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t-lifting Module and t-Supplemented Module

A Graduation Paper Submitted in a Partial Fulfillment of the Requirements for a Bachelor of Science in Mathematics

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April 2019

بسم الله الرحمن الرحيم

﴿ وَقُل رَّبِّ زِدْنِي عِلْمًا ﴾

صدق الله العظيم

سورة طه ـ الآية ١١٤

الاهداء

الى معلم البشرية ومنبع العلم نبينا محمد (صلى الله على محمد واله وسلم)

الى من مهدوا الطريق امامي للوصول الى ذروة العلم

الى كل من مات لتحيا ارضه

الشكر والتقدير

الحمد لله رب العالمين والصلاة والسلام على اشرف خلق الله سيدنا محمد (صلى الله عليه واله وسلم)

نتوجه بالشكر الجزيل لكل من ساهم في انجاز هذا البحث واخراجه الى حيز التنفيذ ، الى كل من كان سبب في تعليمنا وتوجيهنا ومساعدتنا ..

الى مشرف البحث أ. م. د. فرحان داخل شياع حيث بذل جهوده في توجيهنا اثناء عملنا في البحث ..

الى الكادر التدريسي الافاضل في قسم الرياضيات ..

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Introduction

Through this paper all rings are associative with unity and all modules are unitary right modules. We recall some relevant notions and results. A submodule N of an R-module M is essential in M (briefly N $\leq_{ess}$ M) if N ∩ W = (0), W ≤ M implies W = (0) . as generalization of essential the concept of t-essential is introduced by Asgari and Haghany in 2011 they said that a submodule A of "an R-module M is t-essential in M(written $A \leq_{tes} M$) if whenever $A \cap C \leq Z_2(M), C \leq M$ implies $C \leq Z_2(M)$" where $Z_2(M)$ is the second singular submodule of M, and " $Z_2(M)$ is (or Goldi torsion) defined by $Z(\frac{M}{Z(M)}) = \frac{Z_2(M)}{Z(M)}$ where $Z(M) = \{x \in M: xI = (0)$ for some essential ideal of R$\}$. A submodule W of M is called a small submodule of M if whenever $M = W + U$, U is a submodule of M implies $U = M$.

In [5] Truong and Phan studied some properties of e-supplemented and e-lifting modules . in their work introduced e-small submodule in M, a submodule N of M is called e-small in M if $N + L = M$ with $L \leq_{ess} M$ implies $L = M$. Also, they studied e-lifting module , a module M is called e-lifting if for any $N \leq M$, there exista a decomposition $M = A \oplus B$ such that $A \leq N$ and $N \cap B$ e-small in M and they investigated properties of e-lifting. Moreover, they studied e-supplement module, a module M is called e-supplemented if every submodule of M has an e-supplement in M. Let N, L be submodule of M. L is called an e-supplement of N in M, if $M = N + L$ and $N \cap L$ is e-small in L.

This work consists of two chapters. In chapter one we deal with certain knows result about some properties of e-supplemented and e-lifting modules which introduced by Truong and Phan in [5] . In chapter two we define and study t-lifting module and t-supplemented module. A module M is called t-liting if for any N $\leq$ M there exists a decomposition $M = A \oplus B$ such that $A \leq$ N and $N \cap L$ t-small M, a submodule N of M is called to be t-small in M(denoted by $N \ll_t M$) if $N + L = M$ with $L \leq_{tes} M$ implies $L = M$. A module M is called t-supplemented if every submodule of M has a t-supplemented in M. Let N, L be submodules of , L is called t-supplement of N in M if $M = N + L$ and $N \cap L \ll_t L$.

Chapter One
Preliminaries

Definition 1-1 [6]: Let R be a commutative ring with unity and let M be a unitary right R- module. A submodule N of M is said to be essential submodule in M (denoted by $N \leq_{ess} M$) if for any submodule K of M, $N \cap K = 0$ implies that K=0 , or $N \leq_{ess} M$ if $N \cap K \neq 0 \ \forall K \leq M$.

Definition 1-2 [5] : Let R be a ring and let M be a right R-module. A submodule N of M is said to be small submodule (denoted by $N \ll M$) if for any submodule K of M such that N+K = M , implies K=M .

