University of AL-Qadisiyah
College of Education
Department of Mathematics

Some Properties of Fuzzy Banach Algebra

A Research

Submitted to the Council of Mathematical Department/College of Education/University of Al- Qadisiyah As a Partial Fulfillment of the Requirements for the Degree of Bachelor of Science in Mathematics

By

Mustafa Read

2019AC

الإهداء

إلهي لا يطيب الليل إلا بشكرك ولا يطيب النهار إلى بطاعتك.. ولا تطيب اللحظات إلا بذكرك.. ولا تطيب الآخرة إلا بعفوك.. ولا تطيب الجنة إلا برؤيتك

الله ﷻ

الى من بلغ الرسالة وأدى الأمانة.. ونصح الأمة.. الى نبي الرحمة ونور العالمين..

سيدنا محمد صلى الله عليه واله وسلم

الى من كلله الله بالهيبة والوقار.. الى من علمني العطاء بدون انتظار.. الى من أحمل أسمه بكل افتخار.. ارجو من الله أن يمد في عمرك لترى ثماراً قد حان قطافها بعد طول انتظار وستبقى كلماتك نجوم أهتدي بها اليوم وفي الغد والى الأبد..

والدي العزيز

الى ملاكي في الحياة.. الى معنى الحب والى معنى الحنان والتفاني.. الى بسمة الحياة وسر الوجود.. الى من كان دعائها سر نجاحي وحنانها بلسم جراحي إلى أغلى الحبايب

أمي الحبيبة

الى من به أكبر وعليه أعتمد .. الى شمعة متقدة تنير ظلمة حياتي ..

الى من بوجودها أكتسب قوة ومحبة لا حدود لها ..الى من عرفت معها معنى الحياة الى من رعانا وحافظ علينا, الى من وقف الى جانبنا عندما ضللنا الطريق...

شكر وتقدير

من الصعب البوح بكلمة الشكر لأنها لا تحد عطاء اساتذتي الذين تعلمت على ايديهم واخص فيهم بالذكر الجميل والثناء الوفير

استاذي المشرف أ.م.د. بشرى يوسف حسين

فجزاها الله خير جزاء المحسنين

الباحث

الإهداء

إلهي لا يطيب الليل إلا بشكرك ولا يطيب النهار إلى بطاعتك.. ولا تطيب اللحظات إلا بذكرك.. ولا تطيب الآخرة إلا بعفوك.. ولا تطيب الجنة إلا برؤيتك

الله جل جلاله

الى من بلغ الرسالة وأدى الأمانة.. ونصح الأمة.. الى نبي الرحمة ونور العالمين..

سيدنا محمد صلى الله عليه واله وسلم

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Table Of Contents

1. Introduction

[4] M.A Naimark, "Normed Algebras",1^{st} .Ed. Academy of Science U.S.S.R, springer, 1972. [2] E. Kreyszig, Introductory Functional Analysis with Applications, John Wiley & Sons, New York, Chichester, Brisbane, Toronto and Singapore, 1978. [1] R. SAADATI and S.M. VAEZ POUR, "Some Results on Fuzzy BANACH SPACES", Korean, (2005). [3] Sorin Nadaban, "on Fuzzy Normed algebras, University of Timisoara, 2008.

Now , in this paper we study completion of quasi-normal algebra and completion of module algebra. This paper contains two chapter.

Chapter one consist of three sections, in section one we recall all definitions and concepts related to quasi-normed space.

In section two we study all concepts related algebra.

In section three we study all definitions and concepts related of symmetric algebra.

Chapter two concepts of three sections, in section one we study the definition fuzzy normed spaces and some properties with proofs. We start this section by the following definition. In section two at first we define Quotient spaces and give several examples of these spaces, and then we define Quotient spaces. In section three at first we define Fuzzy normed algebras and give several examples of these algebras, and then we define Fuzzy normed algebras.

Section One:

1.1. Basic concepts and Definitions

In this section, we introduce some concepts and definitions related of main subject.

Definition 1.1.1 [4]: (vector space) : let X be a non-empty set of objects in which two operation addition$(+)$ and multiplication by scelars$(\cdot)$ are defined we say that $(X,+,\cdot)$ a vector space if satisfying the following :

1- $x + y \in X, for\ all\ x, y \in X$

2- $x + y = y + x, for\ all\ x, y \in X$

3- $x + (y + z) = (x + y) + z, for\ all\ x, y, z \in X$

4- There exists $0 \in X \ni X + 0 = 0 + x = x$ (Called 0 zero vector)

5- $For\ all\ x \in X,$ there exists $-x \in X$ such that $x + (-x) = (-x) + x = 0.$

6- $a.x \in X, for\ all\ x \in X, a \in \mathbb{R}$

7- $a.(x + y) = a.x + a.y, for\ all\ x, y \in X, for\ all\ a \in \mathbb{R}$

8- $(a + b).x = a.x + b.x, for\ all\ x \in X, for\ all\ a, b \in \mathbb{R}$

9- $(a.b).x = a.(b.x), for\ all\ x \in X, a, b \in \mathbb{R}$

10- $I.x = x, for\ all\ x \in X$ such that I is identity element multiplication.

Definition 1.1.2 [4]: (normed space)**:** Let X be a vector space over field F. A function $\|.\|: X \to \mathbb{R}$ is called norm on X if satisfy in the following conditions:

1- $\| x \| \geq 0, for\ all\ x \in X.$

2- $\| x \| = 0$ if and only if $x = 0.$

3- $\| \lambda \| = |\lambda| \ \| x \|, for\ all x \in X, for\ all\ \lambda \in \mathcal{F}.$

4- $\| x + y \| \leq \| x \| + \| y \|, for\ all\ x, y \in X.$

The pair $(X, \|.\|)$ is called normed space. Such that X is a vector space over the field F and $\|.\|$ a norm on X.

if $F = \mathbb{R}$ then X is a real normed space , while

if $F = \mathbb{C}$ then X is a complex normed space.

Remark : Every sub space of normed space is normed space.

Theorem1.1.3 [4] : If the X is a normed space, then

1- $\| 0 \| = 0$

2- $\| -x \| = \| x \|, for\ all\ x \in X$

3- $\| x - y \| = \| y - x \|, for\ all\ x, y \in X$

4- $| \| x \| - \| y \| | \leq \| x - y \|, for\ all\ x, y \in X$

Proof :

Properties (1.2) are concluded from the definition directly .

4- $| \| x \| - \| y \| | \leq \| x - y \|, for\ all\ x, y \in X$

$$x = (x - y) + y$$

$\| x \| = \| (x - y) + y \| \leq \| x - y \| + \| y \|$

$\| x \| - \| y \| \leq \| x - y \|$ …………..(1)

$\| y \| - \| x \| \leq \| x - y \|$

$-(\| x \| - \| y \|) \leq \| x - y \|$

$\| x \| - \| y \| \geq - \| x - y \|$ ………… (2)

From (1) and (2) we have

$-\| x - y \| \leq \| x \| - \| y \| \leq \| x - y \|$

$| \| x \| - \| y \| | \leq \| x - y \|.$

Remark : Now remember some important inequality

1- (Holder's in equality) If $p , q \in \mathbb{R}$ such that $\frac{1}{p} + \frac{1}{q} = 1$

$$\sum_{i=1} |x_i \, y_i| \leq \left[\sum |x_i|^p\right]^{\frac{1}{p}} \left[\sum_{i=1} |y_i|^q\right]^{\frac{1}{q}}$$

And special if it is $p = 2$ then $q = 2$

$$\sum_{i=1} |x_i \, y_i| \leq \left[\sum |x_i|^2\right]^{\frac{1}{2}} \left[\sum_{i=1} |y_i|^2\right]^{\frac{1}{2}}$$

Called (Cauchy-Schwar's inequality)

2- (Min kowsk's inequality) : If $p \geq 1$

$$\left(\sum_{i=1} |x_i + y_i|^p\right)^{\frac{1}{p}} \leq \left(\sum_{i=1} |x_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1} |y_i|^p\right)^{\frac{1}{p}}$$

Some important examples on normed space

Example 1.1.4 [4] : The vector space R is be normed space if

$\| x \| = |x| \; \forall \; x \in X.$

Solution:

1- Since $|x| \geq 0 \Rightarrow \| x \| \geq 0$

2- $\| x \| = 0 \Leftrightarrow |x| = 0$

$\Leftrightarrow x = 0$

3- let $x \in X$ and $\lambda \in F$

$\| \lambda x \| = |\lambda x| = |\lambda||x| = |\lambda| \| x \|$

4- let $x, y \in X$

$\| x + y \| = |x + y| \leq |x| + |y| = \| x \| + \| y \|$.

