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# On Osculatrary interpolation

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بسم الله الرحمن الرحيم

﴿يُؤتِي الْحِكْمَةَ مَن يَشَاءُ وَمَن يُؤْتَ الْحِكْمَةَ فَقَدْ أُوتِيَ خَيْرًا كَثِيرًا وَمَا يَذَّكَّرُ إِلَّا أُولُو الْأَلْبَابِ﴾

صدق الله العلي العظيم

سورة البقرة الآية (269)

## الإهداء

إلى من جرع الكأس فارغاً ليسقيني قطرة حب
إلى من كلّت أنامله ليقدم لنا لحظة سعادة إلى من حصد الأشواك عن دربي ليمهد لي طريق العلم
إلى القلب الكبير (والدي العزيز)

إلى من أرضعتني الحب والحنان
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الآن تفتح الأشرعة وترفع المرساة لتنطلق السفينة في عرض بحر واسع مظلم هو بحر الحياة وفي هذه الظلمة لا يضيء إلا قنديل الذكريات ذكريات الأخوة البعيدة إلى الذين أحببتهم وأحبوني (أصدقائي)

# شكر وتقدير

الحمد لله يقول الله في محكم كتابه { لئن شكرتم لأزيدنكم } والصلاة والسلام على اشرف خلق الله سيدنا محمد (صلى الله عليه واله وسلم ) القائل: من لم يشكر المخلوق لم يشكر الخالق.

بداية اشكر الله عز وجل الذي ساعدني على اتمام بحثي وتفضل علينا بإتمام هذا العمل.. وبعد

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فجزاه الله خير جزاء المحسنين

الباحث

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# CHAPTER
# ONE

# PRELIMINARIES AND BACKGROUND

## 1.1.INTRODUCTION

In this chapter, we have two classes of preliminary discussions: ODEs and approximation theory ,for they are fundamental compositions in our subject. The former is mainly about the two point second order boundary value problems (TPBVP) and semi-analytic methods for solving TPBVP.

## 1.2. MATHEMATICAL BACKGROUND

We begin by considering some elementary mathematical background material for ODE BVPs in this these .The dynamic behavior of systems is an important subject. A mechanical system involves displacements, velocities, and accelerations. An electric or electronic system involves voltages, currents, and time derivatives of these quantities. An equation that involves one or more derivatives of the unknown function is called an Ordinary Differential Equations.

The order of the equation is determined by the order of the highest derivative. For example, if the first derivative is the only derivative, the equation is called a first-order ODE. In the same way, if the highest derivative is second order, the equation is called a second-order ODE.

The problems of solving an ODE are classified into initial-value problems (IVP ) and boundary-value problems (BVP), depending on how the conditions at the endpoints of the domain are specified. All the conditions of an initial-value problem are specified at the initial point. On the other hand, the problem becomes a boundary-value problem if the conditions are needed for both initial and final points.[1]

The words two-point in the two point boundary value problem, refer to the fact that the boundary condition is evaluated at the solution at the two interval endpoints and unlike for initial value problems (IVPs) where the initial conditions are all evaluated at a single point .Occasionally, problems arise where the boundary condition is also evaluated at the solution at other points in the domain. In these cases, we have a multipoint BVP. As shown in [2], a multipoint problem may be converted to a two-point problem by defining separate sets of variables for each subinterval between the points and adding boundary conditions which ensure continuity of the variables across the whole interval. Like rewriting the original BVP in the compact, rewriting a multipoint problem as a two-point problem may not lead to a problem with the most efficient computational solution[3 ]. We note that a different type of boundary condition may be applied at each end point, for example, periodicity conditions. The reader is referred to Keller for methods of treatment.

Most practically arising TPBVP have separated boundary conditions where the boundary function g may be split into two parts (one for each endpoint:

$$g_a\,(y(a)) = 0\ ,\quad g_b\,(y(b)) = 0\ .$$

Here, $g_a \in R^s$ and $g_b \in R^{n-s}$ for some value s with $1 < s < n$ and where each of the vector functions $g_a$ and $g_b$ are independent. However, there are well-known, commonly arising, boundary conditions which are not separated; for example, consider periodic boundary conditions which.

A boundary-value problem in standard form consists of the second-order linear differential equation :

$$y'' + P(x)\, y' + Q(x)\, y = \phi(x) \qquad \ldots\ldots(1.1)$$

and the boundary conditions

$$\alpha_1\, y(a) + \beta_1 y'(a) = \gamma_1$$
$$\alpha_2\, y(b) + \beta_2\, y'(b) = \gamma_2 \qquad \ldots\ldots\ldots\ (1.2)$$

where P(x), Q(x), and $\phi(x)$ are continuous in [a, b] and $\alpha_1$ , $\alpha_2$, $\beta_1$, $\beta_2$ , $\gamma_1$ ,and $\gamma_2$ are all real constants. Furthermore, it is assumed that $\alpha_1$ and $\beta_1$are not both zero, and also that $\alpha_2$ and $\beta_2$are not both zero. In this thesis we take the following probability of boundary conditions (1.2) : $\alpha_1$ and $\alpha_2$ are equal to zero(Neuman condition ) or $\beta_1$and $\beta_2$ are equal to zero ( Dirichlet condition ) or ( $\alpha_1$and $\beta_2$ or $\alpha_2$ and $\beta_1$) equal to zero (mixed condition ).