Definition 1-3 [6]: A submodule N of M is said to be direct summand (denoted by $N \leq^{\oplus} M$) if for any submodule K of M such that $N \cap K = 0$ then N+K=M.

Definition 1-4 [5]: Let U a submodule of M , a submodule V of M is called a supplement of U if V is a minimal submodule of M with the property U+V=M .

Definition 1-5 [5]: Let M be an R-module and M is called lifting module if $\forall$ $N \leq M$, $\exists$ K , $\acute{K} \leq M$, such that $K \leq N$ and $K \oplus \acute{K} = M$ and $N \cap \acute{K} \ll \acute{K}$.

Definition 1-6 [5]: A submodule N of M is said to be e – small in M (denoted by $N <<_{ess} M$) , if N+L = M with $L \leq_{ess} M$ implies L = M .

Definition 1-7 [5]: A module M is called e-lifting if, for any $N \leq M$, there exists a decomposition $M=A \oplus B$ such that $A \leq N$ and $N \cap B <<_{ess} M$.

Definition 1-8 [5]: A module M is called duo, if every submodule of M is fully invariant.

Definition 1-9 [5]: Let N , L be submodule of M . L is called an e-supplement of N in M if M= N+L and N ∩ L is e-small in L . A module M is called e-supplemented if every submodule of M has an e-supplement in M.

Definition 1-10 [5]: A module M is called amply e-supplemented if, for any submodules A , B of M with M = A+B, there exists an e-supplement P of A such that $P \leq B$.

Definition 1-11 [4]: A module M is said to be π-Projective if, for every two submodules U , V of M with U +V = M, there exists $f \in End(M)$ with $Im(f) \leq U$ and $Im(1 - f) \leq V$.

Definition 1-12 [4] : For R – modules N and A. N is said to be A – projective , if every submodule X of A , any homomorphism $\emptyset: N \to A / X$ can be lifted to a homorphism , $\varphi : N \to A$, that is if $\pi : A \to A / X$, be the natural epimorphism , then there exists a homorphism $\varphi : N \to A$ such that $\pi \circ \varphi = \emptyset$.

Definition 1-13 [4] :A module M is called projective if M is N – projective for every R – module N . If M is M – projective , M is called self – projective .

Definition 1-14 [4]: $Z_2(M)$ or (Goldi torsion) is defined by $Z(M / Z(M)) = Z_2(M) / Z(M)$ where $Z(M) = \{x \in M : xI = (0)$ for some essential ideal of $R\}$ .In fact $Z(M) = \{ x \in M : ann(x) \leq_{ess} R \}$ where $ann(x) = \{\mathbf{r} \in R : xr = 0 \}$.

Definition 1-15 [4]: A module M is called singular (nonsingular) if $Z(M) = M (Z(M)= 0)$.

***** $Z_2(M) = \{x \in M : xI = (0)$ for some t-essential ideal I of $R \}$.

Definition 1-16 [4]: A module M is called Z_2- torsion if $Z_2(M) = M$ and a ring R is called right Z_2 – torsion if $Z_2(R_R) = R_R$.

Definition 1-17 [1]: A submodule A of M is said to be t- essential in M (denoted by $A \leq_{tes} M$) if for every submodule B of M , $A \cap B \leq Z_2(M)$ implies that $B \leq Z_2(M)$.

***** $Z_2(M) = \{ x \in M : ann_R (x) \leq_{tes} R \}$, where $ann_R (x) = \{ r \in R : xr = 0\}$.

Example 1-18 [4]: Consider Z_{12} as Z-module . It is clear that Z_{12} is singular module . Hence Z_{12} is Z_2- torsion , that is $Z_2(Z_{12}) = Z_{12}$.

***** Every essential submodule is t- essential ,but the invers is not true in general.

Example 1-19 [1]: Let $A = (\bar{4}) \leq Z_{12}$. Then for all $B \leq Z_{12}$ and $(\bar{4}) \cap B \leq Z_2(Z_{12}) = Z_{12}$ then $B \leq Z_2(Z_{12}) = Z_{12}$. Hence $(\bar{4}) \leq_{tes} M$, but $(\bar{4})$ is not essential of Z_{12} .