Example 1.1.4 [4]: Let the function $\| . \|: \mathbb{R}^n \longrightarrow \mathbb{R}$

$$\| x \| = (\sum_{i=1}^{n} x_i^2)^{\frac{1}{2}}.$$

For all $x = ((x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ then $\|.\|$ is norm on $\mathbb{R}^n$

Solution :

1- since $x_i^2 \geq 0 \ \forall \ i = 1, \dots, n \Longrightarrow \| x \| \geq 0$

2- $\| x \| = 0 \Leftrightarrow (\sum_{i=1}^{n} x_i^2)^{\frac{1}{2}} = 0$

$$\Leftrightarrow \sum_{i=1}^{n} x_i^2 = 0$$

$$\Leftrightarrow x_i^2 = 0 \ \forall \ i = 1,2, \dots, n$$

$$\Leftrightarrow x_i = 0 \ \forall \ i = 1,2, \dots, n$$

$$x = 0$$

3- let $x \in \mathbb{R}^n, \lambda \in \mathbb{R}$

$\lambda x = \lambda(x_1, \dots, x_n) = (\lambda x_1, \dots, \lambda x_n)$

$$\|\lambda x\| = (\sum_{i=1}^{n} (\lambda x_i)^2)^{\frac{1}{2}} = |\lambda| (\sum_{i=1}^{n} x_i^2)^{\frac{1}{2}} = |\lambda| \|x\|$$

4- let $x, y \in \mathbb{R}^n$

$x + y = (x_1, \dots, x_n) + (y_1, \dots, y_n) = (x_1 + y_1, \dots, x_n + y_n)$

$\| x + y \| = (\sum_{i=1}^{n} (x_i + y_i)^2)^{\frac{1}{2}}$ (by Min Kowk's inequality)

$$\| \ x \ + \ y \ \| \leq (\sum_{i=1}^{n} x_i^2)^{\frac{1}{2}} + (\sum_{i=1}^{n} y_i^2)^{\frac{1}{2}} = \|x\| + \|y\|.$$

Example1.1.5 [4]: Let $\|.\| : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a function defined

$\| \ x \ \| = max \ \{|x_1|, |x_2|, \dots, |x_n|\}$ for all $x = (x_1, \dots, x_n)$ then $\| \, . \|$ is a norm on $\mathbb{R}^n$.

1- since $|x_i| \geq 0, for \ all \ \ i = 1, \dots, n \Longrightarrow \|x\| \geq 0$

2- $\| \ x \ \| = 0 \Leftrightarrow max \, \{|x_1|, \dots, |x_n|\} = 0$

$\Leftrightarrow |x_i| = 0, for \ all \, i = 1,2, \dots, n$

$\Leftrightarrow x_i = 0, for \ all \ \ i = 1,2, \dots, n$

$\Leftrightarrow x = 0$

3- let $\lambda \in \mathbb{R}, x \in \mathbb{R}^n$

$\lambda x = \lambda(x_1, \dots, x_n)$

$= (\lambda x_1, \dots, \lambda x_n)$

$\| \ \lambda x \ \| = max \, \{|\lambda x_1|, \dots, |\lambda x_n|\}$

$= max \, \{|\lambda||x_1|, \dots, |x_n|\}$

$= |\lambda| \ max \, \{|x_1|, \dots, |x_n|\} = |\lambda||x|.$

4- let $x, y \in \mathbb{R}^n$

$x \ + \ y \ = (x_1, \dots, x_n) + (y_1, \dots, y_n) = (x_1 + y_1, \dots, x_n + y_n)$

$\| \ x + y \ \| = max \ \{|x_1 + y_1|, \dots, |x_n + y_n|\}$

$\leq max \ \ \{|x_1| + |y_1|, , , , |x_n| + |y_n|\}$

$\leq max \ \ \{|x_1|, \dots, |x_n|\} + max \ \{|y_1|, \dots, |y_n|\}$

$= \| \ x \ \| + \| \ y \ \|.$

Definition 1.1.6 [4]: Let X be a linear space A quasi-norm is a real valued function on X satisfying the following :

1- $\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$

2- $\|\lambda x\| = |\lambda|\|x\|$ for all $\lambda \in \mathbb{R}$ and all $x \in X$.

3- There is constant $k \geq 1$ such that $\|x + y\| \leq k(\|x\| + \|y\|)$ for all $x , y \in X$. The pair $(X ,\|.\|)$ is called a quasi-normed space if $\|.\|$ is a quasi-norm on X.

$*$ A quasi-Banach space is a complete quasi-normed space .

A quasi-norm $\|.\|$ is called a p-norm $(0 < p \leq 1$ if

$$\|x + y\|^p \leq \|x\|^p + \|y\|^p$$

For all $x, y \in X$. In this case , a quasi-Banach space is called a p-Banach space.

Section Two:

1.2. Normed Algebra

In this section, we recal and definitions related to linear algebra, normed algebra and topological algebra.

Definition 1.2.1 [4]: (linear algebra) : we shall say that X is a linear algebra if X is a linear space with an operation of multiplication $(\cdot)$ (which in general is not commutative) satisfying the following conditions:-

1− $(x\,.\,y)\,.z = x\,.(\,y\,.z\,)$ $\quad for\ all\ x\,,y\,,z\ \in R$

2- $x\,.(y + z\,) = x\,.y + x\,.z$ $\quad for\ all\ x\,,y\,,z\ \in R$

3- $\lambda\,(x\,.\,y) = (\lambda \circ x)\,.\,y = x\,.(\lambda \circ y)$ $\quad for\ all\ x\,,y\,,z\ \in R, \lambda \in F.$

Then $(X,+,\circ,\cdot)$ is algebra space.

Definition 1.2.2 [4]: In element $x\,,y$ in the algebra X are said to commute if $xy = yx$ an algebra is said to be commutative if any two of it's elements commute.

Definition 1.2.3 [4]: let $(X,+,\circ,\cdot)$ is linear algebra and let $\emptyset \neq M \subseteq X$ then $(M,+,\circ,\cdot)$ is called sub algebra if $(M,+,\circ,\cdot)$ is itself algebra.

Definition 1.2.4 [4]: Let M be a commutative sub algebra of the algebra $(X,+,\circ,\cdot)$ and $M \neq X$ then $(M,+,\circ,\cdot)$ is called maximal if is not contained in any element a commutative sub algebra.

Theorem 1.2.5 [4]: Every commutative sub algebra is contained in a maximal commutative sub algebra.

Theorem 1.2.6 [4]: Every element X is contains in a maximal commutative sub algebra.

Definition 1.2.7 [4]: (linear algebra)

Let X be a linear space. Then X is said to be algebra if there exist operation on X $(\cdot : X \times X \to X)$ its called The multiplication operation if the following axioms are realized;

1- $z \cdot (\alpha x + \beta y) = \alpha(z \cdot x) + \beta(y \cdot z), \quad for\ all x, y, z \in X$

$$for\ all\ \alpha, \beta \in R$$

2- $(\alpha x + \beta y) \cdot z = \alpha(x \cdot z) + \beta(y \cdot z), \ for\ all\ x, y, z \in X$

$$for\ all\ \alpha, \beta \in R$$

$$z \cdot (\alpha x) = \alpha\, (z \cdot x) = (\alpha z) \cdot x \quad \Leftrightarrow \beta = 0$$

$$\alpha = 1, \qquad \beta = 1$$

$$(x + y) \cdot z = x \cdot z + y \cdot z$$

$$z \cdot (x + y) = z \cdot x + z \cdot y$$

It is said about algebra X as

1- Associative algebra if

$x \cdot (y \cdot z) = (x \cdot y) \cdot z \qquad\qquad for\ all\ x, y, z \in X$

2- Algebra with identity if there exist $I \in X$

Such that $I \cdot x = x \cdot I$ $, x \in X$, then I is called identity element

3- Commutative algebra if $x \cdot y = y \cdot x$, $x, y \in X$.

