP + QXQ$$

so, by Lemmas 3.3, 3.5, 3.7 and 3.8 it follows immediately that $\Delta(X) = J(X) + I(X) + D(X)$ for all $X \in u$ where $\delta = D + I$ is a derivation and J is an antiderivation from u into itself such that $J(PXP) = 0\ and\ J(QXQ) = 0\ for\ any\ X \in u$.

Let $\delta : u \longrightarrow u$ be a derivation and $J' : u \longrightarrow u$ be an antiderivation such that $\Delta = \delta + J', J'(PXP) = 0$ and $J'(QxQ) = 0$ for any $X \in u$. So $\delta + J = \delta' + J'$ and hence $\delta - \delta' = J - J'$. Therefore, $J - J'$ is an improper antiderivation such that $(J - J')(PXP) = 0$ and

$(J - J')(QxQ) = 0$. Thus, by Lemma 3.4, we have $J = J'$ and hence $\delta = \delta'$. So we have that δ and J are uniquely determined. The proof of Theorem 3.1 is thus complete.

Note that if $J \neq 0$, then J is a proper antiderivation (by Lemma 3.4).

Remark 3.9.

By the above lemmas and the proof of Theorem 3.1, one observes that if $\Delta : u \longrightarrow u$ is a Jordan derivation, then the following are equivalent.

(a) Δ is a derivation.

(b) $P\Delta(QXP)Q = 0 \; and \; Q\Delta(PXQ)P = 0 \; for \; all \; X \in u$.

(c) $\Delta(PUQ) \subseteq PUQ \; and \; \Delta(QUP) \subseteq QUP$.

We have the following corollary, which was proved by a different method in [23].

Corollary 3.10.

Let A, B be unital algebras over the 2-torsion free commutative ring R, M be a unital (A, B)-bimodule that is faithful as a left A-module and also as a right B-module. Let $T = Tri(A, M, B)$ be the triangular algebra. Then every Jordan derivation from T into itself is a derivation.

Proof.

Let $A \oplus B$ be the direct sum of A and B as R-algebras and $E = (1,0)$. Consider $T(A \oplus B, M)$ as defined in introduction. So this trivial extension satisfies all the requirements in Theorem 3.1 and therefore any Jordan derivation on it satisfies condition (b) of Remark 3.9. Therefore, every Jordan derivation on $T(A \oplus B, M)$ is a derivation. By the isomorphism given in the introduction we have the result.

Remark 3.11.

Let $T = Tri(A, M, B)$ be a triangular algebra satisfying the conditions of Corollary 3.10, $P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ be the standard idempotent of T and $Q = 1 - P$. Suppose that N is a unital T-bimodule such that $QNP = \{O\}$ and, let for $N \in N$, the condition $PNPTQ = \{O\}$ implies $PNP = 0$ and the condition $PTQNQ = \{O\}$ implies $QNQ = 0$. Then $(P, 0)$ and $(Q, 0)$ are idempotent of $T(T, N)$ such that

$$(Q, 0)T(T, N)(P, 0) = \{(0, 0)\}.$$

Let $(S, N)\ E\ T(T, N)$ such that

$$(P, 0)(S, N)(P, 0)T(T, N)(Q, 0) = \{(0, 0)\}.$$

So for each $S' \in T$ we have $(P, 0)(S, N)(P, 0)(S', O)(Q, 0) = (0,0)$ and hence $(PSPS'Q, PNPS'Q) = (0,0)$. Therefore, $PSPTQ = \{0\}$ and $PNPTQ = \{0\}$. By assumption, we have $PSP = 0$ and $PNP = 0$. So $(P, 0)(S, N)(P, 0) = 0$. Similarly, if $(P, 0)T(T, N)(Q, 0)(S, N)(Q, 0) = \{(0, 0)\}$, then $(Q, O)(S, N)(Q, 0) = 0$. Therefore

$$T(T, N) \cong \begin{pmatrix} (P, 0)T(T, N)(P, 0) & (P, 0)T(T, N)(Q, 0) \\ 0 & (Q, 0)T(T, N)(Q, 0) \end{pmatrix}.$$

Thus, $T(T, N)$ is a triangular algebra. So by Corollary 3.10 every Jordan derivation from $T(T, N)$ into itself is a derivation.

Let A be a unital algebra over R and M be a unital A-bimodule.

An R-linear map δ from A into M is a Jordan derivation (derivation) if and only if the R-linear map $\Delta\colon T(A,M) \longrightarrow T(A,M)$, given by $\Delta(A,M) = (0,\delta(A))$, is a Jordan derivation (derivation). From this result and Remark 3.11, we have the next corollary which is a generalization of Corollary 3.10.

Corollary 3.12.

Let $T = Tri(A,M,B)$ be a triangular algebra satisfying the conditions of Corollary 3.10 and N be a unital T-bimodule as in the Remark 3.11. Then every Jordan derivation from T into N is a derivation.

We now provide an example of trivial extension which satisfies

conditions of Theorem 3.1, but is not a triangular algebra.

Example 3.13.

Let R be a 2-torsion free commutative ring with unity and A be the R-algebra of $2 \, x \, 2$ lower triangular matrices over R. We make R into an A-bimodule by defining $RA = RA_{22}$ and $AR = A_{11}R$ for all $R \in R, A \in A$. Let $E = E_{11}$. Then the conditions of Theorem 3.1 hold for $T(A,R)$ but this trivial extension is not a triangular algebra because the map $\Delta : T(A,R) \longrightarrow T(A,R)$ defined by $\Delta(A,R) = (RE_{21}, A_{21})$ is a proper antiderivation, while by the above corollary, triangular algebras have no nonzero proper antiderivation. (We denote E_{ij} for the matrix units, for all $i,j.$)

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