Proposition 1-20 [1]: The following statements are equivalent for a submodule A of an R – module M **.**

1. A is t- essential in M **.**
2. $A + Z_2(M) / Z_2(M)$ is essential in $M / Z_2(M)$**.**
3. $A + Z_2(M)$ is essential in M **.**
4. M / A is Z_2 – torson **.**

Chapter Two

t-Lifting Module and t-Supplemented Module

Definition 2-1 [5]: A submodule N of M is said to be e – small in M (denoted by $N <<_{ess} M$) , if N+L = M with $L \leq_{ess} M$ implies L = M.

Lemma 2-2 [8]: Let M be a right R- module. Then

1. If $N <<_{ess} M$ and $K \leq N$, then $K <<_{ess} M$ and $N / K <<_{ess} M / K$.

2. Let $N <<_{ess} M$ and M=X+N. Then $M = X \oplus Y$ for some semisimple submodule Y of M.

3. If $K <<_{ess} M$ and $f : M \rightarrow N$ is a homomorphism , then $f(K) <<_{ess} N$. In Particular, if $K <<_{ess} M \leq N$, then $K <<_{ess} N$.

4. Let $K_1 \leq M_1 \leq M$, $K_2 \leq M_2 \leq M$ and $M = M_1 \oplus M_2$.

Then $K_1 \oplus K_2$ is e- small in $M_1 \oplus M_2$ if and only if $K_1 <<_{ess} M_1$ and $K_2 <<_{ess} M_2$.

Definition 2-3 [5]: A module M is called e-lifting if, for any $N \leq M$, there exists a decomposition $M = A \oplus B$ such that $A \leq N$ and $N \cap B <<_{ess} M$.

Lemma 2-4 [8]: The following condition are equivalent for a module M.

(1) M is e-lifting .

(2) for any $N \leq M$, there exists a decomposition $N = A \oplus B$ such that A is a direct summand of M and $B \ll_{ess} M$.

(3) For every submodule N of M**,** there exists a direct summand A of M such that $A \leq N$ and $N/A \ll_{ess} M/A$**.**

Proof (1) → (2) by definition **.**

(2) → (3). Let N be a submodule of M**.** we have a decomposition $N = A \oplus B$,where A is a direct summand of M and $B \ll_{ess} M$ by (2)**.**Let $\pi : M \longrightarrow M/A$ be the natural map**.** since $B \ll_{ess} M$ **,** $\pi(B) \ll_{ess} M/A$ by Lemma **2-2,** i.e**.** $N/A \ll_{ess} M/A$ **.**

(3)→(1) . For every submodule N of M **,** there exists a decomposition $M = A \oplus B$ such that **A** $\leq N$ and $N/A \ll_{ess} M/A$ by (3) **.** So $N = A \oplus (N \cap B)$ **.**Then $M/A \cong B$ and $N/A \cong N \cap B$**.**

It follows from $N/A \ll_{ess} M/A$ that $N \cap B \ll_{ess} B$ **.** Hence**,** $N \cap B \ll_{ess} M$**.**

Lemma 2-5 [5]: Every direct summand of an e- lifting module is also an e- lifting module**.**

Proof: Let M be an e- lifting module and N be a direct summand of M with $M = N \oplus L$ for some submodule L of M**.** Let $A \leq N$**.** There exists a direct summand K of M Such that $K \leq A$**,** $M = K \oplus T$ and $A \cap T \ll_{ess} T$**.** Then $N = K \oplus (N \cap T)$**,** and by Lemma **2-2,** $A \cap T \ll_{ess} N \cap T$**.**

Definition 2-6 [5]: A module M is called distributive if its lattice of submodules is a distributive lattice **,** that is $A \cap (B+C) = (A \cap B) + (A \cap C)$ for any submodules A**,** B and C of M**.**

**we give sufficient conditions for a factor module of an e- lifting module to be e- lifting.

Proposition 2-7 [3]: Let M be an e- lifting module and X $\leq$M .If one of the following conditions are satisfied **:**

1-for every direct summand K of M**,**(K+X) / X is a direct summand of M / X .