Definition 1.2.8 [4]: (Algebra with identity)

An algebra X is called an algebra with identity if X contains an element which satisfies the condition $ex = xe = x\ for\ all\ x \in X$.

The element e itself which condition (I) is called an identity of the algebra X.

Theorem 1.2.9 [4]: Every algebra X with out identity can be considered as a sub algebra of an algebra with X identity.

Definition 1.2.10 [4]: An element $y \in X$ is called a left quasi – inverse of the element $e + x$ in X $e + y$ is a left inverse of the element $e + x$ in X that mean if $(e + y)\ (e + x) = e$.

Definition 1.2.11 [4]: The center of algebra X is the set of those element $a \in X$ which commutative with all the element of X, The center a commutative sub algebra of the algebra X.

Definition 1.2.12 [4]: A set I_1 of element of the algebra X is called a left ideal X if

1- $I_1 \neq X$

2- I_c is a sub space of the linear space X

3- If $x \in I$,$a \in X$ then $a\,x \in I$

Theorem 1.2.13 [4]: An element x of the algebra with identity has a left (right) invers if and only if it is not contained in any left (right) ideal

Theorem 1.2.14 [4]: Every left (right) ideal of the algebra with identity is contained in a maximal left (right) ideal.

Theorem 1.2.15 [4]: An element x of an algebra with identity has a left (right) inverse if and only if it is not contained in any maximal left (right) ideal.

Theorem 1.2.16 [4]: Every two-sided ideal of an algebra with identity is contained in a maximal left (right) ideal.

Theorem 1.2.17 [4]: Every regular (right, left , two-said) ideal can be extended to a maximal (right, left, respectively, two-said) ideal (which is obviously regular also).

Theorem 1.2.18 [4]: An element X in the algebra X has a left quasi-inverse if and only if for arbitrary maximal regular left ideal M, there exist element such that $x + y + yx \in M$.

Theorem 1.2.19 [4]: An element X in the algebra X dose not have a left a quasi-inverse if and only if $I_1 = \{z + z_x\}$, $z \in X$.

Definition 1.2.20 [4]: An element x_o in the algebra X with identity is said to be generalized nilpotent if $(\mathrm{e} + yx_o)^{-1}$ exist for an arbitrary element $y \in X$. the set of all generalized nilpotent element in the algebra X is called it's (Jacobson /radical).

Theorem 1.2.21 [4]: The radical of an algebra with identity coincides with intersection of all it's maximal left ideal.

Theorem 1.2.22 [4]: The intersection of all maximal left ideal coincides with the intersection of all maximal right ideal and is the radical of the algebra.

Definition 1. 2.23 [4]: An algebra is said to be semi simple if it radical consist of only the zero element.

Definition 1.2.24 [4]: An element x_o is said to be generalized nilpotent $x\, x_o + zx_o$ has a left quasi-inverse for arbitrary $z \in X$ and arbitrary numbers X in this definition X is no large necessarily an algebra with identity.

Definition 1.2.25 [4]: A mapping $X \to X'$ of the algebra X into an arbitrary algebra X' if $x \to x'$, $y \to y'$ imply that $\lambda x \to \lambda x'$,

$x+y \to x'+y'$, $xy \to x'y'$ if is the image of algebra X , then the homomorphism is called a homomorphism of X on to X'.

Definition 1.2.26 [4]: Two algebra X and X' are said to be isomorphic if there exist isomorphism of X onto X'.

Theorem 1.2.27 [4]: Under a homomorphism of the algebra X into the algebra X' , the inverse image I of the zero of X is a two sides ideal in X.

Definition 1.2.28 [4]: (topological algebra)

X is called a topological algebra if :

1- X is an algebra.

2- X is a locally convex topological linear space.

3- The product xy is a continuous function of each of the factors x, y provided other factor is fixed .

Definition 1.2.29 [4]: A mapping $x \to x'$ of the topological algebra X into the topological algebra X' is called a continuous homomorphism if :

1- $x \to x'$ is a homomorphism of the algebra X in to the algebra X'.

2- $x \to x'$ is continuous in mapping of the topological space X into the topological space X'.

Definition 1.2.30[4]: A sub set $A \subseteq X$ is said to be a closed sub algebra of the algebra X if :

1- A is a sub algebra of the algebra X.

2- A is a closed sub space of the topological space X.

Theorem 1.2.31 [4]: If A is a sub algebra of the algebra X then it's closer $\frac{\bar{A}}{\bar{R}_1}$ is closed sub algebra of X.

Theorem 1.2.32 [4]: The algebra $R_1(s)$ is the closer of the algebra $R_a(s)$: $R_a(s) = \overline{R_a(s)}$.

Theorem 1.2.33 [4]: The closer of a commutative sub algebra of the topological algebra is commutative.

Theorem 1.2.34 [4]: A maximal commutative sub algebra of a topological algebra is closed.

Theorem 1.2.35 [4]: The center z of a topological algebra X is closed commutative sub algebra in X.

Theorem 1.2.36 [4]: The closer of a (left , right , two-said) ideal in a topological algebra which dose not coincide with the entire algebra is also (left, right, two-said) ideal in this algebra.

Definition 1.2.37 [4]: (normed algebra) :

X is called normal algebra if

1- X is algebra.

2- X is normed space.

3- For any two element $x, y \in X$

$$|xy| = |x||y|.$$

4- if X is contains an identity e , then $|e| = 1$

The norm in a normed X defines a topology in X in natural manner recall that in this topology, the open balls $|x - x_o| < r$ with center at x_o from a neighborhood basis of the element $x_o \in X$.

Definition 1.2.38 [2]: Let $(X, \|.\|)$ be a quasi-normed space.

The quasi-normed space $(X, \|.\|)$ is called a quasi-normed algebra if X is an algebra and there is a constant $c > 0$ such that $\|xy\| \leq C\|x\|.\|y\|$ for all $x, y \in X$.

A quasi-Banach algebra is a complete quasi-normed algebra.

$*$ If the quasi-norm $\|.\|$ is a p-norm then the quasi-Banach algebra is called a p-Banach algebra.

Definition 1.2.39 [2]: Let $(A,|.|)$ be a Banach algebra and X a module over A quasi-norm is a real-valued function on X satisfying the following:

1- $\|x\| \geq 0, for\ all\ x \in X$ and $\|x\| = 0$ if and only if $x = 0$.

2- $\|\lambda x\| = |\lambda|\|x\|, for\ all\ \lambda \in A$ and all $x \in X$.

3- There exist constant $k \geq 1$ s.t $\|x + y\| \leq k\ \|x\| + \|y\|, for\ allx, y \in X$.

The pair $(X, \|.\|)$ is called a quasi-normed module over A if $\|.\|$ is a quasi-norm on X.

A quasi-banach module over A is a complete quasi-normed module over A.

Definition 1.2.40 [4]: Let $(X, \|.\|_x)$ and $(y, \|.\|_y)$ be a quasi-normed algebras.

1- A mapping $L: X \to Y$ is said to be isometric or an isometric if for all $x, y \in X$.

$\|Lx - Ly\|_y = \|x - y\|_x$.

2- The algebra X is said to be isometric with algebra Y if there exist a bijective isometry of X on to Y . The algebra X and Y are called isometric algebras.

Definition 1.2.41 [2]: (Banach space)

Said that the normative space X is a complete space if it is all sequential of couchy is converge in X.

Full normative space is called Banach space .

Example 1.1.42 [2]: The space $F^n = \{x:(x_1,\ x_2, \dots x_n), x_i \in F,$ for all $i = 1,2, \dots n\}$ with norm

$$\|x\| = \left(\sum_{i=1}^{n} |x_i|^2\right)^{\frac{1}{2}} \forall\, x = (x_1, x_2, \dots, x_n) \in F^n$$

It is a Banach space .

Solution :

It is clear F^n is a normed space.

Let $\{x_m\}$ is elementary sequence in a F^n

$x_m \in F^n \Longrightarrow x_m = \{x_{im}, \dots, x_{nm}\}$.

