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On the eignvalue of the composition operator C_ϕ

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By

Mohammed jamal

Supervised by

Aqeel Mohammad Hussein

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Abstract

Let U denate the unit ball in the complix plane, the Hardy space H^2 is the set of functions $f(z)=\sum_{n=0}^{\infty} f^{\wedge}(n)z^n$ holomarphic on U such that $\sum_{n=0}^{\infty} |f^{\wedge}(n)|^2 < \infty$ with $f^{\wedge}(n)$ denates then the Taylor coefficient of f.

Let ψ be a holomarphic self-map of U, the composetion operater C_ψ induced by ψ is defined on H^2 by the equation

$$C_\psi f = f \circ \psi \quad (f \in H^2)$$

We have studied the eignvalue of the composetion operater induced by the bijective map ϕ and discussed the adjoint of the composetion operater induced by the bijective map ϕ. We have look also at some known properties on composetion operaters and tried to see the analogue properties in order to show how the results are changed by changing the function ψ in U.

In order to make the work accessible to the reader, we have included some known results with the details of the proofs for some cases and proofs for the properties .

المستخلص

ليكن U يرمز إلى كرة الوحدة في المستوى العقدي، إن فضاء هاردي H^2 هو مجموعة كل الدوال

$f(z) = \sum_{n=0}^{\infty} f^{\wedge}(n)\, z^n$ التحليلية على U بحيث أن $\sum_{n=0}^{\infty} \left| f^{\wedge}(n) \right|^2 < \infty$, $f^{\wedge}(n)$ يرمز إلى معاملات تيلر .

لتكن $\psi: U \to U$ دالة تحليلية على U ، المؤثر التركيبي المتولد من ψ يعرف على فضاء هاردي H^2 بواسطة:

$$C_{\psi} f = f \circ \psi \quad (f \in H^2).$$

درسنا في هذا البحث القيم الذاتية للمؤثر التركيبي المتولد من الدالة المتقابلة ϕ حيث ناقشنا المؤثر المرافق للمؤثر التركيبي المتولد من الدالة المتقابلة ϕ. بالإضافة إلى ذلك نظرنا إلى بعض النتائج المعروفة وحاولنا الحصول على نتائج مناظرة لنتمكن من ملاحظة كيفية تغير النتائج عندما تتغير الدالة التحليلية ψ .

ومن أجل جعل مهمة القارئ أكثر سهولة، عرضنا بعض النتائج المعروفة عن المؤثرات التركيبية وعرضنا براهين مفصلة وكذلك برهنا بعض النتائج .

Contents

المقدمة

هذا البحث يشمل فصلين . في الفصل الأول سوف نتناول الدالة المتقابلة ϕ وخواصها ، ونناقش النقاط الصامتة الداخلية والخارجية للدالة ϕ أيضا وكذلك نناقش أيضا الدوران المحوري حول الأصل للدالة ϕ وكذلك نناقش أيضا هل الدالة ϕ قطع ناقص وكذلك نناقش أيضا هل الدالة ϕ تحويل كسوري خطيّ .

في الفصل الثاني، سوف نتناول المؤثر التركيبي C_ϕ المتولد بالدالة ϕ وخواصه ، وكذلك نناقش أيضا المرافق للمؤثر التركيبي C_ϕ المتولد بالدالة ϕ وكذلك نناقش أيضا هل المؤثر التركيبي C_ϕ مؤثر تركيبي قابل للعكس وكذلك نناقش أيضا القيم الذاتية للمؤثر التركيبي C_ϕ

Introduction

This search consists of two chaptors .In chaptor one ,we are going to study the bijective map ϕ and proparties of ϕ, and also discuss the interior and exterior fixid points of ϕ and also discuss ϕ is a rototion around the origin and ϕ is elliptic and ϕ is a linear fractional transformation .

In chaptor two, we are going to study the Composetion Operatar C_ϕ induced by the map ϕ and proparties of C_ϕ, and also discuss the adjoint of Composetion Operatar C_ϕ induced by the map ϕ and also discuss C_ϕ is an invertible composetion operatar and discuss C_ϕ is an idempotent composition operatar and discuss the eign value of C_ϕ

Definition(1.1) :

Let U = $\{ z\in C:|z|<1 \}$ is a unit ball in complex plane C and $\partial U=\{ z\in C:|z|=1 \}$ is a boundary of U

Definition(1.2):

For $\gamma \in U$, define $\phi: U \to U$ and $\phi(z)=\dfrac{4z}{4\gamma z-4}$ $(\gamma, z\in U)$.