2- M is a distributive module **.**

3- for any $e^2 = e \in$ End **(**M**),** eX $\subset$ X**,** and in particular **,** X is a fully invariant submodule of M **,** then M / X is an e- lifting module **.**

Proof:1- Let A / X $\leq$ M / X **.** Since M is e- lifting **,** there exists a direct summand K of M such that K $\leq$ A and A / K $<<_{ess}$ M / K by Lemma **2-4.** By hypothesis **,** (K+X)/X is a direct summand of M / X**.** Clearly**,** (K+X) / X $\leq$ A **/** X **.** Since A / K $<<_{ess}$ M / K**,** A / (K+X) $<<_{ess}$ M / (K+X) by Lemma **2-2.** Hence**,** M / X is e- lifting **.**

2- Let M=K$\oplus$L **.**We have M / X = ((K+X) / X) + ((L+X) / X) and X=X+ (K$\cap$L) = (X+K) $\cap$ (X+L) **.** So M / X = ((K+X) / X) $\oplus$ ((L+X) / X) **.** Then by (1) **,** M / X is e- lifting **.**

3- Let M = K$\oplus$ L **.** Then K = e M and L = (1 – e)M for some $e^2 = e \in$ End (M) **.** By hypothesis**,** eX $\leq$ X and (1-e)X = X**.** Hence**,** e X = X $\cap$ K and (1 - e)X = X$\cap$L **.**

Therefore , $X = (X \cap K) \oplus (X \cap L)$. Now $(K+X) / X = (K\oplus (X \cap L)) / X$ and $(L+X) / X=(L \oplus (X\cap K)) / X$. Hence, $M = K+X+L+X= (K \oplus (X\cap L))+L + X$ implies that $M / X = (K \oplus (X \cap L)) = X+ (L+X) / X$. Since $(k \oplus (X \cap L)) \cap (L + X) = (X \cap L) \oplus (X \cap K) = X$, $M / X = (k \oplus (X \cap L)) / X\oplus (L + X) / X$. Thus, by part 1 . M / X is an e- lifting module.

Lemma 2-8 [4] : The following conditions are equivalent for a module $M =M'\oplus M''$.

1- M' is M''- projective.

2- For every submodule N of M with $M=N+ M''$, there exists a submodule N' of N such that $M= N' \oplus M''$.

**[1] A direct sum of two e-lifting modules is not sure an e-lifting module . Indeed , let $R=Z_8$, then $2Z_8 / 4Z_8$ and R_R are e –lifting modules , but $(2Z_8 / 4Z_8) \oplus R_R$ is not e-lifting .Now, we show sufficient conditions for a direct sum of two e-lefting modules to be e-lifting.

Theorem 2-9 [8]: Let $M=M_1 \oplus M_2$. If M_1 and M_2 are e-lifting modules such that M_1 is quasi- projective and M_2- projective, then M is an e- lifting module.

Proof: Let N be a submodule of M. Since M_1 is an e-lifting module , there exists $K \leq M_1 \cap (N+M_2)$ such that $M_1= K \oplus K'$ and $K' \cap (N+M_2) \ll_{ess} M_1$.Therefore $M= K \oplus K' \oplus M_2= N+(K'\oplus M_2)$. Since M_1 is quasi-projective and M_2-projective, K is $K' \oplus M_2$-projective . By Lemma **2-8,** there exists a submodule N_1 of N such that $M= N_1 \oplus (K' \oplus M_2)$.It follows that $N \cap (L+ K') = L \cap (N+ K')$ for any submodule L of M_2 . On the other hand , M_2 is e-lifting, there is a submodule X of $M_2 \cap (N+ K')=N \cap (M_2 \oplus K')$ such that $M_2=X \oplus Y$ and $Y \cap (N+ K') \ll_{ess} M_2$ for some $Y \leq M_2$. Hence $M= (N_1 \oplus X) \oplus (Y \oplus K')$.We have $N_1 \oplus X \leq N$ and $N \cap (Y \oplus K')=Y \cap (N+ K')$. But $Y \cap (N + K') \ll_{ess} Y$. Then $N \cap (Y \oplus K') \ll_{ess} Y \oplus K'$ .Thus M is an e-lifting module .

Corollary 2-10 [7]: If M_1 is a semisimple module and M_2 is an e-lifting module , and they are relatively projective with M_1, then $M=M_1 \oplus M_2$ is an e-lifting module.

**A module M is called duo, if every submodule of M is fully invariant.

Proposition 2-11 [5]: Let $M= M_1 \oplus M_2$ be a duo module . If M_1 and M_2 are e-lifting modules, then M is also an e-lifting module.