The boundary-value problem is said to be homogeneous if both the differential equation and the boundary conditions are homogeneous (i.e.,$\phi(x) \equiv 0$ and $\gamma_1 = \gamma_2 = 0$ ).Otherwise the problem is non homogeneous .Thus a homogeneous boundary-value problem has the form

$$y'' + P(x)\, y' + Q(x)\, y = 0 \qquad \ldots\ldots.. \ (1.3)$$
$$\alpha_1\, y(a) + \beta_1\, y'(a) = 0$$
$$\alpha_2\, y(b) + \beta_2\, y'(b) = 0$$

A boundary-value problem is solved by first obtaining the general solution to the differential equation, using any of the appropriate methods presented heretofore, and then applying the boundary conditions to evaluate the arbitrary constants.

## 1.3. APPROXIMATION THEORY [4]

The primary aim of a general approximation is to represent non-arithmetic quantities by arithmetic quantities so that the accuracy can be ascertained to a desired degree. Secondly, we are also concerned with the amount of computation required to achieve this accuracy. A complicated function f(x) usually is approximated by an easier function of the form $\varphi(x; a_0, \ldots, a_n)$ where $a_0, \ldots, a_n$ are parameters to be determined so as to characterize the best approximation of f. Depending on the sense in which the approximation is realized, there are three types of approaches:

1. Interpolatory approximation: The parameters $a_i$ are chosen so that on a fixed prescribed set of points $x_i$, i = 0, 1, . . ., n, we have

$$\varphi(x_i; a_0, \ldots, a_n) = f(x_i) = f_i. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1.4)$$

Sometimes, we even further require that, for each i, the first $r_i$ derivatives of φ agree with those of f at $x_i$ .

2. Least-square approximation: The parameters $a_i$ are chosen so as to

$$\text{Minimize } \| f(x) - \psi(x; a_0, \ldots, a_n) \|_2. \quad \ldots\ldots\ldots\ldots\ldots\ldots(1.5)$$

3. Min-Max approximation: the parameters $a_i$ are chosen so as to

$$\text{Minimize } \| f(x) - \varphi(x; a_0, \ldots, a_n) \|_\infty. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1.6)$$

**Note**

The approximation functions depend on a set of parameters $\{a_i\}_{i=0}^{n}$, there are many ways to choose these parameters. The most obvious ones are algebraic polynomials.

**Definition 1.4** [5]

We say $\varphi$ is a linear approximation of f if $\varphi$ depends linearly on the parameters $a_i$, that is, if $\varphi(x_i; a_0, . . , a_n) = a_0\varphi_0(x_0) + \; . . + a_n \varphi_n (x_n)$ (1.7) where $\varphi_i(x)$ are given and fixed functions .Now we introduce the theorem of Weierstrass which can be considered as one of the foundations of the approximation theory .

**Theorem 1.5** (Weierstrass Approximation Theorem)

Let f(x) be a continuous function on [a, b]. For any $\epsilon > 0$, there exist an integer n and a polynomial $P_n(x)$ of degree n such that max

$$x\in[a,b] \; |f(x) - P_n(x)| < \epsilon. \qquad \ldots\ldots\ldots\ldots\ldots\ldots(1.8)$$

In this thesis, we shall consider only the interpolatory approximation. From (1.8) it follows that one can always find a polynomial that is arbitrarily close to a given function on some finite interval. This means that the approximation error is bounded and can be reduced by the choice of the adequate polynomial. Unfortunately Theorem 1.6 is not a constructive one, i.e. it does not present a way how to obtain such a polynomial. i.e. the interpolation problem can also be formulated in another way, viz. as the answer to the following question: How to find a .good. representative of a function that is not known explicitly, but only at some points of the domain of interest .In this thesis we use Osculatory Interpolation since has high order with the same given points in the domain .

## 1.6. Osculatory Interpolation[6]

Given $\{x_i\}$, i = 1, . . .k and values $f_i^{(0)}$, . . . , $f_i^{(ri)}$ ,where $r_i$ are nonnegative integers and $f_i = f(x_i)$.We want to construct a polynomial P(x) such that $P^{(j)}(x_i) = f_i^{(j)}$ …………… (1.9)

for i = 1, . . . , k and j = 0, . . . , $r_i$. Such a polynomial is said to be an sculatory interpolating polynomial of a function f .

