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الإهداء

إلهي لا يطيب الليل إلا بشكرك ولا يطيب النهار إلى بطاعتك.. ولا تطيب اللحظات إلا بذكرك.. ولا تطيب الآخرة إلا بعفوك.. ولا تطيب الجنة إلا برؤيتك

الله جل جلاله

الى من بلغ الرسالة وأدى الأمانة.. ونصح الأمة.. الى نبي الرحمة ونور العالمين..

سيدنا محمد صلى الله عليه واله وسلم

الى من كلله الله بالهيبة والوقار.. الى من علمني العطاء بدون انتظار.. الى من أحمل أسمه بكل افتخار.. ارجو من الله أن يمد في عمرك لترى ثماراً قد حان قطافها بعد طول انتظار وستبقى كلماتك نجوم أهتدي بها اليوم وفي الغد والى الأبد..

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الى ملاكي في الحياة.. الى معنى الحب والى معنى الحنان والتفاني.. الى بسمة الحياة وسر الوجود.. الى من كان دعائها سر نجاحي وحنانها بلسم جراحي إلى أغلى الحبايب

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الى من به أكبر وعليه أعتمد .. الى شمعة متقدة تنير ظلمة حياتي ..
الى من بوجودها أكتسب قوة ومحبة لا حدود لها ..الى من عرفت معها معنى الحياة الى من رعانا وحافظ علينا, الى من وقف الى جانبنا عندما ضللنا الطريق...

الذي نقول له بشراك قول رسول الله صلى الله عليه واله وسلم:

((ان الحوت في البحر, والطير في السماء, ليصلون على معلم الناس الخير))

كما انني اتوجه بخالص الشكر, الى من علمنا التفاؤل والمضي الى الامام, الى من رعانا وحافظ علينا, الى من وقف الى جانبنا....

الاستاذة رجاء جفات شاهين

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(رجاء جفات شاهين)

فجزاها الله خير جزاء المحسنين

الباحث

ABSTRACT

In this research .we show that any Jordan higher derivation on a triangular algebra is a higher derivation

1. Introduction

Let $\mathcal{R}$ be a commutative ring with identity and $\mathcal{A}$ be an algebra over $\mathcal{R}$. An $\mathcal{R}$-linear mapping d from into $\mathcal{A}$ an $\mathcal{A}$-bimodule $\mathcal{M}$ is called a Jordan derivation if $d(x^2) = d(x)x + xd(x)$ for all $x \in \mathcal{A}$ and is said to be a derivation if $d(xy) = d(x)y + xd(y)$for all $x, y \in \mathcal{A}$.It is called a Jordan triple derivation if$d(xyx) = d(x)yx + xd(y)x + xyd(x)$ for all $x, y \in \mathcal{A}$. Obviously, every derivation is a Jordan derivation. However, the converse statement is in general not true. It is well known that every Jordan derivation on a 2-torsion free algebra is a Jordan triple derivation .The structures and properties of (Jordan-, Jordan triple-) derivations on matrix algebras and operator algebras were extensively studied in [1,2,4,10,12,13]. In this related area, there is a natural question which motivates many researchers' interest: what kind of algebras can enable a Jordan derivation on themselves to be a derivation? Her stein in [9] proved that every Jordan derivation on a 2-torsion free prime ring is a derivation. This result was extended to the case of semi prime rings by Cusack [7]. He showed that any Jordan derivation on a 2-torsion free semi prime ring is also a derivation .Furthermore, it was shown in[3] that every Jordan triple derivation on a 2-torsion free semi prime ring is a derivation .

Let$\mathcal{A}, \mathcal{B}$ be unital algebras over a commutative ring $\mathcal{R}$ and $\mathcal{M}$ be a unital $(\mathcal{A}, \mathcal{B})$-bimodule, which is faithful as a left$\mathcal{A}$-module and also as a right $\mathcal{B}$-module .The $\mathcal{R}$-algebra

$$\text{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B}) = \left\{ \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} : a \in \mathcal{A}, m \in \mathcal{M}, b \in \mathcal{B} \right\}$$

Under the usual matrix operations is said to be a triangular algebra. For convenience, we write $\mathcal{U}$ =Tri$(\mathcal{A},\mathcal{M},\mathcal{B})$.Cheung [4] first introduced the notion of triangular algebras, and then investigated the commuting mappings and derivations on triangular algebras [5,6]. Benkovic [2] studied Jordan derivations on triangular matrices over commutative rings and found that any Jordan derivation from the algebra of all upper triangular matrices into its arbitrary bimodule is the sum of a derivation and an antiderivation. This result was developed by$Z$hang and Yu in [13] by showing that every Jordan derivation of triangular algebras is a derivation. The main objective of this note is to generalize Zhang and Yu's result to the case of Jordan higher derivations .