Proof: Assume M_1 and M_2 are e-lifting modules . Take any submodule L of M. Then $L= (L \cap M_1) \oplus (L \cap M_2)$. For each $i \in \{1,2\}$,there exists a direct summand D_i of M_i such that $M_i=D_i \oplus D_i'$ with $D_i \leq L \cap M_i$ and $L \cap D'_i \ll_{ess} D'_i$ .Therefore $M= (D_1 \oplus D'_1) \oplus (D_2 \oplus D'_2) = (D_1 \oplus D_2) \oplus (D'_1 \oplus D'_2)$.We have $D_1 \oplus D_2 \leq L$ and $L \cap (D'_1 \oplus D'_2) \ll_{ess} D'_1 \oplus D'_2$.

Lemma 2-12 [5]: Let M= N + L .The following conditions are equivalent:

(1) $N \cap L \ll_{ess} L$.

(2) If for any submodule K of L with $K \leq_{ess} L$ and M= N+K, then K=L.

Proof: (1)→(2). If M= N+K , where $K \leq L$ and $K \leq_{ess} L$, then L =(L ∩ N) + K. Since $L \cap N \ll_{ess} L$, L=K. This is a contradiction.

(2)→(1). If L = (N ∩ L) + K, where $K \leq L$ and $K \leq_{ess} L$, then M= N+L = N+K.

By **(2)** , K=L . So $N \cap L \ll_{ess} L$.

Definition 2-13 [5]: Let N , L be submodule of M . L is called an e-supplement of N in M if M= N+L and N ∩ L is e-small in L . A module M is called e-supplemented if every submodule of M has an e-supplement in M.

* It is clear to see that every e-lifting module is e-supplemented .The next example shows an e-supplemented module that is not e-lifting.

Example 2-14 [5]: Let R = Z_8 .Since R is perfect, every R-module is supplemented. Thus M = R ⊕ (2R / 4R) is e-supplemented and not e-lifting.

Lemma 2-15 [8]: Let N and L be submodules of a module M such that N+L has an e-supplement H in M and N ∩ (H + L) has an e-supplement G in N . Then H + G is an e-supplement of L in M .

Proof : Let H be an e- supplement of N+L in M and G be an e-supplement of $N \cap (H+L)$ in N .Then $M = (N+L)+H$ with $(N+L) \cap H \ll_{ess} H$ and $N = [N \cap (H+L)] + G$ with $(H+L) \cap G \ll_{ess} G$. Since $(H+G) \cap L \leq H \cap (L+G) + G \cap (L+H)$, H+G is an e-supplement of L in M .

*Let M be a module . Denote $Rad_e(M) = \bigcap\{N \leq_e M \setminus N \text{ is maximal in } M\}$.Then $Rad_e(M) = \sum\{N \setminus N \ll ess\ M\}$. From the definition of an e-supplemented module , we get the following properties .

Lemma 2-16 [7]: Let M be an e-supplemented module .Then :

(1) $M / Rad_e(M)$ is a semisimple module .

(2) If L a submodule of M with $L \cap Rad_e(M) = 0$, then L is semisimple .

Proof : (1), Let $Rad_e(M) \leq N \leq M$. There exists $X \leq M$ such that $M = N+X$ and $N \cap X \ll_{ess} X$. So $N \cap X \ll_{ess} M$ .Then $M / Rad_e(M) = N / Rad_e(M) + (X + Rad_e(M)) / Rad_e(M) = N / Rad_e(M) \oplus (X + Rad_e(M)) / Rad_e(M)$ because $N \cap (X + Rad_e(M)) = (N \cap X) + Rad_e(M) = Rad_e(M)$.

(2) It is clear by **(1)** , since $L \cong L \oplus Rad_e(M) / Rad_e(M) \leq M / Rad_e(M)$.

Proposition 2-17 [7]: Let M be an e-supplemented module .Then $M = M_1 \oplus M_2$, where M_1 is a semisimple module and $M_2 \leq M$ with $Rad_e(M_2) \leq_{ess} M_2$.