In this paper we use two-point osculatory interpolation [6]. Essentially this is a generalization of interpolation using Taylor polynomials and for that reason osculatory interpolation is sometimes referred to as two-point Taylor interpolation. The idea is to approximate a function y(x) by a polynomial P(x) in which values of y(x) and any number of its derivatives at given points are fitted by the corresponding function values and derivatives of P(x).

In this thesis we are particularly concerned with fitting function values and derivatives at the two end points of a finite interval, say [0,1],wherein a useful and succinct way of writing a osculatory interpolant $P_{2n+1}(x)$ of degree 2n + 1 was given for example by Phillips [14] as:

$$P_{2n+1}(x)=\sum_{j=0}^{n} \{y^{(j)}(0)\, q_j(x)+(-1)^j\, y^{(j)}(1)\, q_j(1-x)\} \qquad \text{…………(1.10)}$$

$$q_j(x) =( x^j/j!)(1-x)^{n+1} \sum_{s=0}^{n-j} \binom{n+s}{s} x^s= Q_j(x)/j! \qquad \text{...………..(1.11)}$$

so that (1.10) with (1.11) satisfies

$$y^{(r)}(0)= P_{2n+1}^{(r)}(0) \ , \quad y^{(r)}(1)= P_{2n+1}^{(r)}(1) \ , \qquad r=0,1,2,\dots,n.$$

implying that $P_{2n+1}(x)$agrees with the appropriately truncated Taylor series for y(x) about x = 0 and x = 1.The error on [0, 1] is given by

$R_{2n+1}$=y(x)-$P_{2n+1}$(x)= $\dfrac{(-1)^{n+1}x^{(n+1)}(1-x)^{n+1}y^{(2n+2)}(\varepsilon)}{(2n+2)!}$ where $0 \langle \varepsilon \langle 1$ and $\mathbf{y}^{(2n+2)}$ is assumed to be continuous.

The osculatory interpolant for $P_{2n+1}$(x) may converge to y(x) in [0,1] irrespective of whether the intervals of convergence of the constituent series intersect or are disjoint .The important consideration here is whether $R_{2n+1} \rightarrow 0$ as $n \rightarrow \infty$ for all x in [0,1]. In the application to the boundary value problems in this thesis such convergence with n is always confirmed numerically .We observe that (1.10) fits an equal number of derivatives at each end point but it is possible and indeed sometimes desirable to use polynomials which fit different numbers of derivatives at the end points of an interval. As an example of a two-point osculatory interpolant we may take n = 2 so that (1.10) with (1.11)becomes the quintic

$$P_5(x)=(1-x)^3(1+3x+6x^2)y(0)+x^3(10-15x+6x^2)y(1)+x(1-x)^3(1+3x)y'(0)-x^3(1-x)(4-3x)y'(1)+1/2x^2(1-x)^3y''(0)+1/2x^3(1-x)^2y''(1)$$

Satisfying

$P_5(0)=y(0)$ , $P'_5(0)=y'(0)$ , $P''_5(0)=y''(0)$ .
$P_5(1)=y(1)$ , $P'_5(1)=y'(1)$ , $P''_5(1)=y''(1)$ .

Finally we observe that (1.17) can be written directly in terms of the Taylor coefficients $a_i$ and $b_i$ about x = 0 and x = 1 respectively, as

$$P_{2n+1}(x)=\sum_{j=0}^{n} \{ a_j Q_j(x) + (-1)^j b_j Q_j(1-x) \}. \quad \ldots .(1.12)$$

# CHAPTER
# TWO
# On Osculatrary interpoiation

## 2.1. INTRODUCTION

The most general form of the problem to be considered is:

$$y'' = f(x, y, y'), \quad x \in [a, b],$$

with boundary conditions : $y(a) = A$ , $y(b) = B$

there is no loss in generality in taking $a = 0$ and $b = 1$, and we will sometimes employ this slight simplification. We view f as a generally nonlinear function of y and y', but for the present, we will take $f = f(x)$ only. For such a problem to have a solution it is generally necessary either that $f(x) \neq 0$ hold, or that $A \neq 0$ at one or both ends of the interval. When $f(x) \equiv 0$, and $A = 0$ , $B=0$ the BVP is said to be homogeneous and will in general have only the trivial solution, $y(x) \equiv 0$.[25].In this chapter we introduce a new technique for the qualitative and quantitative analysis of nonhomogeneous TPBVP using two-point polynomial interpolation,

## 2.2. SOLUTION OF TWO POINT SECOND-ORDER BOUNDARY VALUE PROBLEMS

We consider the boundary value problem

$$y'' + f(x, y, y') = 0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots (2.1)$$

$$g_i(y(0), y(1), y'(0), y'(1)) = 0, \quad i=1, 2 \quad \ldots\ldots (2.2)$$

where f , $g_1$, $g_2$ are in general nonlinear functions of their arguments and $g_1$ and $g_2$ are given in three kinds.