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CHAPTER ONE

Definition(1.1) (Jordan-derivation):

is called a Jordan-derivation if An -linear mapping θ from A

$\theta\ (a \circ b) \ = \ \theta\ (a) \circ b \ + \ a \circ \ \theta\ (b)$,for all $a, b \in \ A$

Definition(1.2) (Module):

Let $(R, +, \cdot)$ be a ring and let $(M, +)$ abelion group then $(M, +)$ is called Left-Module if there is a mapping $\cdot : R \times M \rightarrow M$ Such that

1) $r \cdot (M_1 + M_2) = rM_1 + rM_2 \ \ni r \in R,\ M \in M$
2) $(r_1 + r_2) \cdot M = r_1 M + r_2 M \in M \ni r_1, r_2 \in R, M \in M$
3) $(r_1 \cdot r_2) \cdot M = r_1 \cdot (r_2 \cdot M) \in M \ni r_1, r_2 \in R, M \in M.$

Definition(1.3) (Bi-Module):

Is an abelion group that is both a left and a right Module such that the left and right multiplication compliable.

Definition(1.4) (Triangular algebra):

An algebra that contains a multiplicative identity element.

Definition(1.5): (Higher derivation)

Let $\mathbb{N}$ be the set of all nonnegative integers and let $D = (d_i)_{i\in\mathbb{N}}$ be a family of additive mappings on an algebra $\mathcal{A}$ such that $d_0 = id_{\mathcal{A}}$. D is called a higher derivation if for each $n \in \mathbb{N}$,

$$d_n(xy) = \sum_{i+j=n} d_i(x)d_j(y)$$

For all $x, y \in \mathcal{A}$.

Definition(1.6): (Jordan higher derivation)

Let $\mathbb{N}$ be the set of all nonnegative integers and let $D = (d_i)_{i\in\mathbb{N}}$ be a family of additive mappings on an algebra $\mathcal{A}$ such that $d_0 = id_{\mathcal{A}}$. D is called a Jordan higher derivation if for each $n \epsilon \mathbb{N}$.

$$d_n(x^2) = \sum_{i+j=n} d_i(x)d_j(x)$$

For all $x, y \in \mathcal{A}$.

Definition(1.7): (Jordan triple higher derivation)

Let $\mathbb{N}$ be the set of all nonnegative integers and let $D = (d_i)_{i\in\mathbb{N}}$ be a family of additive mappings on an algebra $\mathcal{A}$ such that $d_0 = id_{\mathcal{A}}$. D is called a Jordan triple higher derivation if for each $n \in \mathbb{N}$.

$d_n(xyx) = \sum_{i+j+k=n} d_i(x)d_j(y)d_k(x)$ For all $x, y \in \mathcal{A}$.

CHAPTER TWO

Main Results

Lemma2.1.

Let $\mathcal{A}$ be an associative algebra over a 2-torsion free commutative ring and $D = (d_i)_{i\in\mathbb{N}}$ be a Jordan higher derivation from into itself. Then$x, y, z \in \mathcal{A}$ and each $n \in \mathbb{N}$,we have ,we have

(a) $d_n(xy + yx) = \sum_{i+j=n}(d_i(x)d_j(y) + d_i(y)d_j(x)$;

(b) $d_n(xyx) = \sum_{i+j+k=n} d_i(x)d_j(y)d_k(x)$;

(c) $d_n(xyz + zyx) = \sum_{i+j+k=n}(d_i(x)d_j(y)d_k(z) + d_i(z)d_j(y)d_k(x)$.

Before proving the main result, we need several lemmas.

Lemma 2.2.

If $D = (d_i)_{i\in\mathbb{N}}$ is a Jordan higher derivation on $\mathcal{U}$,then $d_n(P) = Pd_n(P)$ and $d_n(Q) = d_n(Q)Q$ for each $n \in \mathbb{N}$.

Proof.