Let $\in > 0\ \exists\ k \in Z^+$ such that

$|x_m - x_L| < \in \forall\, m, L > k$

1- $\|x_m - x_L\|^2 < \epsilon^2 \forall\, m, L > k \Longrightarrow x_m - x_L = (x_{1m} - x_{1L}, \dots, x_{nm} - x_{nL})$

2-

$$\|x_m - x_L\|^2 = \sum_{i=1}^{n} |x_{im} - x_{iL}|$$

From (1) , (2) we get

$$\sum_{i=1}^{n} |x_{im} - x_{iL}|^2 < \epsilon^2 \forall\, m, L \geq k$$

$|x_{im} - x_{iL}| < \epsilon^2 \forall m, L \geq k \Longrightarrow |x_{im} - x_{iL}| < \in \forall\, m, L \geq k$

For you , the van $\{x_m\}$ is couchy sequence. In a F forally

Since F is complete space $\Longrightarrow$ To each i there exist $x_i \in F$ such that $x_{im} \rightarrow x_1 \Longrightarrow x = (x_1, \dots, x_n) \Longrightarrow x \in F^n$ To proof $x_m \longrightarrow x$

Let $\in > 0 \ \forall\, m > k$ then

$$\|x_m - x\|^2 = \sum_{i=1}^{n} |x_{im} - x_i|^2 < \epsilon^2$$

$$\|x_m - x\| < \in \forall\, m > L$$

$\therefore \{x_m\}$ is converg $\Longrightarrow F^n$ complete space

$\because F^n$ normed space $\Longrightarrow F^n$ Banach space.

Section Three:

1.3. Symmetric Algebra

In this section, we introduce some definitions and concepts related of symmetric algebra.

Definition 1.3.1. [4]: R is called a symmetric algebra if:

1) R is an algebra

2) an operation is defined in R which assigns to each element x in R the element x^* in R in such a way that the following conditions are satisfies:-

a) $(\lambda x + \lambda Y)^* = \lambda x^* + \mu y^*$

b) $x^{**} = x$

c) $(xy)^* = x^* y^*$

An element x is said to be Hermitian if $x^* = x$.

Theorem 1.3.2 [4]: Every element x of a symmetric algebra can be uniquely represented in the form $x = x_1 + ix_2$, where x_1, x_2 are Hermitian elements.

In fact, if such a representation holds, then $x^* = x_1 - ix_2$consequently

$$x_1 = \frac{x + x^*}{2}, \quad x_2 = \frac{x - x^*}{2i}$$

Thus, this representation is unique. Conversely, the elements x_1, x_2 defined by equalities (1) are Hermitian and $x = x_1 + ix_2$.

These elements x_1, x_2 will be called the Hermitian components of the element X an element x is called normal if $x^* x = x x^*$.

Theorem 1.3.3. [4]: Every element of the form x^*x is Hermitian.

In fact, in virtue of c and b). $(x^*x)^* = x^*x^{**} = x^*x$.

Theorem 1.3.4. [4]: The identity e is a Hermitian element. In fact $e^* = e^*e$ is a Hermitian element. Consequently, $e^* = e$.

If R is asymmetric algebra without identity and R' is the algebra obtained from R by adjunction of the identity, then setting $(\lambda e + x)^* = \lambda e + x^*$ for $x \in R$.

Theorem 1.3.5. [4]: If x^{-1} exists, then $(x^*)^{-1}$ also exists and $(x^*)^{-1} = (x^{-1})^*$

Theorem 1.3.6.[4]: If R is a maximal commutative symmetric sub algebra containing a normal element x and if x^{-1} exists, then $x^{-1} \in R$. In fact since x and x^* commute with all elements in R, x^{-1} and $x^* = (x^{-1})^*$

Definition 1.3.7. [4]: The mapping $x \longrightarrow x'$ of a symmetric algebra R into the symmetric algebra R' is called a symmetric homomorphism if

β) $x \longrightarrow x'$ is a homomorphism

α) $x \longrightarrow x'$ implies that $x^* \longrightarrow x'^*$.

Theorem 1.3.8. [4]: The radical of a symmetric two- sided ideal.

Example 1.3.9. [4]:

1) The algebra $C(T)$ is a symmetric algebra if we set $x^* = \overline{X(t)}$ for $X = X(t)$ (where the vinculum denotes conjugate complex number).

2) Suppose R is a Hilbert space. the algebra $R(R)$ that mean $R(x)$ with $X = R$ is a symmetric algebra if involution is under stood to be passage over to the adjoint operator.

3) The algebra W is asymmetric algebra if we set

$x^* = \sum_{n=-\infty}^{\infty} \bar{C}_{-n} e^{int}$ for $x = \sum_{n=-\infty}^{\infty} C_n e^{int}$

Definition 1.3.10. [4]: (Positive functional)

A linear functional f in the symmetric algebra R is said to be real- valued if f assumes real value on all Hemitian elements of the algebra R.

Theorem 1.3.11. [4]: Every linear functional in a symmetric algebra can be represented in the form $f = f_1 + if_2$ where f_1, f_2 are real valued functional. Namely it suffices to set

$$f_1(x) = [f(x) + \overline{f(x^*)}],\ f_2(x) = \frac{1}{2i}[f(x) - \overline{f(x^*)}].$$

Then f_1, f_2 are real valued functional and $f(x) = f_1(x) + if_2(x)$ these functional f_1, f_2 are called the real components.

Theorem 1.3.12. [4]: If f is a real- valued functional then $f(x^*) = \overline{f(x)}$ for an arbitrary $x \in R$. In fact setting $x = x_1 + ix_2$ where x_1, x_2 are Hermitian we have $f(x^*) = f(x_1 - ix_2) = \overline{f(x_1) + if(x_2)} = \overline{f(x)}$.

Inasmuch as $f(x_1), f(x_2)$ are real- valued by assumption. A linear functional f is said to be positive if $f(x^*x) > 0$ for an arbitrary element x of the algebra R.

Theorem 1.3.13. [4]: For every positive functional f in the symmetric algebra R.

1) $f(y^*x) = f\overline{(x^*y)}$

2) $|f(y^*x)|^2 \leq f(y^*y)f(x^*x)$

3) $f((\lambda x + uy) \cdot (\lambda x + uy)) > 0$

4) $|\lambda|^2 f(x^*x) + \lambda\bar{u}f(y^*x) + \lambda u f(x^*y) + |u|^2 f(y^*y) \geq 0.$

Theorem 1.3.14. [4]: Every positive functional t in a symmetric algebra R with identity is real and $|f(x)^2| \leq f(e)f(x^*x)$.

Theorem 1.3.15. [4]: Suppose R is a symmetric algebra without identity and that R' is the symmetric algebra obtained from R by adjunction of the identity. A positive functional f in R can be extended to a positive functional in R' if and only if f is real and satisfies inequality

$|f(x)^2| \leq cf(x^*x)$ for all $x \in R$ where c is some constant.

Theorem 1.3.16 [4]: If in a symmetric normed algebra R

a) $|x^*| = |x|$

b) There exists a set $\{e_x\}$ approximating the identity, then every continuous positive functional in R can be extended to a positive functional in R'.

Definition 1.3.17. [4]: R is called a normed symmetric algebra if

a) R is a normed algebra

b) R is a symmetric algebra

c) $|x^*|$.

Definition 1.3.18. [4]: A binary operation $*: [0,1] \times [0,1] \longrightarrow [0,1]$ is said to be a *continuous t-norm* if $([0,1],*)$ is a topological monoid with unit 1 such that $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ $(a, b, c, d \in [0,1])$.

It is clear that if we define $a * b = ab$ or $a * b = min(a, b)$ then $*$ is a continuous t-norm.

Section One:

2.1. Definition

In this section we study the definition fuzzy normed spaces and some properties with proofs. We start this section by the following definition.

Definition 2.1.1 [1]: The 3-tuple $(X, M, *)$ is said to be a fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set on

$X^2 \times (0, \infty)$ satisfying the following conditions for all $x, y, z \in X$ and $t, s > 0$,

(i) $M(x,y,0) > 0$,

(ii) $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$,

(iii) $M(x, y, t) = M(y, x, t)$,

(iv) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$ for all $t, s > 0$, (v) $M(x, y, .) : (0, \infty) \longrightarrow [0,1]$ is continuous.