1- $y(0)=a_0$ $,y(1)=b_0$ ……(2.2a), and we say this kind Dirichlet condition(value specified).

2- $y'(0) = a_1$ $,y'(1) = b_1$ ….(2.2b) , and we say this kind Neumann condition(Derivative specified).

3- $c_0y'(0)+c_1y(0)=a$ , $d_0y'(1)+d_1y(1)= b$ ….(2.2c), where $c_0,c_1,d_0,d_1$ are all positive constants not all are zero but $c_1,d_0$ are equal to zero or $c_0,d_1$are equal to zero and we say this kind Mixed condition (Gradient & value).

The simple idea behind the use of two-point polynomials is to replace y(.x) in problem (2.1)–(2.2), or an alternative formulation of it, by a $P_{2n+1}$ which enables any unknown boundary values or derivatives of y(x) to be computed .The first step therefore is to construct the $P_{2n+1}$ . To do this we need the Taylor coefficients of y (x) at x = 0:

$y = a_0 + a_1 x + \sum_{i=2}^{\infty} a_i x^i$ ….(2.3a) into (2.1)and equate coefficients of powers of x. The resulting system of equations can be solved to obtain $a_i$ $(a_0, a_1)$for all $i \geq 2$. Also we need the Taylor coefficients of y (x) at x = 1. Using MATLAB throughout we simply insert the series forms:

$$y = b_0 + b_1 (x-1) + \sum_{i=2}^{\infty} b_i (x-1)^i \quad \text{……..(2.3b)}$$

into (2.1) and equate coefficients of powers of (x−1). The resulting system of equations can be solved to obtain $b_i$ $(b_0, b_1)$ for all $i \geq 2$.The notation implies that the coefficients depend only on the indicated unknowns $a_0$, $a_1$, $b_0$, $b_1$. The algebraic manipulations needed for this process .We are now in a position to construct a $P_{2n+1}(.x)$ from (2.3) of the form (1.10) and use it as a replacement in the problem (2.1)–(2.2). Since we have only the four unknownsto compute for any n we only need to generate two equations from this procedure as two equations are already supplied by the boundary conditions (2.2).An obvious way to do this would be to satisfy the equation (2.1) itself at two selected points $x = c_1$, $x = c_2$ in [0,1] so that the two required equations become

$$P''_{2n+1}(c_i) + f\{P_{2n+1}(c_i),\, P'_{2n+1}(c_i),\, c_i\} = 0, \quad i=1,2. \quad \ldots\ldots\ldots\ldots(2.4)$$

An alternative approach is to recast the problem in an integral form before doing the replacement. Extensive computations have shown that this generally provides a more accurate polynomial representation for a given n. We therefore use this alternative formulation throughout this thesis although we should keep in mind that the procedure based on (2.4) is a viable option and shares many common features with the approach outlined below. Of the many ways we could provide an integral formulation we adopt the following. We first integrate (2.1) to obtain

$$y'(x) - a_1 + \int_0^x f(y(s), y'(s), s)\, ds = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(2.5)$$

and again to find

$$y(x) - a_0 - x\, a_1 + \int_0^x (x-s)\, f(y(s), y'(s), s)\, ds = 0 \quad \ldots\ldots\ldots\ldots\ldots(2.6)$$

where $a_0 = y(0)$ and $a_1 = y'(0)$. Putting x = 1 in (2.5) and (2.6) then gives

$$b_1 - a_1 + \int_0^1 f(y(s), y'(s), s)\, ds = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2.7)$$

and

$$b_0 - a_0 - a_1 + \int_0^1 (1-s)\, f(y(s), y'(s), s)\, ds = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(2.8)$$

where $b_0 = y(1)$ and $b_1 = y'(1)$.

The precise way we make the replacement of y(x) with a $P_{2n+1}(x)$ in (2.7) and (2.8)depends on the nature of f(y,y',x) and will be explained in the examples which follow. In any event the important point to note is that once this replacement has been made, the equations (2.2), (2.7) and (2.8) constitute the four equations we require to determine the set $\{a_0, b_0, a_1, b_1\}$. As we shall see the fact that the number of unknowns is independent of the number of derivatives fitted represents perhaps the most important feature of the method. We make the following points at this stage :

(i) In the majority of cases where the boundary conditions are simple enough the system of algebraic equations may be reduced a priori to a system in two unknowns, since the boundary condition can be substituted directly into the integral formulations, which MATLAB can be utilized to solve. That is, if we have the BC(2.2a),then we have only the unknown pair $\{a_1, b_1\}$and is known the required polynomial can be constructed. For the benefit of the reader the entire procedure for Examples in section 2.3 .And if, we have the BC(2.2b), then we have only the unknown pair $\{a_0, b_0\}$and is known the required polynomial can be constructed. Also if, we have the BC(2.2c), then we have only the unknown pair $\{a_0, b_1\}$or $\{a_1, b_0\}$ and is known the required polynomial can be constructed.