When $n = 0$, there is nothing to prove . When $n = 1$, this is result follows from [13, Lemma 2.2]. We assume that the relations $d_m(P) = pd_m(P)$ and $d_m(Q) = d_m(Q)Q$ are valid for all $m < n$.

According to the definition of Jordan higher derivation, we get

$$d_n(P) = d_n(P)P + Pd_N(P) + \sum_{\substack{i+j=n \\ ij\geq 1}} d_i(P)d_j(P). \qquad (2.1)$$

On the other hand, we have that $Pd_1(P)P = Qd_1(P)Q = 0$ by the proof of [13, Lemma2.2]. We assert that $Pd_n(P)P = 0$ for each $\in \mathbb{N}$. Indeed , we assume that $Pd_m(P)P = 0$ for all $m < n$. Combining this assumption with(2.1) yields that

$$d_n(P) = d_n(P)P + Pd_n(P) \qquad (2.2)$$

Left multiplication by P and right multiplication by P in(2.2) leads to

$$Pd_n(P)P = 0.$$

Similarly, we also have That $Qd_n(P)Q = 0$. Note the fact that $QAP = 0$ for all $A \in \mathcal{U}$. Therefore

$$d_n(P) = (P + Q)d_n(P)(P + Q)$$

$$= \quad = Pd_n(P)P + Pd_n(P)Q + Qd_n(P)P + Qd_n(P)Q$$

$$Pd_n(P)P + Pd_n(P)Q$$

$$= Pd_n(P)(P + Q)$$

$$= Pd_n(P)$$

By the fact $d_n(I) = 0$ for all $n > 0$,we obtatin

$$d_n(Q) = -d_n(P) = -Pd_n(P)(P + Q) = -Pd_n(P)Q$$

This gives

$$d_n(Q) = d_n(Q)Q.$$

Lemma2.3.

With notations as above, then for any ,we have

(a) $d_n(PA) = \sum_{i+j=n} d_i(P)d_j(A)$;

(b) $d_n(AQ) = \sum_{i+j=n} d_i(A)d_j(Q)$;

(C) $d_n(AP) = \sum_{i+j=n} d_i(A)d_j(P)$;

(d) $d_n(QA) = \sum_{i+j=n} d_i(Q)d_j(A)$.

Proof.

(a) By Lemma2.1 (c) and the fact $QAP = 0$ we get

$$d_n(PAQ) = d_n(PAQ + QAP)$$

$$= \sum_{i+j+k=n} (d_i(P)d_j(A)d_k(Q) + d_i(Q)d_j(A)d_k(P)).$$

Furthermore, it follows Lemma2.2 that

$$d_n(PAQ) = \sum_{i+j+k=n} (d_i(P)d_j(A)d_k(Q) + d_i(Q)d_j(A)d_k(P))$$

$$= \sum_{i+j+k=n} d_i(P)d_j(A)d_k(Q).$$

Applying Lemma2.1 (b) and the above relation yields

$$d_n(PA) = d_n(PAP + PAQ)$$

$$= \sum_{i+j+k=n} d_i(P)d_j(A)d_k(P) + \sum_{i+j+k=n} d_i(P)d_j(A)d_k(Q)$$

$= \sum_{i+j+k=n} d_i(P)d_j(A)d_k(P+Q)$ [Note that$d_k(I) = 0$ for$k > 0$]

$$= \sum_{i+j=n} d_i(P)d_j(A).$$

(b) Similarly, we have

$$d_n(AQ) = d_n(QAQ + PAQ)$$

$$= \sum_{i+j+k=n} d_i(Q)d_j(A)d_k(Q) + \sum_{i+j+k=n} d_i(P)d_j(A)d_k(Q)$$

$$= \sum_{i+j+k=n} d_i(Q + P)d_j(A)d_k(Q)$$

$$= \sum_{i+j+k=n} d_j(A)d_k(Q)$$

(C) As for the equality (c). we have

$$d_n(AP) = d_n(A - AQ) = d_n(A) - \sum_{i+j=n} d_i(A)d_j(Q)$$

$$= d_n(A) - d_n(A)Q - \sum_{\substack{i+j=n \\ j>0}} d_i(A)d_j(Q)$$

$$= d_n(A) - d_n(A)Q - \sum_{\substack{i+j=n \\ j>0}} d_i(A)d_j(I - P)$$

$$= d_n(A)P + \sum_{\substack{i+j=n \\ j>0}} d_i(A)d_j(P)$$

$$= \sum_{i+j=n} d_i(A)d_j(P)$$

(d) Let us see the last one.