Proof : Let M_1 be a submodule of M such that $Rad_e(M) \oplus M_1 \leq_{ess} M$. Since M is e-supplemented , there exists a submodule M_2 of M such that $M = M_1 + M_2$ and $M_1 \cap M_2 \ll_{ess} M_2$. Hence , $M_1 \cap M_2$ is a submodule of $Rad_e(M)$. It follows that $M = M_1 \oplus M_2$ and $Rad_e(M) = Rad_e(M_2)$ is essential in M_2 **.**By Lemma **2-16,** M_1 is a semisimple module **.**

Lemma 2-18 [5]: Let M_1 **,** U be submodules of M and M_1 be an e-supplemented module **.** If $M_1 + U$ has an e-supplement in M **,** then so does U**.**

Proof : Since $M_1 + U$ has an e-supplement in M **,**there exists $X \leq M$ such that $X + (M_1 + U) = M$ and $X \cap (M_1 + U) \ll_{ess} X$ **.** For $(X + U) \cap M_1$ **,** since M_1 is an e-supplemented module **,** there exists $Y \leq M_1$ such that $(X + U) \cap M_1 + Y = M_1$ and $(X + U) \cap Y \ll_{ess} Y$ **.**We have $X + U + Y = M$ and $(X + U) \cap Y \ll_{ess} Y$ **,** that is Y is an e-supplement of X + U in M **.** Next **,** we will show that X +Y is an e-supplement of U in M **.** It is clear that $(X + Y) + U = M$**,** so it suffices to show that $(X + Y) \cap U \ll_{ess} X + Y$ **.** Since $Y + U \leq M_1 + U$ **,** $X \cap (Y+U) \leq X \cap (M_1 + U) \ll_{ess} X$ **.** Thus $(X + Y) \cap U \leq X \cap (Y+U) + Y \cap (X+U) \ll_{ess} X + Y$ by Lemma **2-2,** as required**.**

Proposition 2-19 [5]: Let $M = M_1 + M_2$ **.** If M_1 and M_2 are e-supplemented modules **,** then M is an e-supplemented module **.**

Proof : Let U be a submodule of M **.** Since $M_1 + M_2 + U = M$ trivially has an e-supplement in M **,** $M_2 + U$ has an e-supplement in M by Lemma **2-18 .** Thus U has an e-supplement in M by Lemma **2-18** again**.** So M is an e-supplemented module **.**

Corollary 2-20 [5]: Let M $=\sum_{i=1}^{k} M_i$ **.** If M_1 , M_2 , **….,** M_k are e-supplemented modules **,** then M is an e-supplemented module **.**

Corollary 2-21 [5]: Let M $=\bigoplus_{i=1}^{k} M_i$ **.**Then M is an e-supplemented module if and only if M_1 **,** M_2 **, . . . ,** M_k are e-supplemented modules **.**

Proposition 2-22 [4]: If M is an e-supplemented module **,** then every finitely M-generated module is an e-supplemented module **.**

Proof : From corollary **2-20 ,** we know that every finite sum of e-supplemented modules is an e-supplemented module **.** Next **,** we will show that every factor module of an e-supplemented module is again an e-supplemented module **.** Let M be an e-supplemented module and M / N any factor module of M **.** For any submodule L of M containing N **,** since M is an e-supplemented module **,** there exists $K \leq M$ such that L + K = M and $L \cap K \ll_{ess} K$ **.**Thus M / N = L / N + (N + K) / N and (L / N) ∩ ((N + K) / N) = (N + (L ∩ K))/ N $\ll_{ess}$ (N + K) / N **,** that is (N+K) / N is an e-supplement of L / N in M / N **,** as required **.**

⁎ A module M is called amply e-supplemented if**,** for any submodules A **,** B of M with M = A+B**,** there exists an e-supplement P of A such that $P \leq B$ **.**

⁎ we consider the relation of e-supplemented modules and amply e-supplemented modules **.**

Proposition 2-23 [5]: Let M be an amply e-supplemented module . Then any homomorphic image of M is an amply e-supplemented module .

Proof : Assume M is an amply e-supplemented module and $f : M \to N$ is any epimorphism . We want to show that N is amply e-supplemented . Let N = A + B . Then $M = f^{-1}(A) + f^{-1}(B)$. Since M is amply e-supplemented, there exists a submodule X of M such that $M = f^{-1}(A) + X$, $f^{-1}(A) \cap X \ll X \leq f^{-1}(B)$. Now, $N = A + f(X)$ and $A \cap f(X) = f(f^{-1}(A) \cap X) \ll_{ess} f(X)$. Clearly $f(X) \leq B$.