(ii) The method offers a certain amount of flexibility. For example we could choose to satisfy at two internal points or we could use alternative integral formulations. The fact remains that whatever strategy we adopt produces a quickly convergent sequence of values of the set $\{a_0, a_1, b_0, b_1\}$ as n increases.

(iii) Throughout we assess the accuracy of the procedure by examining the convergence with n. Using a symbolic computational facility such as MATLAB, computing the required convergent is not an issue. Where possible we can also run checks on our solutions using shooting with MATLAB codes.

(iv) We compare our method with the other method. We now consider a number of examples designed to illustrate the convergence, accuracy, implementation and utility of the method. In what follows the use of bold digits in the tables is intended to give a rough visual indication of the convergence.

**Remark**

1- All computations in the following examples were performed in the MATLAB environment, Version 7, running on a Microsoft Windows 2003 Professional operating system .

2- In the following examples when analytical solutions are known so that we can measure the error of a solution. When analytical solutions are not known, we compare our results to values computed by other methods .

## 2.3. EXAMPLES

In this section we introduce some examples illustrates suggested method:

### 2.3.1. A linear problem

Linear boundary value problems (BVPs) can be used to model several physical phenomena. For example, a common problem in civil engineering concerns the deflection of a beam of rectangular cross section subject to uniform loading, while the ends of the beam are supported so that they undergo no deflection. This problem is linear second-order TPBVP[6] .Now, we give many other examples, we first consider the linear problem with Dirichlet BC :

**Example 1**

$$y'' - 4(y - x) = 0 \quad , \quad y(0) = 0 \;, \quad y(1) = 2 \;\ldots\; (2.9)$$

has exact solution [6] : $e^2 (e^4 - 1)^{-1} (e^{2x} - e^{-2x}) + x$

Here (2.7) and (2.8) become

$$b_1 - a_1 + 2 - 4\int_0^1 y(s)\,ds = 0 \qquad (2.10)$$

$$8/3 - a_1 - 4\int_0^1 (1-s)\,y(s)\,ds = 0 \qquad (2.11)$$

and the coefficients : $a_2$ ,$b_2$, $a_3$ , $b_3$ ,…can be found from (2.3a) and (2.3b) .Ab initio inclusion of the boundary conditions in (2.9) has reduced the number of unknowns to two, namely $\{a_1, b_1\}$, which are computed by solving (2.10) and (2.11) with y(s) replaced by a $P_{2n+1}(s)$. The results for n = 2, 3,4 are displayed in Table 1. We can see that there is clear convergence with n to the 'exact' values which are obtained using MATLAB boundary value software. Table 2 gives the compare between the suggested method and other methods and figure 1 gives the accuracy of the method.

TABLE 1: The result of the methods for n= 2, 3, 4 of example1

| | | $P_5$ | $P_7$ | $P_9$ | |
|---|---|---|---|---|---|
| $a_1$ | | 1.5511387164 | 1.5514458006 | 1.5514410832 | |
| $b_1$ | | 3.0749482402 | 3.0746246085 | 3.0746294890 | |
| **X** | **Y:exact** | $P_5$ | $P_7$ | $P_9$ | $\|Y-P_9\|$ |
| 0.25 | 0.3936766919 | 0.3937912461 | 0.3936753464 | 0.3936767011 | 0.000000009170200 |
| 0.5 | 0.8240271368 | 0.8244047619 | 0.8240204194 | 0.8240272117 | 0.000000074822647 |
| 0.75 | 1.3370861339 | 1.3372355396 | 1.3370844322 | 1.3370861455 | 0.000000011556981 |
| **S.S.E = 5.81608482230229E-15** | | | | | |

Then from table 1 and the relation (1.10)and (1.11) in the previous chapter we have:

**$P_5$=.121739*x^5-.662526e-1*x^4+.393375*x^3+1.55114*x**

**$P_7$=.114177e-1*x^7-.905686e-2*x^6+.804532e-1*x^5-.189035e-2*x^4 +.367631*x^3 +1.55145*x**

**$P_9$=.628069e-3*x^9-.652397e-3*x^8+.774834e-2*x^7-.403156e-3*x^6 + 0.0736107*x^5+0.367627*x^3 +1.55144*x**

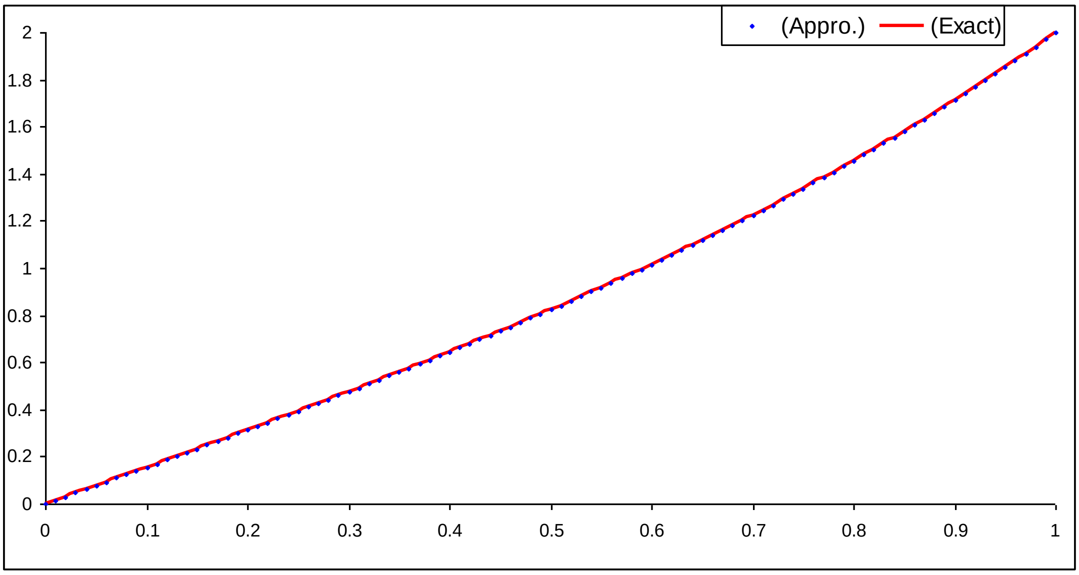


**Figure1:Comparison between the exact solution and semi-analytic method $P_9$**

Now we give the comparison between the solution of suggested method and solution of other methods in the following table :

TABLE 2: A Comparison between $P_9$ and other methods of example 1

| X | Y | $\Phi_1$ by using linear shooting method | $\Phi_2$ by sing linear Finite-Difference method | $P_9$ by using Oscillatory interpolation | $\|Y-P_9\|$ |
|---|---|---|---|---|---|
| 0.25 | 0.393676692 | 0.3936767 | 0.39367669 | 0.3936767011 | 0.0000000092 |
| 0.5 | 0.824027137 | 0.8240271 | 0.82402714 | 0.8240272117 | 0.0000000748 |
| 0.75 | 1.337086134 | 1.337086 | 1.33708613 | 1.3370861455 | 0.0000000116 |

## Example 2:

$$y''=\{-y'+(x+2)\,y-(2-(x+1)^2)\,e^1\ln(2)+2\,e^x\}/(x+1)$$

with BC: $y(0) = 0$ , $y(1) = 0$

Exact solution [6] $y(x) = e^x \ln(x+1) - (e^1 \ln(2))\,x$ .

The results of solution given in the following table :

Table 3: The result of the methods for n = 2, 3, 4 of example2

| | | $P_5$ | $P_7$ | $P_9$ | |
|---|---|---|---|---|---|
| $a_1$ | | -0.88416918 | -0.88416925 | -0.88416940 | |
| $b_1$ | | 1.35913519 | 1.35914462 | 1.35914055 | |
| **x** | **Y** | $P_5$ | **P7** | **P9** | $\lvert Y-P_9 \rvert$ |
| 0.25 | -0.1845203549 | -0.1845324937 | -0.1845173602 | -0.1845205972 | 0.000000242273476 |
| 0.5 | -0.2735857444 | -0.2735962538 | -0.2735782173 | -0.2735865577 | 0.000000813249308 |
| 0.75 | -0.2284204067 | -0.2284181840 | -0.2284189827 | -0.2284205035 | 0.000000096793373 |
| **S.S.E=7.29439830 658099E-13** | | | | | |

Therefore: $\mathbf{P_9 = 0.0015x^9 - 0.0091x^8 + 0.0257x^7 - .0410x^6 + 0.737x^5 + 0.3333x^3 + 0.5000x^2 - 0.8842x}$

The accuracy of the solution given in the following figure:

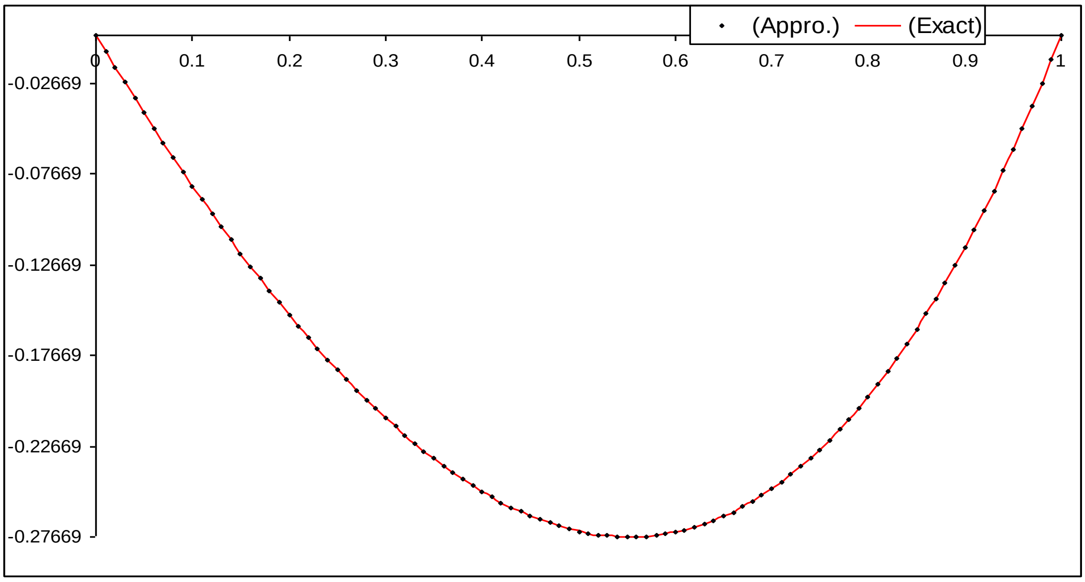


**Figure2: A comparison between exact and approximate solution of example2**

Now we give the comparison between the solution of suggested method and solution of other methods in the following table :

Table4: A Comparison between $P_9$ and other methods of example 2

| $x_i$ | Y:exact | $\Phi_1$ by using cubic splines | $\Phi_2$ by another numerical solution | $P_9$ by using Osculatory interpolation | \| Y-$P_9$ \| |
|---|---|---|---|---|---|
| 0.25 | -0.1845203549 | -0.1845191 | -0.1846134 | -0.1845205970 | 0.0000002423 |
| 0.5 | -0.2735857444 | -0.2735833 | -0.2737099 | -0.2735865580 | 0.0000008132 |
| 0.75 | -0.2284204067 | -0.2284186 | -0.2285169 | -0.2284205030 | 0.0000000968 |

Now we give the linear problem with Neumann BC :

**Example 3**

$$y'' - \pi^2 y + 2\pi^2 \sin(\pi x) = 0 \quad \text{BC: } y'(0) = \pi \ , \ y'(1) = -\pi$$

With Exact solution [6] : $\sin(\pi x)$

In this case equation (2.3) gives $\{a_i , b_i\}$ where i= 2,3,…,so we have four unknown $\{a_0 , a_1 , b_0 , b_1\}$ then BC give $a_1$ ,$b_1$,thus solving (2.7)and (2.8) to obtain $a_0$ , $b_0$ .The result of method given in the following table :

Table 5 :The result of the methods for n = 2, 3, 4 of example 3

| | | $P_5$ | $P_7$ | $P_9$ | |
|---|---|---|---|---|---|
| $a_0$ | | 0.0071286292 | 0.0002701079 | 0.0000066077 | |
| $b_0$ | | 0.0071286292 | 0.0002701079 | 0.0000066077 | |
| **X** | **Y** | $P_5$ | $P_7$ | $P_9$ | $\|Y-P_9\|$ |
| 0 | 0.0000000000 | 0.0071286292 | 0.0002701079 | 0.0000066077 | 0.000006607700484 |
| 0.1 | 0.3090169943 | 0.3156038133 | 0.3092862669 | 0.3090237892 | 0.000006794799366 |
| 0.2 | 0.5877852522 | 0.5911087921 | 0.5879637222 | 0.5877906400 | 0.000005387754140 |
| 0.3 | 0.8090169943 | 0.8069586888 | 0.8089626630 | 0.8090164488 | 0.000000545560411 |
| 0.4 | 0.9510565162 | 0.9440928773 | 0.9507420034 | 0.9510478432 | 0.000008673077361 |
| 0.5 | 1.0000000000 | 0.9910749818 | 0.9995720325 | 0.9999874662 | 0.000012533840896 |
| 0.6 | 0.9510565162 | 0.9440928773 | 0.9507420034 | 0.9510478432 | 0.000008673077361 |
| 0.7 | 0.8090169943 | 0.8069586888 | 0.8089626630 | 0.8090164488 | 0.000000545560411 |
| 0.8 | 0.5877852522 | 0.5911087921 | 0.5879637222 | 0.5877906400 | 0.000005387754140 |
| 0.9 | 0.3090169943 | 0.3156038133 | 0.3092862669 | 0.3090237892 | 0.000006794799367 |
| 1 | 0.0000000000 | 0.0071286292 | 0.0002701079 | 0.0000066077 | 0.0000066077005 |
| **S.S.E = 5.45854779363699E-10** | | | | | |