$$d_n(QA) = d_n(A - PA) = d_n(A) - \sum_{i+j=n} d_i(P)d_j(A)$$

$$= d_n(A) - Pd_n(A) - \sum_{\substack{i+j=n \\ i>0}} d_i(P)d_j(A)$$

$$= d_n(A) - Pd_n(A) - \sum_{\substack{i+j=n \\ i>0}} d_i(I - Q)d_j(A)$$

$$= Qd_n(A) + \sum_{\substack{i+j=n \\ i>0}} d_i(Q)d_j(A)$$

$$= \sum_{i+j=n} d_i(Q)d_j(A).$$

Lemma2.4.

For any $A, X \in \mathcal{U}$,we have

(a) $d_n(APXQ) = \sum_{i+j=n} d_i(A)d_j(PXQ)$;

(b) $d_n(PXQA) = \sum_{i+j=n} d_i(PXQ)d_j(A)$;

Proof.

(a) Since $QAP = 0$ for all $\in \mathcal{U}$,$AP = IAP = (P + Q)AP = PAP$ for all $A \in \mathcal{U}$.

$d_n(APXQ) =$ By Lemma2.1 (a) and Lemma 2.3 (a) it follows that

$d_n((PA)(PXQ) + (PXQ)(PA))$

$$= \sum_{i+j=n} d_i(PA)d_j(PXQ) + \sum_{i+j=n} d_i(PXQ)d_j(PA)$$

$$= \sum_{i+j=n} \sum_{e+f=i} \sum_{g+h=j} d_e(P)d_f(A)d_g(P)d_h(XQ)$$

$$+ \sum_{i+j=n} \sum_{k+l+m=i} d_k(P)d_l(x)\, d_m(Q). \sum_{s+t=j} d_s\, d_t$$

$$= \sum_{i+j=n} d_i(PAP)d_j(XQ)$$

$$= \sum_{i+j=n} d_i(AP)d_j(XQ)$$

$$= \sum_{i+j=n} \sum_{e+f=i} d_e(A)d_f(P)d_j(XQ)$$

$$= \sum_{i+j=n} d_i(A)d_j(PXQ).$$

(b) Since$QAP = 0$ for all$A \in \mathcal{U}$,$QA = QAI = QA(P+Q) = QAQ$ for all$A \in \mathcal{U}$.

we get from Lemmas 2.1 (a) and 2.3 (b) that

$$d_n(PXQA) = d_n\big((AQ)(PXQ) + (PXQ)(AQ)\big)$$

$$= \sum_{i+j=n} d_i(AQ)d_j(PXQ) + \sum_{i+j=n} d_i(PXQ)d_j(AQ)$$

$$= \sum_{i+j=n} \sum_{e+f=i} \sum_{g+h=j} d_e(PX)d_f(Q)d_g(A)d_h(Q)$$

$$= \sum_{i+j=n} d_i(PX)d_j(QAQ)$$

$$= \sum_{i+j=n} d_i(PX)d_j(QA)$$

$$= \sum_{i+j=n} \sum_{e+f=j} d_i(PX)d_e(Q)d_f(A)$$

$$= \sum_{i+j=n} d_i(PXQ)d_j(A)$$

Now we are in a position to state the main result of this paper.

Theorem 2.5.

Let $\mathcal{A}, \mathcal{B}$ be unital algebras over a 2-torsion free commutative ring $\mathcal{R}$, and let $\mathcal{M}$ be a unital $(\mathcal{A}, \mathcal{B})$–bimodule which is faithful as a left $\mathcal{A}$–module and also as a right $\mathcal{B}$–module .Then any Jordan higher derivation from $\mathcal{U}$ to itself is a higher derivation .

Proof.

Let $D = (d_i)_{i\in\mathbb{N}}$ be a Jordan higher derivation of $\mathcal{U}$. By [13, Theorem 2.1] we immediately get that $d_1(AB) = d_1(A)B + Ad_1(B)$ for all A,B ∈ $\mathcal{U}$ .We now assume that

$$d_m(AB) = \sum_{i+j=m} d_i(A)d_j(B)$$

For all A,B ∈ $\mathcal{U}$ and for all $1 \leq m < n$.