Proposition 2-24 [5]: Let M be a module . If every submodule of M is an e-supplemented module, then M is an amply e-supplemented module .

Proof : Let L , N ≤ M and M = N + L . By assumption , there is H ≤ L such that $(L \cap N) + H = L$ and $(L \cap N) \cap H = N \cap H \ll_{ess} H$. Thus $L = H + (L \cap N) \leq H + N$ and hence $M = (N + L) \leq H + N$. Therefore, M = H + N as desired.

Corollary2-25 [5]: The following statements are equivalent for a ring R.

1* Every module is amply e-supplemented .

2* Every module is e-supplemented .

***[5]** A module M is said to be π-Projective if, for every two submodules U , V of M with U + V = M, there exists $f \in End(M)$ with $Im(f) \leq U$ and $Im(1 - f) \leq V$.

Theorem 2-26 [4]: Let M be a module .If M is a π-projective e-supplemented module , then M is an amply e-supplemented module.

Proof : Let A , B be submodules of M such that M = A + B. By the hypothesis M is π-projective, there exists an endomorphism e of M such that e(M) ≤ A and (1 – e)(M) ≤ B. Note that (1 – e)(A) ≤ A .Let C be an e-supplement of A in M. Then M = e(M) + (1– e)(M) = e(M) + (1 – e)(A + C) ≤ A + (1 – e)(C) ≤ M, so M = A +(1– e)(C). We see that (1 – e)(C) is a submodule of B. Let y ∈ A ∩ (1 – e)(C). Then y ∈ A and y = (1– e)(x) = x – e(x) for some x ∈ C . Next, x = y + e(x) ∈ A, so y ∈ (1– e)(A ∩ C). But, A ∩ C $\ll_{ess}$ C, which gives A ∩ (1–e)(C) = (1 –e)(A∩ C) $\ll_{ess}$ (1–e)(C). Thus (1–e)(C) is an e-supplement of A in M .Thus M is an amply e-supplemented module.

Definition 2-27 [1]: A submodule N of M is said to be t-small in M (denoted by $N \ll_t M$) if N+L = M with $L \leq_{tes} M$ implies L = M .

Definition 2-28 [1]: A module M is called t-lifting if for any N ≤ M then exists a decomposition M = A ⊕ B such that A ≤ N and $N \cap B \ll_t M$.

Definition 2-29 [1]: Let N , L be a submodule of M , L is called t-supplement of N in M if M = N+L then $N \cap L \ll_t L$. A module M is called t-supplemented of every submodule of M has a t-supplement in M .

Lemma 2-30 [2]:1- If $N \ll_t M$ and $K \leq N$ then $K \ll_t M$ and $N / K \ll_t M / K$. **2-** $N + L \ll_t M$ if and only if $N \ll_t M$ and $L \ll_t M$ **.**

3-If $K \ll_t M$ and f **:**$M \to N$ is a homomorphism then $f(K) \ll_t N$**.** In particular if $K \ll_t M \leq N$ **,** then $K \ll_t N$ **.**

4- Let $K_1 \leq M_1 \leq M$**,** $K_2 \leq M_2$ and $M = M_1 \oplus M_2$ then $K_1 \oplus K_2 \ll_t M_1 \oplus M_2$ if and only if $K_1 \ll_t M_1$ and $K_2 \ll_t M_2$**.**

Proof 1: Suppose $L \leq_{tes} M$ and $L+K = M$ **,** then $N+L = M$ **,** thus $L=M$ for $N \ll_t M$ **,** so $K \ll_t M$ **.**

If $L \leq M$ with $L / K \leq_{tes} M / K$ and $L / K + N / K = M / K$ **,** then $N + L = M$ and $L \leq_{tes} M$ **.** Hence $L = M$ and $L / K = M / K$ **.** Thus $N / K \ll_t M / K$ **.**

Proof 2 : Suppose $N + L \ll_t M$**,** to prove $N \ll_t M$ and $L \ll_t M$**.** Let $N + L \ll_t M$ and $N \leq N + L$ and $L \leq N + L$ **,** by (1) $N \ll_t M$ and $L \ll_t M$ **.**