Then from table 5 and the relation (1.10)and (1.11) in the previous chapter we have :

**$P_5$= 3.17677*x^4-6.35354*x^3+.351784e-1*x^2+3.14159*x+.712863e-2**

**$P_7$=-.900000e-18*x^7-1.11814*x^6+3.35442*x^5-.211490*x^4-.16771* x^3 +.133293e-2*x^2+3.14159*x+.270108e-3**

**$P_9$=-.406544e-17*x^9+.205015*x^8-20059*x^7+.114551*x^6+2.52655* x^5+.268188e-4*x^4-5.16771*x^3+.326077e-x^2+3.14159*x+.660770e-5**

The accuracy of the solution given in the following figure:

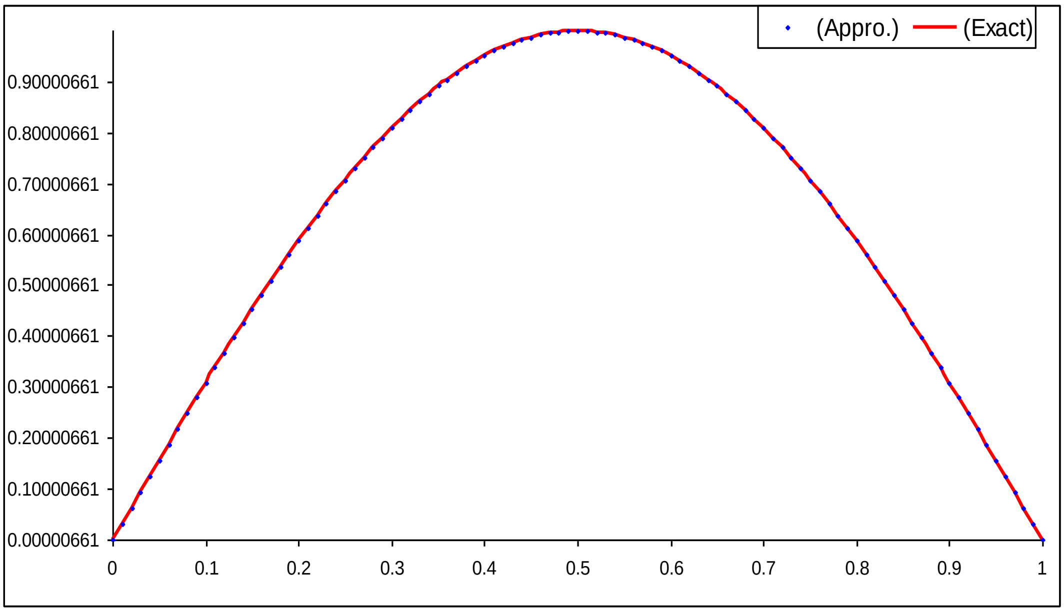


**Figure3: A comparison between exact and approximate solution of example3**

Now we give the comparison between the solution of suggested method and solution of other methods in the following table:

Table 6: A Comparison between $P_9$ and other methods of example 3

| $x_i$ | Y exact | $\Phi_1(x_i)$ by using B-splines | $\Phi_2(x_i)$ by using piecewise linear Rayleigh-Rit | $P_9$ by using Osculatory interpolation | $\lvert Y-P_9\rvert$ |
|---|---|---|---|---|---|
| 0 | 0.0000000000 | 0 | 0.0000000000 | 0.0000000000 | 0.0000000000 |
| 0.1 | 0.3090169943 | 0.30901644 | 0.3102866742 | 0.3090187881 | 0.0000017937 |
| 0.2 | 0.5877852522 | 0.58778549 | 0.5902003271 | 0.5877867479 | 0.0000014956 |
| 0.3 | 0.8090169943 | 0.80901687 | 0.8123410598 | 0.8090132782 | 0.0000037161 |
| 0.4 | 0.9510565162 | 0.95105667 | 0.9549641896 | 0.9510450787 | 0.0000114376 |
| 0.5 | 1.0000000000 | 1.00000002 | 1.0041087710 | 0.9999848327 | 0.0000151673 |
| 0.6 | 0.9510565162 | 0.95105713 | 0.9549641893 | 0.9510450787 | 0.0000114376 |
| 0.7 | 0.8090169943 | 0.80901773 | 0.8123410398 | 0.8090132782 | 0.0000037161 |
| 0.8 | 0.5877852522 | 0.5877869 | 0.5902003271 | 0.5877867479 | 0.0000014956 |
| 0.9 | 0.3090169943 | 0.3090181 | 0.3102866742 | 0.3090187881 | 0.0000017937 |
| 1 | 0.0000000000 | 0 | 0.0000000000 | 0.0000000000 | 0.0000000000 |

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