Proposition (1.3):

ϕ is bijective .

Proof:

Since $\phi(z)=\dfrac{4z}{4\gamma z-4}$ $(\gamma, z\in U)$

Suppose $\phi(z_1)=\phi(z_2)$, that is $\dfrac{4z_1}{4\gamma z_1-4}=\dfrac{4z_2}{4\gamma z_2-4}$, therefare $16\gamma z_1 z_2-16z_1=16\gamma z_1 z_2-16z_2$, therefare $-16z_1=-16z_2$, hance $z_1= z_2$. Thas ϕ is injective .

Let $y=\phi(z)$, that is $y=\dfrac{4z}{4\gamma z-4}$, therefare $4\gamma zy-4y=4z$, than $4\gamma zy-4z=4y$, hance

$$z=\frac{4y}{4\gamma y-4}, \mu(z)=\mu\left(\frac{4y}{4\gamma y-4}\right)=\frac{4\left(\frac{4y}{4\gamma y-4}\right)}{4\gamma\left(\frac{4y}{4\gamma y-4}\right)-4}=\frac{\frac{16y}{4\gamma y-4}}{\frac{16\gamma y-16\gamma y+16}{4\gamma y-4}}=\frac{16y}{16}=y,$$ for every $y\in U$,

there exists $z\in U$ such that $\phi(z)=y$ .Thus ϕ is surjective . Hence ϕ is bijective.

Definition(1.4) :

A point p $\in$ C is a fixid point for the function ψ, if $\psi(p)=p$.

Proposition (1.5) :

ϕ has two fixid points $z_1=0, z_2=\dfrac{2}{\gamma}$

Proof :

Let $\phi(z)=z$, that is $\frac{4z}{4\gamma z-4`}=z$, therefore $4\gamma z^2-4z=4z$, hence $4\gamma z^2-8z=0$. ϕ has two fixed points $z_1=0, z_2=\frac{2}{\gamma}$

Definition(1.6):

Let $\psi:U\to U$ be holomarphic map on U with a fixid point r, then :

1) r is interior fixid point for ψ if $r\in U$
2) r is exterior fixid point for ψ if $r\not\in U$

Proposition (1.7):

0 is interior fixid point and $\frac{2}{\gamma}$ is exterior fixid point of ϕ.

Proof :

Since ϕ has two fixid points $z_1=0$, $z_2=\frac{2}{\gamma}$, $|z_1|=|0|=0<1$, $z_1\in U$, than $z_1=0$ is interior fixid point of ϕ. Since $\gamma\in U$, then $|\gamma|<1<2$, thus $|\gamma|<2$, therefore $\left|\frac{2}{\gamma}\right|=\frac{2}{|\gamma|}>1$, hence $|z_2|=\left|\frac{2}{\gamma}\right|>1$. Thus $z_2=\frac{2}{\gamma}$ is exterior fixid point of ϕ .

Proposition (1.8) :

$$\phi^{-1}(z)=\frac{4z}{4\gamma z-4}=\phi(z)$$

Proof :

Let y = $\phi^{-1}(z)$ than $z=\phi(y)$ therefare $z=\frac{4y}{4\gamma y-4}$, hance $4\gamma yz-4z=4y$, thas $4\gamma yz-4y=4z$ than $y=\frac{4z}{4\gamma z-4}$. Hance $\phi^{-1}(z)=\frac{4z}{4\gamma z-4}=\phi(z)$

Remark(1.9) :

$$\phi'\left(\bar{\gamma}\right)=\frac{-1}{\left(|\gamma|^2-1\right)^2} \ , \ \phi'(0)=-1 .$$

Definition(1.10) :

Let $\psi:U\to U$ be holomarphic map on U . We say that ψ is a rototion around the origin if there exists $\sigma\in\partial U$ such that $\psi(z)=\sigma z \ (z\in U)$

Proposition (1.11):

If $\gamma=0$, ϕ is a rototion around the origin

Proof:

Since $\gamma=0$, $\phi(z)=\dfrac{4z}{4\gamma z-4}=-z=\sigma z, \sigma=-1, |\sigma|=1\in\partial U$, then by (1.10) ϕ is a rototion around the origin .