Definition 2.1.2 [1]: A sequence $\{x_n\}_n$ in a fuzzy metric space $(X, M, *)$ is a *Cauchy sequence* if and only if for each $0 < \in < 1$ and $t > 0$ there exist $n_0 \in N$ such that for all $n, m \geq n_0$ we have,

$$M(x_n, x_m, t) > 1 - e .$$

A fuzzy metric space is said be *complete* if and only if every Cauchy sequence is convergent.

At last we state the following lemma which will be used later.

Definition 2.1.3 [1]: The 3-tuple $(X, N,*)$ is said to be a *fuzzy normed space* if X is a vector space, $*$ is a continuous t-norm and N is a fuzzy set on $X \times (0,\infty)$ satisfying the following conditions for every $x, y \in X$ and $t, s > 0$:

(i) $N(x,t) > 0$,

(ii) $N(x,t) = 1 \; iff \; x = 0$,

(iii) $N(\alpha x,t) = N(x,t/|\alpha|), for \; all \; \alpha \; 6 = 0$,

(iv) $N(x,t) * N(y,s) \leq N(x + y,t + s)$,

(v) $N(x,.) : (0,\infty) \longrightarrow [0,1]$ is continuous,

(vi) $limt \rightarrow \infty \, N(x,t) = 1$.

Lemma 2.1.4 [1]: Let N be a fuzzy norm. Then

(i) $N(x,t)$ is non-decreasing with respect to t for each $x \in X$.

(ii) $N(x - y,t) = N(y - x,t)$. We need the proof of above lemma.

Proof:

Let $t < s$. Then $k = s - t > 0$ and we have

$N(x,t) = N(x,t) * 1$

$= N(x,t) * N(0,k)$

$\leq N(x,s)$.

This proves the (i). To prove (ii) we have

$N(x - y,t) = N((-1)(y - x),t)$

$= N\left(y - x, \frac{t}{-1}\right)$

$= N(y - x,t)$.

Lemma 2.1.5 [1]: Let $(X, N,*)$ be a fuzzy normed space. If we define

$M(x, y, t) = N(x - y, t)$,

then M is a fuzzy metric on X, which is called the fuzzy metric induced by the fuzzy norm N.

Lemma 2.1.6 [1]: A fuzzy metric M which is induced by a fuzzy norm on a fuzzy normed space $(X, N,*)$ has the following properties for all $x, y, z \in X$ and every scalar $\alpha \neq 0$:

(i) $M(x + z, y + z, t) = M(x, y, t)$.

(ii) $M(\alpha x, \alpha y, t) = M\left(x, y, \frac{t}{|\alpha|}\right)$

Proof:

$$M(x + z, y + z, t) = N((x + z) - (y + z), t)$$
$$= N(x - y, t) = M(x, y, t)$$

Also,

$$M(\alpha x, \alpha y, t) = N(\alpha x - \alpha y, t)$$

$=N\left(x - y, \frac{t}{|\alpha|}\right)$

$=M\left(x, y, \frac{t}{|\alpha|}\right)$.

Example 2.2.7 [1]: Let $(x, \|.\|)$ be a normed space. We define $a * b = ab$ or $a * b = min(a, b)$ and

$$N(x, t) = \frac{ki^n}{ki^n + m\|x\|} \qquad k, m, n \in R .^+$$

Then $(X, N,*)$ is a fuzzy normed space. In particular if $k = n = m = 1$ we have

$$N(x,t) = \frac{t}{t + \|x\|}$$

which is called the standard fuzzy norm induced by norm $\|.\|$.

Example 2.2.8 [1]: Let $(x, \|.\|)$ be a normed space. We define $a * b = ab$ and

$$N(x,t) = (\exp\frac{\|x\|}{t})^{-1}$$

for $x \in X$ and $t \in (0,\infty)$. Then $(X, N,*)$ is a fuzzy normed space.

Definition 2.2.9 [1]: Let $(X, N,*)$ be a fuzzy normed space. We define the open ball $B(x,r,t)$ and the closed *ball* $B[x,r,t]$ with center $x \in X$ and radius $0 < r < 1, t > 0$ as follows:

$$B(x,r,t) = \{y \in X : N(x - y,t) > 1 - r\}$$

$$B[x,r,t] = \{y \in X : N(x - y,t) \geq 1 - r\}.$$

Lemma 2.2.10 [1]: If $(X, N,*)$ is a fuzzy normed space, then

(a) The function $(x,y) \rightarrow x + y$ is continuous,

(b) The function $(\alpha, x) \rightarrow \alpha x$ is continuous.

Proof:

If $x_n \rightarrow x$ and $y_n \rightarrow y$, then as $n \rightarrow \infty$,

$$N((x_n + y_n) - (x + y), t) \geq N\left(x_n - x, \frac{1}{2}\right) * N\left(y_n - y, \frac{1}{2}\right) \rightarrow 1$$

This proves (a).

Now if $x_n \rightarrow x$, $\alpha_n \rightarrow \alpha$ and $\alpha_n \neq 0$ then

$$N(\alpha_n x_n - \alpha x, t) = N(\alpha_n(x_n - x) + x(\alpha_n - \alpha), t)$$

$$\geq N\left(\alpha_n(x_n - x), \frac{t}{2}\right) * N\left(x(\alpha_n - \alpha), \frac{t}{2}\right)$$

$$= N\left(x_n - x, \frac{t}{2\alpha_n}\right) * N\left(x, \frac{t}{2(\alpha_n - \alpha)}\right) \to 1 \,,$$

as $n \to \infty$, and this proves (b).

Definition 2.1.11 [1]: The fuzzy normed space $(X, N, *)$ is said to be a *fuzzy Banach space* whenever X is complete with respect to the fuzzy metric induced by fuzzy norm.

Section Two:

2.2 Quotient spaces

In this section at first we define Quotient spaces and give several examples of these spaces, and then we define Quotient spaces.

Definition 2.2.1[1]: Let $(X, N,*)$ be a fuzzy normed space, M be a linear manifold in X and let $Q : X \to X/M$ be the natural map, $Qx = x + M$. We define

$$N(x + M, t) = \sup\{N(x + y, t) : y \in M\}, \ t > 0.$$

Theorem 2.2.2 [1]: If M is a closed subspace of fuzzy normed space X and $N(x + M, t)$ is defined as above then

(a) N is a fuzzy norm on X/M.

(b) $N(Qx, t) \geq N(x, t)$.

(c) If $(X, N,*)$ is a fuzzy Banach space, then so is $(X/M, N,*)$.

Proof:

It is clear that $N(x + M, t) \geq 0$. Let $N(x + M, t) = 1$. By definition there is a sequence $\{x_n\}$ in M such that $N(x + x_n, t) \to 1$. So $x + x_n \to 0$ or equivalently $x_n \to (-x)$ and since M is closed so $x \in M$ and $x + M = M$, the zero element of X/M. On the other hand we have,

$$\begin{aligned} N((x + M) + (y + M), t) &= N((x + y) + M, t) \\ &\geq N((x + m) + (y + m), t) \\ &\geq N(x + m, t_1) * N(y + m, t_2) \end{aligned}$$

for $m,n \in M, x,y \in X$ and $t_1 + t_2 = t$. Now if we take *sup* on both sides, we have,

$$N((x + M) + (y + M),t) \geq N(x + M,t_1) * N(y + M,t_2).$$

Also we have,

$N(\alpha(x + M),t) = N(\alpha x + M,t)$

$= \sup \{N(\alpha x + \alpha y,t) : y \in M\}$

$= \sup\{N(x + y,t/|\alpha|) : y \in M\}$

$= N\left(x + M, {}^{t}/_{|\alpha|}\right)$

Therefore $(X,N,*)$ is a fuzzy normed space.