It follows from Lemma 2.4 (a) that

$$d_n(ABPXQ) = \sum_{i+j=n} d_i(AB)d_j(PXQ)$$

$$= d_n(AB)PXQ + \sum_{\substack{i+j=n \\ j\geq 1}} d_i(AB)d_j(PXQ) \qquad (2.3)$$

for all $A, B, X \in \mathcal{U}$. On the other hand ,by Lemma 2.4 (a) and the fact $BP = PBP$ we have

$$d_n(ABPXQ) = d_n(AP(BPX)Q)$$

$$= \sum_{i+j=n} d_i(A)d_j(BPXQ)$$

$$= \sum_{i+j=n} \sum_{e+f=j} d_i(A)d_e(B)d_f(PXQ) \qquad (2.4)$$

$$= \sum_{i+j=n} d_i(A)d_j(B)PXQ + \sum_{\substack{i+j+e=n \\ e\geq 1}} d_i(A)d_j(B)d_e(\mathrm{PXQ}).$$

For all $, B, X \in \mathcal{U}$. Combining(2.3)with (2.4)and using the induction hypothesis, we arrive at

$$[d_n(AB) - \sum_{i+j=n} d_i(A)d_j(B)\ ]P\mathcal{U}Q = 0 \qquad (2.5)$$

For all $A, B \in \mathcal{U}$ .Since $P\mathcal{U}Q$ is a faithful left $P\mathcal{U}P$–module, we get

$$P[d_n(AB) - \sum_{i+j=n} d_i(A)d_j(B)\ ]P = 0 \qquad (2.6)$$

$$Q[d_n(AB) - \sum_{i+j=n} d_i(A)d_j(B)\ ]Q = 0 \qquad (2.7)$$

For all $A, B \in \mathcal{U}$. Applying Lemma 2.3 yields

$$d_n(PABQ) = \sum_{i+j+K=n} d_i(P)d_j(AB)d_k(\mathrm{Q})$$

$$= Pd_n(AB)Q + \sum_{\substack{i+j+k=n \\ j<n}} d_i(P)d_j(AB)d_k(\mathrm{Q}) \qquad (2.8)$$

For all A,B ∈ $\mathcal{U}$ By Lemmas 2.1 (a) and 2.3 it follows that

$$d_n(PABQ) = d_n(PA)(BQ) + (BQ)(PA))$$

$$= \sum_{i+j=n} d_i(PA)d_j(BQ)$$

$$= \sum_{i+j=n} \sum_{e+f=i} \sum_{g+h=j} d_e(P)d_f(A)d_g(B)d_h(Q) \qquad (2.9)$$

$$P\left(\sum_{i+j=n} d_i(A)d_j(B)\right)\mathrm{Q} + \sum_{\substack{e+f+g+h=n \\ f+g<n}} d_e(P)d_f(A)d_g(\mathrm{B})d_h(\mathrm{Q})$$

For all $A, B \in \mathcal{U}$, combining (2.8) and (2.9) , we have by the induction hypothesis that

$$P\left[d_n(AB) - \sum_{i+j=n} d_i(A)d_j(B)\right]Q = 0 \qquad (2.10)$$

For all A,B ∈ $\mathcal{U}$. The relations (2.6),(2.7)and (2.10) lead to

$$d_n(AB) = \sum_{i+j=n} d_i(A)d_j(B)$$

for all $A, B \in \mathcal{U}$ .This shows that $D = (d_i)_{i\in\mathbb{N}}$is a higher derivation from $\mathcal{U}$ to itself ,which is the desired result.

As a direct consequence of Theorem 2.5, we obtain.

Corollary2.6.

With the same assumptions as in Theorem 2.5. Then any Jordan derivation on $\mathcal{U}$ is a derivation .

Let $D = (d_i)_{i\in\mathbb{N}}$ be a Jordan triple higher derivation of the triangular algebra . Note that the fact $d_1(I) = 0$. A straightforward induction shows that $d_m(I) = 0$ for all $m \geq 1$. Thus D becomes a Jordan higher derivation of $\mathcal{U}$ .Therefore we get the following corollary .

Corollary 2.7.

With the same assumptions as in Theorem 2.5 ,then any Jordan triple higher derivation on $\mathcal{U}$ is also a higher derivation .