Theorem (1.12) :

If $\psi:U\to U$ is holomarphic map on U, then ψ is elliptic if and only if ψ bijective and ψ has interior fixid point.

Proposition (1.13) :

ϕ is elliptic

Proof :

From (1.3), ϕ is bijective, and from (1.7) ϕ has interior fixid point . By (1.12) ϕ is elliptic.

Definition(1. 14):

A linear fractionil transformation is a mapping of the form $\tau(z)=\frac{az+b}{cz+d}$ where a , b , c and d are complix numbers, and sometime denate it by $\tau_A(z)$ where A is the non–singular 2×2 complix matrix $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

<u>Proposition (1.15)</u> :

ϕ is a linear fractionil transformation .

<u>Proof</u> :

Since $\phi(z)=\frac{4z}{4\gamma z-4}=\frac{az+b}{cz+d}$ such that $a=4, b=0, c=4\gamma, d=-4$ and a , b , c , and d are complex numbers and $A=\begin{bmatrix} 4 & 0 \\ 4\gamma & -4 \end{bmatrix}$, hence by (1.14) ϕ is a linear fractional transformation .

<u>Definition(2.1)</u>:

Let U denate the unit ball in the complix plane, the Hardy space H^2 is the set of functions $f(z)=\sum_{n=0}^{\infty} f\,\hat{}(n)\, z^n$ holomarphic on U such that $\sum_{n=0}^{\infty} |f\,\hat{}(n)|^2 < \infty$ with $f\,\hat{}(n)$ denates then the Taylor coefficient of f .

Remark (2.2) :

We can define an anner product of the Hardy space functions as follows:

$f(z) = \sum_{n=0}^{\infty} f^{\wedge}(n)\, z^n$ and $g(z) = \sum_{n=0}^{\infty} g^{\wedge}(n)\, z^n$, then the anner product of f and g is:

$\langle f, g\rangle = \sum_{n=0}^{\infty} f^{\wedge}(n)\, \overline{g^{\wedge}(n)}$

Definition (2.3) :

Let $\alpha \in U$, define $K_\alpha(z) = \frac{1}{1-\overline{\alpha}z}\ (z \in U)$. Since $\alpha \in U$, than $|\alpha| < 1$. The geometric series $\sum_{n=0}^{\infty} |\alpha|^{2n}$ is converge . Thas $K_\alpha \in H^2$ and $K_\alpha = \sum_{n=0}^{\infty} (\overline{\alpha})^n z^n$

Definition(2.4) :

Let $\psi : U \to U$ be holomarphic map on U, the composetion operater C_ψ induced by ψ is defined on H^2 as follows $C_\psi f = f \circ \psi \quad (f \in H^2)$

Definition(2.5) :

Let T be a bounded operator on a Hilbert space H, then the norm of an operator T is defined by $\|T\| = \sup\{\|Tf\| : f \in H, \|f\| = 1\}$.

Littlewood's Subordination Principle (2.6) :

If $\psi : U \to U$ is holomarphic map on U with $\psi(0) = 0$,then $f \circ \psi \in H^2$ and $\|f \circ \psi\| \le \|f\|$ for each $f \in H^2$.

The goal of this theorem $C_\psi : H^2 \to H^2$.

Definition(2.7) :

The composetion operater C_ϕ induced by ϕ is defined on H^2 as follows $C_\phi f = f \circ \phi$, for all $f \in H^2$.

Proposition(2.8) :

If $\phi(z) = \frac{4z}{4\gamma z - 4}$, then $f \circ \phi \in H^2$ and $\|f \circ \phi\| \le \|f\|$ for each $f \in H^2$.