To prove (b) we have,

$N(Qx,t) = N(x + M,t)$

$= \sup\{N(x + y,t) : y \in M\}$

$\geq N(x,t).$

Let $\{x_n + M\}$ be a Cauchy sequence in X/M. Then there exists $\epsilon_n > 0$ such that $\epsilon_n \to 0$ and,

$$N\big((x_n + M) - (x_{n+1} + M),t\big) \geq 1 - e_n .$$

Let $y_1 = 0$. We choose $y_2 \in M$ such that,

$$N(x_1 - (x_2 - y_2),t) \geq N\big((x_1 - x_2) + M,t\big) * (1 - \epsilon_1).$$

But $N((x_1 - x_2) + M,t) \geq (1 - \epsilon_1)$. Therefore,

$$N(x_1 - (x_2 - y_2),t) \geq (1 - \epsilon_1)(1 - \epsilon_1).$$

Now suppose $y_n - 1$ has been chosen, $y_n \in M$ can be chosen such that

$$N\big((x_{n-1} + y_{n-1}) - (x_n + y_n),t\big)$$
$$\geq N\big((x_{n-1} - x_n) + M,t\big) * (1 - \epsilon_{n-1}) .$$

and therefore,

$$N\big((x_{n-1}+y_{n-1})-(x_n+y_n),t\big) \geq (1-\epsilon_{n-1}) * (1-\epsilon_{n-1}).$$

Thus, $\{x_n + y_n\}$ is a Cauchy sequence in X. Since X is complete, there is an x_0 in X such that $x_n + y_n \dashrightarrow x_0$ in X. On the other hand

$$x_n + M = Q(x_n + y_n) \dashrightarrow Q(x_0) = x_0 + M.$$

Therefore every Cauchy sequence $\{x_n + M\}$ is convergent in X/M and so X/M is complete and $(X/M, N,*)$ is a fuzzy Banach space.

Theorem 2.2.3 [1]: Let M be a closed subspace of a fuzzy normed space X. If $x \in X$ and $e \in \big[0, N(x+M,t)\big)$, then there is an x^0 in X such that, $x^0 + M = x + M \; and \; N(x^0,t) > N(x+M,t) * \epsilon$.

Proof:

By (1.4) there always exists $y \in M$ such that,

$$N(x+y,t) > N(x+M,t) * \epsilon .$$

Now it is enough to put $x^0 = x + y$.

Theorem 2.2.4 [1]: Let M be a closed subspace of a fuzzy normed space (X, N, t). If a couple of the spaces $X, M, X/M$ are complete, so is the third one.

Proof:

If X is a fuzzy Banach space, so are X/M and M. Therefore all that needs to be checked is that X is complete whenever both M and X/M are complete. Suppose M and X/M are fuzzy Banach spaces and let $\{x_n\}$ be a

Cauchy sequence in X. Since

$$N((x_n - x_m) + M, t) \geq N(x_n - x_m, t)$$

whenever $m, n \in N$, the sequence $\{x_n + M\}$ is Cauchy in X/M and so converges to $y + M$ for some $y \in M$. So there exists a sequence ϵ_n such that $\epsilon_n \to 0$ and

$N\big((x_n - y) + M, t\big) > 1 - \epsilon_n$ for each $t > 0$.

Now by last theorem there exists a sequence $\{y_n\}$ in X such that $y_n + M = (x_n - y) + M$ and

$$N(y_n, t) > \big((x_n - y) + M, t\big) * (1 - \epsilon_n).$$

So $\lim_n N(y_n, t) \geq 1$ and $\lim_n y_n = 0$. Therefore $\{x_n - y_n - y\}$ is a Cauchy sequence in M and thus is convergent to a point $z \in M$ and this implies that $\{x_n\}$ converges to $z + y$ and X is complete.

Theorem 2.2.5 [1]: (Open mapping theorem) If T is a continuous linear operator from the fuzzy Banach space $(X, N_1, *)$ onto the fuzzy Banach space $(X, N_2, *)$, then T is an open mapping.

Section Three:

2.3. Fuzzy normed algebras

In this section at first we define Fuzzy normed algebras and give several examples of these algebras, and then we define Fuzzy normed algebras.

Definition 2.3.1 [3]: It is called fuzzy nonned algebra-the quadruplet $(X, N, *, o)$ if we have

(A 1) $*, o$ are continuous t-norms;

(A2) X is an algebra;

(A3) $(X, N, *)$ is a fuzzy normed linear space;

(A4) $N(xy, ts) \geq N(x,t)\ o\ N(y,s)$ for all $x, y \in X$, for all $t, s \geq 0$.

If $(X, N, *)$ is a fuzzy Banach space, then $(X, N, *, o)$ will be called fuzzy Banach algebra.

Example 2.3.2. [3]: Let $(X, \|\cdot\|)$ be a normed algebra, $*, o$ be continuous t-norms and

$N : X \times [0,\infty) \to [0,1]$ defined by $N(x,t) = \begin{cases} 0 & , t \leq \|x\| \\ 1 & , t > \|x\| \end{cases}$.

Then $(X, N, *, o)$ is a fuzzy-normed algebra.

Proof.

It is easy to check (N1)-(N3) and (N5). We verify the condition (N4). Let $x, y \in X$, $t, s \in [0,\infty)$. If $\|x+y\| \geq t+s$, then $t \leq \|x\|$ or $s \leq \|y\|$ (contrarily $t > \|x\|$ and $s > \|y\|$, thus $t+s > \|x\| + \|y\| \geq \|x+y\|$, contradiction). If $t > \|x\|$, then $N(x,t) = 0$. If $s \leq \|y\|$, then $N(y,s) = 0$.

Thus $N(x,t) * N(y,s) = 0$. Therefore the inequality $N(x + y, t + s) \geq N(x,t) * N(y,s)$ holds. If $\|x\| + \|y\| < t + s$, then $N(x + y, t + s) = 1$ and the inequality $N(x + y,\ t + s) \geq N(x,t) * N(y,s)$ holds.

It remains to verify (A4). Let $x, y \in X, t, s \in [0,\infty)$. If $\|xy\| \geq ts$, then $t \leq \|x\|$ or $s \leq \|y\|$ (contrarily $t > \|x\|$ and $s > \|y\|$, thus $ts > \|x\| \cdot \|y\| \geq \|xy\|$, contradiction). If $t \leq \|x\|$, then $N(x,t) = 0$ If $s \leq \|y\|$, then $N(y,s) = 0$. Thus $N(x,t)\, o\, N(y,s) = 0$. Therefore the inequality $N(xy, ts) \geq N(x,t)\, o\, N(y,s)$ holds. If $\|xy\| \geq ts$, then $N(xy, ts) = 1$ and the inequality $N(xy, ts) \geq N(x,t)\, o\, N(y,s)$ holds.

Example 2.3.3. [3]: Let $(X, \|\cdot\|)$ be a normed algebra and $N: X \times [0,\infty) \to [0,1]$ defined by

$$N(x,t) = \begin{cases} \dfrac{t}{t + \|x\|}, & t > 0 \\ 0\ , & t = 0 \end{cases}.$$

Then $(X, N, \wedge, \cdot)$ is a fuzzy Harmed algebra.

Proof.

By [4]' $(X, N, \wedge)$ is a fuzzy normed linear space. It remains to verify (A4), that is

$N(xy, ts) \geq N(x,t)\, .\, N(y,s)$, for all $x, y \in X$, for all $t, s \in [0,\infty)$.

For $t = 0$ or $s = 0$ the inequality is obvious. For $t \neq 0$ and $s \neq 0$, the inequality is equivalent to

$$\frac{\mathrm{ts}}{\mathrm{ts} + \|xy\|} \geq \frac{\mathrm{t}}{\mathrm{t} + \|x\|} \cdot \frac{\mathrm{s}}{\mathrm{s} + \|y\|},$$

namely $ts + t\|y\| + s\|x\| + \|x\| \cdot \|y\| \geq ts + \|xy\|$, which is evidently true.

<u>Example 2.3.4. [3]:</u> Let $(X, \|\cdot\|)$ be a normed algebra and $N: X \times [0,\infty) \rightarrow [0,1]$ defined by

$$N(x,t) = \begin{cases} \dfrac{t}{t + \|x\|}, & t > 0 \\ 0\,, & t = 0 \end{cases}.$$

Then $(X, N, \cdot, \cdot)$ is a fuzzy normed algebra.