Suppose $N \ll_t M$ and $L \ll_t M$ **,** to prove $N + L \ll_t M$ **.** Let $N \ll_t M$ and $L \ll_t M$ **,** $N + L + K = M$ with $K \leq_{tes} M$ **,** since $N \ll_t M$ then $L + K = M$ also $L \ll_t M$ then $K = M$ **.** Hence $N + L \ll_t M$ **.**

Proof 3: Suppose $A \leq_{tes} N$ and $A + f(K) = N$ **,** $f^{-1}(A) \leq_{tes} M$ by **[**Asqar **,** 2014 **],** and $f^{-1}(A) + K = M$ and since $K \ll_t M$ **,** we have $f^{-1}(A) = M$ then $A = f(M) = N$ **.**

Proof 4 : Let $K_1 \oplus K_2 \ll_t M$ **,** to prove $K_1 \ll_t M$ and $K_2 \ll_t M$ **,** Let $K_1 + T_1 = M_1$ and $K_2 + T_2 = M_2$ with $T_1 \leq_{tes} M_1$ and $T_2 \leq_{tes} M_2$ **.** $K_1 + T_1 + K_2 + T_2 = M_1 + M_2$ $\rightarrow K_1 + K_2 + T_1 + T_2 = M_1 + M_2$ then $T_1 + T_2 = M_1 + M_2$ and $T_1 = M_1$ **,** $T_2 = M_2$ **.**

Let $K_1 \ll_t M$ and $K_2 \ll_t M$ to prove $K_1 \oplus K_2 \ll_t M$**,** $K_1 + K_2 + T = M_1 + M_2$ such that $T = T_1 + T_2$ **,** $T_1 \leq_{tes} M_1$ and $T_2 \leq_{tes} M_2$ **.** $K_1 + T_1 + K_2 + T_2 = M_1 + M_2$ **,** then $K_1 + T_1 = M_1$ and $K_2 + T_2 = M_2$**,** but $K_1 \ll_t M$ **,** then $T_1 = M_1$ and $K_2 \ll_t M_2$ **,** then $T_2 = M_2$ **,** we got $T_1 + T_2 = M_1 + M_2$ thus $K_1 \oplus K_2 \ll_t M_1 + M_2$ **.**

Lemma 2-31 [2]: The following statements are equivalent **:**

1- M is t- lifting **.**

2- For any $N \leq M$ **,** there exist $N = A \oplus B$ such that $A \leq^{\oplus} M$ and $B \ll_t M$ **.**

3- For each submodule N of M **,** there exist $A \leq^{\oplus} M$ such that $A \leq N$ and $N / A \ll_t M / A$ **.**

Proof : 1→2 by definition**.**

2→3 Let $N \leq M$ **,** we have a decomposition $N = A \oplus B$ **,** where $A \leq M$ and $B \ll_t M$ by (2) π**:**$M \rightarrow M/A$ be the natural map **,** since $B \ll_t M$ **,** $\pi(B) \ll_t M$ by Lemma **2-30 ,** i.e $N / A \ll_t M / A$ **.**

3→1 For each $N \leq M$ there exists a decomposition $M = A \oplus B$ such that $A \leq N$ and $N / A \ll_t M / A$ by (3)**.** So $N = N \cap M = N \cap (A \oplus B) = (N \cap A) \oplus (N \cap B) = A \oplus$ (N

$\cap$B) then M / A $\cong$ B and N / A $\cong$ N $\cap$ B **.** It follows form N / A $\ll_t$ M / A that N $\cap$ B $\ll_t$ B **.** Hence N $\cap$ B $\ll_t$ M **.**

Lemma 2-32 [1]: Every direct summand of a t – lifting module is also a t-lifting module **.**

Proof : Let M be a t-lifting module and N $\leq^{\oplus}$M with M = N $\oplus$ L for some L $\leq$M **,** let A $\leq$ N **,** there exist K $\leq^{\oplus}$M such that K $\leq$ A **,** M = K $\oplus$ T and A $\cap$ T $\ll_t$ T **,** then N = N $\cap$M **,** N = N $\cap$(K $\oplus$ T)= (N $\cap$K) $\oplus$ (N $\cap$T) = K $\oplus$ (N $\cap$T) and by Lemma **2-30** A $\cap$T $\ll_t$ N $\cap$T **.**

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