Proof :

Since $\phi : U \to U$ is holomorphic map on U with $\phi(0) = 0$ by (1.5), then by (2.6) $f \circ \phi \in H^2$ and $\|f \circ \phi\| \le \|f\|$ for each $f \in H^2$, hence $C_\phi : H^2 \to H^2$

Remark (2.9) :

1) One can easily show that $C_\kappa C_\psi = C_{\psi \circ \kappa}$ and hence $C_\psi^n = C_\psi C_\psi \cdots C_\psi$
$= C_{\psi \circ \psi \circ \cdots \circ \psi} = C_{\psi_n}$

2) C_ψ is the identity operator on H^2 if and only if ψ is identity map from U into U and holomorphic on U.

3) It is simple to prove that $C_\kappa = C_\psi$ if and only if $\kappa = \psi$.

Definition(2.10):

Let T be an operater on a Hilbert space H , The operater T^* is the adjoint of T if $\langle Tx, y \rangle = \langle x, T^* y \rangle$ for each $x, y \in H$.

Theorem (2.11) :

$\{K_\alpha\}_{\alpha \in U}$ forms a danse subset of H^2.

Theorem (2.12) :

If $\psi : U \to U$ is holomarphic map on U, then for all $\alpha \in U$, then $C_\psi^* K_\alpha = K_{\psi(\alpha)}$

Definition(2.13):

Let H^{∞} be the set of all bounded holomarphic map on U .

Definition(2.14):

Let $g \in H^{\infty}$, the Toeplits operator T_g is the operater on H^2 given by :

$(T_g f)(z) = g(z)\ f(z) \quad (f \in H^2,\ z \in U)$

Theorem (2.15) :

If $\psi : U \to U$ is holomarphic map on U, then $C_{\psi} T_g = T_{g \circ \psi} C_{\psi} \quad (g \in H^{\infty})$

Remark (2.16) :

For each $f \in H^2$, it is well- know that $T_h^* f = T_{\bar{h}} f$, such that $h \in H^{\infty}$.

Proposition(2.17) :

$C_{\phi}^* = T_g\ C_{\lambda}\ T_h^*$ where $h(z) = 1 - \gamma z$, $g(z) = 1$, $\lambda(z) = \bar{\gamma} - z$

Proof :

By (2.16), $T_h^* f = T_{\bar{h}} f$ for each $f \in H^2$. Hence for all $\alpha \in U$,

$$\left\langle T_h^* f, K_{\alpha} \right\rangle = \left\langle T_{\bar{h}} f, K_{\alpha} \right\rangle = \left\langle f, T_{\bar{h}}^* K_{\alpha} \right\rangle \cdots\cdots (2-1)$$

On the other hand ,

$$\left\langle T_h^* f,\ K_{\alpha} \right\rangle = \left\langle f, T_h K_{\alpha} \right\rangle = \left\langle f, h(\alpha) K_{\alpha} \right\rangle \cdots\cdots (2-2)$$

From (2-1)and (2-2) one can see that $T_{\bar{h}}^* K_{\alpha} = h(\alpha)\ K_{\alpha}$. Hence $T_h^* K_{\alpha} = \overline{h(\alpha)}\ K_{\alpha}$.

Calculation give:

$$C_{\phi}^* K_{\alpha}(z) = K_{\phi(\alpha)}(z)$$

$$= \frac{1}{1-\overline{\phi(\alpha)}\, z} = \frac{1}{1-\dfrac{4\overline{\alpha} z}{4\overline{\gamma}\overline{\alpha}-4}}$$

$$= \frac{1}{\dfrac{4\overline{\gamma}\overline{\alpha}-4-4\overline{\alpha}z}{4\overline{\gamma}\overline{\alpha}-4}} = \frac{4\overline{\gamma}\overline{\alpha}-4}{-4+4\overline{\alpha}(\overline{\gamma}-z)} = \frac{\overline{(1-\gamma\alpha`)}}{1-\overline{\alpha}(\overline{\gamma}-z)}$$

$$= \overline{(1-\gamma\alpha)} \cdot (1) \cdot \frac{1}{1-\overline{\alpha}(\overline{\gamma}-z)}$$

$$= \overline{h(\alpha)} \cdot T_g\, K_\alpha(\lambda(z)) = T_g\, \overline{h(\alpha)}\, K_\alpha(\lambda(z))$$

$$= T_g\, \overline{h(\alpha)}\, C_\lambda\, K_\alpha(z) = T_g\, C_\lambda\, \overline{h(\alpha)}\, K_\alpha(z)$$

$$= T_g\, C_\lambda\, T_h^*\, K_\alpha(z)$$, therefare

$$C_\phi^*\, K_\alpha(z) = T_g\, C_\lambda\, T_h^*\, K_\alpha(z)\ .$$

But $\overline{\{K_\alpha\}_{\alpha\in U}} = H^2$, then $C_\phi^* = T_g\, C_\lambda\, T_h^*$