<u>Proof.</u>

We will prove that $(X, N, \cdot)$ is a fuzzy normed linear space. According to [4], conditions (N1)-(N3) and (N5) are verified. It remains to prove (N4), that is,

$N(x + y, t + s) \geq N(x,t) \cdot N(y,s)$, for all $x, y \in X$, for all $t, s \geq 0$.

Indeed, for $t = 0$ or $s = 0$ the inequality is obviously true. For $t \neq 0$ and $s \neq 0$, the inequality is equivalent to

$$\frac{\mathrm{t + s}}{\mathrm{t + s} + \|x + y\|} \geq \frac{\mathrm{t}}{\mathrm{t} + \|x\|} \cdot \frac{\mathrm{s}}{\mathrm{s} + \|y\|},$$

namely $(t + s)(t + \|x\|)(s + \|y\|) \geq ts(t + s + \|x + y\|)$, which is equivalent to

$$ts(\|x\| + \|y\|) + s^2\|x\| + t^2\|y\| + (t + s)\|x\|\|y\| \geq ts\|x + y\|.$$

Because $ts(\|x\| + \|x\|) \geq ts\|x + y\|$ and all the other terms from the left member are positive, the inequality follows. Therefore $(X, N, \cdot)$ is a fuzzy normed linear space. Moreover, conditions (A1)-(A4) are satisfied similar to the proof from the previous example. It follows $(X, N, \cdot, \cdot)$ is a fuzzy normed algebra.

Theorem 2.3.5. [3]:

A fuzzy normed algebra $(X, N, *, \circ)$ is with continuous product if and only if

For all $\alpha \in (0,1), \exists \beta = \beta(a) \in (0,1), \exists M = M(\alpha) > 0$ such that

For all

$x, y \in X$, For all $s, t > 0 : N(x, s) > \beta, N(y, t) > \beta \Longrightarrow N(xy, Mst) > \alpha$

Proof.

($\Longrightarrow$) Let $\alpha \in (0,1)$ and $V := \{u \in X : N(u, 1) > \alpha\}$ be an open neighbourhood of zero. As $X \times X \ni (x, y) \to x \cdot y \in X$ is continuous at (0,0), there exist $\epsilon_1 = \epsilon_1(\alpha) > 0$, $\epsilon_2 = \epsilon_2(\alpha) > 0$, $\gamma_1 = \gamma_1(\alpha) \in (0,1)$, $\gamma_2 = \gamma_2(\alpha) \in (0,1)$ such that

For all $u_1, u_2 \in X : N(u_1 \epsilon_1) > \gamma_1 N(u_2 \epsilon_2) > \gamma_2$ we have that $N(u_1 u_2, 1) > \alpha$.

Let $\beta = max\{\gamma_1, \gamma_2\} \in (0,1)$, $M = \frac{1}{\epsilon_1 \epsilon_2} > 0$. Let $x, y \in X$, $s, t > 0$ such that $N(x, s) > \beta$, $N(y, t) > \beta$ Then $N(x/s, 1) > \beta \geq \gamma_1$ and $N(y/t, 1) > \beta \geq \gamma_2$. Let $u_1 = \frac{\epsilon_1 x}{s}, u_2 = \frac{\epsilon_2 y}{t}$ We note that

$$N(u_1, \epsilon_1) = N(u_1/\epsilon_1, 1) = N(x/s, 1) > \gamma_1,$$

$$N(u_2 \epsilon_2) = N(u_2/\epsilon_2, 1) = N(u/t, 1) > \gamma_2 \cdot$$

Hence $N(u_1 u_2, 1) > \alpha$, i.e., $N\left(\frac{\epsilon_1 x}{s} \cdot \frac{\epsilon_2 y}{t}, 1\right) > \alpha$, namely $N\left(xy, \frac{st}{\epsilon_1 \epsilon_2}\right) > \alpha$.Thus $N(xy, Mst) > \alpha$.

($\Longleftarrow$) First we will prove that for each $y_o \in X$, the mapping

$$X \in x \mapsto y_o \in X$$

is continuous.

Let $\epsilon > 0$, $\alpha \in (0,1)$. Thus there exist $\beta = \beta(\alpha) \in (0,1)$, $M = M(\alpha) > 0$ such that

$$N(x,s) > \beta, N(y,t) > \beta \Longrightarrow N(xy, Mst) > \alpha.$$

As $\lim_{t\to\infty} N(y_0,t) = 1$, there exists $t_0 > 0$ such that $N(y_0,t_0) > \beta$. Let $\delta = \delta(\alpha,\epsilon) = \frac{\epsilon}{t_0 M}$ and $\beta(\alpha,\epsilon) = \beta$. Let

$x \in X$ such that $N(x,\delta) > \beta$. As $N(y_0,t_0) > \beta$, we obtain that $N(xy_0, Mt_0\delta) > \alpha$, namely $N(xy_0,\epsilon) > \alpha$.

Similarly, we can establish that, for each $x_0 \in X$, the mapping

$$X \ni y \mapsto x_0 y \in X$$

is continuous.

Now, we will prove that $(X, N, *, \circ)$ is with continuous product. Let $x_n \longrightarrow x_0$, $y_n \longrightarrow y_0$. Thus $x_n y_0 \longrightarrow x_0 y_0$ and $x_0 y_n \longrightarrow x_0 y_0$. Hence $\lim_{n\to\infty} N(x_n y_0 - x_0 y_0, s) = 1$, $\lim_{n\to\infty} N(x_0 y_n - x_0 y_0, t) = 1$ for all $s, t > 0$. Therefore

$$N(x_n y_n - x_0 y_0, t)$$

$$= N((x_n - x_0)(y_n - y_0) + (x_n - x_0)y_0 + x_0(y_n - y_0), t)$$

$$\geq N\left((x_n - x_0)(y_n - y_0), \frac{t}{3}\right) * N\left((x_n - x_0)y_0, \frac{t}{3}\right) * N\left(x_0(y_n - y_0), \frac{t}{3}\right)$$

$$\geq \left(N\left(x_n - x_0, \sqrt{\frac{t}{3}}\right) \circ N\left(y_n - y_0, \sqrt{\frac{t}{3}}\right)\right) * N\left((x_n - x_0)y_0, \frac{t}{3}\right)$$

$$* N\left(x_0(y_n - y_0), \frac{t}{3}\right) \longrightarrow 1$$

Hence $x_n y_n \longrightarrow x_0 y_0$

Lemma 2.3.6. [3]: Any continuous t-norm $*$ satisfies:

For all $\gamma \in (0,1), \exists\, \alpha, \beta \in (0,1)$ such that $\alpha * \beta = \gamma$.

Proof.

Let $\gamma \in (0,1)$. Choose $\alpha > \gamma$ Let $g : [0,1] \longrightarrow [0,1]$ defined by $g(y) = \alpha * y$. As $*$ is continuous, we have that g is continuous. As $g(0) = \alpha * 0 = 0$ and $g(1) = \alpha * 1 = \alpha$, for $\gamma \in (0, \alpha)$ there exists $\beta \in (0,1)$ such that $g(\beta) = \gamma$, namely $\alpha * \beta = \gamma$.

Theorem 2.3.7. [3]: Any fuzzy normed algebra $(X, N, *, \circ)$ is with continuous product.

Proof.

Let $\alpha \in (0,1)$. Then there exists $\epsilon > 0$ such that $\alpha + \epsilon \in (0,1)$. As $\circ$ is a continuous t-norm, by the previous lemma, we obtain that there exist $\beta_\alpha, \gamma_\alpha \in (0,1)$ such that $\alpha + \epsilon = \beta_\alpha \circ \gamma_\alpha$. We suppose that $\beta_\alpha \geq \gamma_\alpha$ (the case $\beta_\alpha \leq \gamma_\alpha$ is similar). We choose $M = M(\alpha) = 1$. Let $x, y \in X,\ s, t > 0$ such that $N(x, s) > \beta_\alpha,\ N(y, t) > \beta_\alpha$ Then

$$N\,(xy, M\,st) \geq N\,(x, s) \circ N\,(y, t) \geq \beta_\alpha \circ \beta_\alpha \geq \beta_\alpha \circ \gamma_\alpha = \alpha + \epsilon\ > \alpha.$$

Definition 2.3.8. [3]: The fuzzy normed algebra $(X, N, *, \circ)$ is called with multiplicatively continuous product if

For all $\alpha \in (0,1)$, for all $x, y \in X$, for all $s, t > 0 :\ N(x, s) >\ \alpha, N(y, t) > \alpha \Longrightarrow N(xy, st) \geq \alpha$.

Example 2.3.9. [3]: (Fuzzy normed algebra with multiplicatively continuous product). The fuzzy normed algebra $(X, N, *, \circ)$ from Example (2.4.2) is with multiplicatively continuous product.

Proof.

Indeed, let $\alpha \in (0,1)$, $x, y \in X$, $s, t > 0$ such that $N(x,s) > \alpha$, $N(y,t) > \alpha$. Then $N(x,s)1$, $N(y,t) = 1$. Thus $\|x\| < s, \|y\| < t$. Therefore $\|xy\| \leq \|x\|.\|y\| < st$. Hence $N(xy, st) = 1 > \alpha$.

Example 2.3.10. [3]: (Fuzzy normed algebra which is not with multiplicatively continuous product). We consider the fuzzy normed algebra from Example 2.4.3, where $X = \mathbb{R}$. and the norm on X is the absolute value $|\cdot|\cdot$ Then $(\mathbb{R}, N, \wedge, \cdot)$ is not with multiplicatively continuous product.

Proof.

Indeed, for $\alpha = \frac{1}{5}$, $x = \frac{5}{2}s, y = \frac{5}{2}t, s, t > 0$ we have that $N(x,s) = \frac{s}{s+|x|} = \frac{s}{s+\frac{5}{2}s} = \frac{7}{2} > \frac{1}{5}$ and $N(y,t) > \frac{1}{5}$. But $N(xy, st) = \frac{st}{st+|xy|} = \frac{st}{st+\frac{25}{4}st} = \frac{49}{4} < \frac{1}{5}$. Thus $(\mathbb{R}, N, \wedge, \cdot)$ is not with multiplicatively

continuous product.

Proposition 2.3.11. [3]: Let $(X, N, *, \circ)$ be a fuzzy normed algebra such that $\alpha \circ \alpha \geq \alpha$ for all $\alpha \in (0,1)$. Then $(X, N, *, \circ)$ is with multiplicatively continuous product.

Proof.

Let $\alpha \in (0,1).$, $x, y \in X, s, t > 0$ such that $N(x,s) > \alpha$, $N(y,t) > \alpha$. Then

$$N(xy, st) \geq N(x,s) \circ N(y,t) \geq \alpha \circ \alpha \geq \alpha.$$

Remark 2.3.12. [3]: The condition $\alpha \circ \alpha \geq \alpha$. for all $\alpha \in (0,1)$ from the previous proposition is sufficient but not necessary.

Indeed, the algebra $(X, N, *, \cdot)$ from Example 2.4.2 is with multiplicatively continuous product, although $\circ = \cdot$ does not verify $\alpha \circ \alpha \geq \alpha$ for all $\alpha \in (0,1)$.

Proposition 2.3.13. [3]: Let $(X_1, N_1, *, \circ)$ and $(X_2, N_2, *, \circ)$ be two fuzzy normed algebras. If t-norm $*'$ dominates both $*$ and $\circ$, then $((X_1 \times X_2), N, *', \circ)$ is a fuzzy normed algebra, where $N\big((x_1 \times x_2), t\big) = N_1(x_1, t) *' N_2(x_2, t)$.

Proof.

According to [13], it remains to be proved that: r :

$$N((x_1y_1, x_2y_2), st) \geq N((x_1, x_2), S) \circ N((y_1, y_2), t),$$

for all $x_1, x_2 \in X_1,\ y_1, y_2 \in Y_2,$ for all $S, t \in (0, \infty)$.

We have

$$N((x_1y_1, x_2y_2), st) = N_1(x_1, y_1, st) \ *' N_2(x_2, y_2, st)$$

$$\geq [N_1(x_1, s) \circ N_1(y_1, t)] *' [N_2(x_2, s) \circ N_2(y_2, t)]$$

$$\geq [N_1(x_1, s) *' N_2(x_2, s)] \circ [N_1(y_1, t) *' N_2(y_2, t)]$$

$$= N((x_1, x_2), s) \circ N((y_1, y_2), t)$$

for all $x_1, x_2 \in X_1,\ y_1, y_2 \in Y_2,$ for all $S, t \in (0, \infty)$

Proposition 2.3.14. [3]: Let $*$ be a t-norm satisfying $\alpha * \alpha \geq \alpha$ for all $\alpha \in (0; 1)$ and let $(X_1, N_1, *, \circ)$and $(X_2, N_2, *, \circ)$ be two fuzzy normed algebras with multiplicatively continuous product. If $*'$ is a t-norm that dominates both $*$ and $\circ$ then $((X_1 \times X_2), N, *', \circ)$ is a fuzzy normed algebra with multiplicatively continuous product.

Proof.

Let $\alpha \in (0,1), (x_1, x_2) \in X_1 \times X_2$, and $(y_1, y_2) \in X_1 \times X_2, s, t > \alpha$ such that $N((x_1, x_2), s) > \alpha$ and $N((y_1, y_2), t) > \alpha$. Then we have successively:

$$N((x_1y_1,x_2y_2),st) = N_1(x_1,y_1,st) *' N_2(x_2,y_2,st)$$

$$\geq [N_1(x_1,s) \circ N_1(y_1,t)] *' [N_2(x_2,s) \circ N_2(y_2,t)]$$

$$\geq [N_1(x_1,s) *' N_2(x_2,s)] \circ [N_1(y_1,t) *' N_2(y_2,t)]$$

$$= N((x_1,x_2),s) \circ N((y_1,y_2),t)$$

$$\geq \alpha \circ \alpha$$

$$\geq \alpha$$

Example 2.3.15. [3]: Let $(X,N,*,\circ)$ be a fuzzy normed algebra with multiplicatively continuous product and let $S \subset X$ be a linear closed sub algebra of X. Then $(S,N,*,\circ)$ is a fuzzy normed algebra with multiplicatively continuous product.

Example 2.3.16. [3]: (Cartesian product of fuzzy normed algebras with multiplicatively continuous product that is not with multiplicatively continuous product). Let $(X,N,\cdot,\cdot)$ be the fuzzy normed algebra from Example 2.4.2, where $X = \mathbb{R}$. The fuzzy normed algebra $(X \times X, N',\cdot,\cdot)$, where

$N'\big((x_1,x_2),t\big) = N(x_1,t).N(x_2,t)$, for all $(x_1,x_2) \in X \times X$, for all $t > 0$ is not with multiplicatively continuous product.

Proof.

Taking into account that

$$N'\big((x_1,x_2),t\big) = N(x_1,t).N(x_2,t)$$

$$\begin{cases} 1, & t \geq |x_1| \\ 0, & \text{for the rest,} \end{cases} . \begin{cases} 1, & t \geq |x_2| \\ 0, & \text{for the rest,} \end{cases}$$

$$\begin{cases} 1, & t \geq \max\{|x_1|,|x_2|\}, \\ 0, & \text{for the rest,} \end{cases}$$

for $\alpha = \frac{1}{2}, x_1 = x_2 = y_1 = y_2 = \frac{1}{2}, s = t = \frac{3}{5}$, we obtain

$N'\big((x_1, x_2), t\big) = 1 > \frac{1}{2}, N'((y_1, y_2), s) = 1 > \frac{1}{2}, N'((x_1 y_1, x_2 y_2), st) = 0 < \frac{1}{2}$.

References

1. R. SAADATI and S.M. VAEZ POUR, "Some Results on Fuzzy BANACH SPACES", Korean, (2005).

2. E. Kreyszig, Introductory Functional Analysis with Applications, John Wiley & Sons, New York, Chichester, Brisbane, Toronto and Singapore, 1978.

3. Sorin Nadaban, "on Fuzzy Normed algebras, University of Timisoara, 2008.

4. M.A Naimark, "Normed Algebras", 1^{st} .Ed. Academy of Science U.S.S.R, springer, 1972.