Theorem (2. 18) :

If $\psi : U \to U$ is holomorphic map on U, then C_ψ is an invertible operator on H^2 if and only if ψ bijective and $C_\psi^{-1} = C_{\psi^{-1}}$

Proposition(2.19) :

C_ϕ is an invertible composetion operater on H^2

Proof :

Since ϕ as bijective by (1.3) , and by (2.18) C_ϕ as an invertible operator on H^2 .

Definition (2.20) :

Let $\psi : U \to U$ and holomarphic on U, the eigenvalue equation for the composetion operater is define by $C_\psi f = \kappa\, f$ or $f \circ \psi = \kappa\, f$.

Theorem (2.21):

Let $\psi : U \to U$ and holomarphic on U, and that fixis the point $p \in U$ and suppose that $C_\psi f = \kappa f$ far some non-constant $f \in H^2$ and some $\kappa \in \mathbb{C}$. Than $\kappa = (\psi'(p))^n$ far some n = 0, 1, 2, ...

Proposition(2.22) :

If $\mu \in U$, then $(-1)^n$ is an eigenvalue of C_ϕ for some n = 0 , 1 , 2 , …..

Proof :

Since $\phi(z) = \frac{4z}{4\gamma z - 4}, \phi'(z) = \frac{(4\gamma z - 4)(4) - (4z)(4\gamma)}{(4\gamma z - 4)^2} = \frac{-16}{(4\gamma z - 4)^2}$, and since ϕ fixid the point $0 \in U$,

and by (2.21) $\kappa = (\phi'(0))^n = (-1)^n$ is an eigenvalue of C_ϕ for some n = 0 , 1 , 2 , …..

REFERENCES

[1] Ahlfors, L.V. ,"**Complex Analysis**", Sec , Ed., McGraw-Hill Kogakusha Ltd ,(1966).

[2] Appell, M.J., Bourdon , P.S. & Thrall, J.J. ," **Norms of Composition Operators on the Hardy Space**" , Experimented Math ., pp.111-117, (1996).

[3] Berberian, S.K., " **Introduction to Hilbert Space**" ,Sec. Ed .,Chelesa publishing Com., New York , N.Y., (1976).

[4] Bourdon, P.S.& Shapiro, J.H.,"**Cyclic Phenomena for Composition Operators**", Math. Soc., (596), 125, (1999).

[5] Cowen ,C.C."**Linear Fraction Composition Operator on** H^2"Integral Equations Operator Theory ,11, pp. 151 -160, (1988).

[6] Deddnes, J.A. "**Analytic Toeplits and Composition Operators** ", Con . J. Math. , vol(5), pp. 859- 865, (1972).

[7] Duren, P.L.," **Theory of** H^p **Space** ", Academic press ,New York ,(1970).

[8] Halmos , P.R .,"**A Hilbert Space Problem Book ",** Springer-Verlag,New York (1982).

[9] Nordgren, E.A., "**Composition operator** ", Can.J.Math.20,pp.442-449,(1986).

[10] Radjavi ,H & Ros (5), pp. 859- 865, (1972).enthal, P.," **Invariant Subspace**" ,
Springer-Verlage, Berlin , Heidelberg , New York , (1973).

[11] Schwartz , H.J.," **Composition Operator on** H^2 ",Ph .D.thesis.Univ.of Toled, (1969).

[12] Shapiro, J.H., " **Composition operators and Classical Function Theory** ", Springer- Verlage, New York, (1993).

[13] Shapiro, J.H., " **Lectures on Composition operators and Analytic Function Theory** ". www.mth.mus.edu./~shapiro/pubrit/Downloads/computer/complutro. pdf

[14] Shapiro, J.H.," **Composition operators and Schroders Functional Equation** ",
Contemporary Math., 213, pp.213-228, (1998).