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On Extending Modules

A research

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بِسْمِ اللَّهِ الرَّحْمَنِ الرَّحِيمِ

أمن هو قانتٌ آناءَ الليلِ ساجداً وقائماً يحذرُ الاخرةَ ويرجو رحمةَ

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Dedication

I dedicate this humble to cry resounding silence in the sky to the martyrs of Iraq wounded. Also, I dedicate my father treasured, also I dedicate to my supervisor **Dr. Tha'ar Younis Ghawi**. Finally, to everyone who seek knowledge, I dedicate this humble work.

Acknowledgment

And later the first to thank in this regard is the Almighty God for the blessing that countless, including the writing of this research modest. Then thanks and gratitude and deepest gratitude to my teacher **Dr. Tha'ar Younis Ghawi**, *which prefer to oversee the research and guidance in spite of concern, and not to ask him reordering Almighty to help him I also wish to express my thanks to the staff of the department of mathematics.*

CONTENTS

Introduction

Throughout, all rings are associative with identity and a modules are left unitary. In this work we will study the concep of closed submodules which is weaker than the concept of direc summand. By using this concept we study the class of extendin modules, where an R-module M is called extending if every close submodule of M is a direct summand. Many results about thes two concepts are given, also many relationships with othe related concepts are introduced.

CHAPTER ONE

Background of Modules

Definition 1.1 [1] A module M is said to be semisimple if $\forall N \leq M \exists K \leq M \ni N \oplus K = M$.

Definition 1.2 Let M be an R module A subset X of M is called basis of M iff :

(1) X is generated M , i.e. $M = \langle X \rangle$.

(2) X is linearly independent , that is for every finite subset$\langle x_1, x_2, \dots, x_n \rangle$ of X with $\sum_{i=1}^{n} X_i \propto_i = 0$, $\forall \propto_i \in R$ $then$ $\propto_i = 0$, $\forall 1 \leq i \leq n$.

Definition 1.3 An R-module M is said to be free if satisfy the following condition :

(1) M has basis.

(2) $M = \oplus_{\forall i \in I} A_i \wedge \forall i \in I\ [A_i \equiv R_R]$.

Example 1.4 Z as Z-module is a free module.

Example 1.5 Z as Z-module is free since ⟨1⟩=Z

$\langle 1 \rangle = \{1.a | a \in Z\} = \{\dots, -3, -2, -1,0,1,2,3, \dots\}$

And $\forall \propto \in Z, \propto .1 = 0 \Rightarrow \propto = 0$.

Zoren's lemma 1.6 If A is non-empty partial order set such that every chain in A has an upper bound in A, then A has maximal element.

Modular law 1.7 [4] If $A, B, C \leq M \wedge B \leq C$, $then$ $(A + B) \cap C = (A \cap C) + (B \cap C) = (A \cap C) + B$.

Theorem 1.8 If $\propto: M \to N$, $\beta N \to K$ modular homomorphism on R-ring then $\ker(\beta \propto) = \propto^{-1} (\ker(\beta))$.

Proof. Let $x \in \ker(\beta \propto) \to \beta \propto (x) = 0' \to \left(\propto (x)\right) = 0' \to \propto (x) \in \ker(\beta) \to x \in \propto^{-1} (\ker(\beta))$. So $\ker(\beta \propto) \subseteq \propto^{-1} (\ker(\beta))$ … (1)

Let $x \in \propto^{-1} (\ker(\beta)) \to \propto (x) \in \ker(\beta) \to \left(\propto (x)\right) = 0' \to \beta \propto (x) = 0' \to x \in \ker(\beta \propto)$. So $\propto^{-1} (\ker(\beta)) \subseteq \ker(\beta \propto)$ … (2)

Form (1),(2)→ $\ker(\beta \propto) = \propto^{-1} (\ker(\beta))$.

Theorem 1.9 [4] If $\propto: M \to N$, $\beta: N \to K$ modular homomorphism on R-ring then if $A \leq M$ $then$ $\propto^{-1} \left(\propto (A)\right) = A + \ker(\propto)$.

Proof. Let $x \in \propto^{-1} (\propto (A)) \to \propto (x) \in \propto (A)$.

Then $\exists b \in A \ \ni \propto (x) = \propto (b)$

$\to \propto (x - b) = 0' \to x - b \in \ker(\propto)$, $then \ \exists k \in \ker(\propto) \ni x - b = k$

$\to x = b + k \to x \in A + \ker(\propto)$ $[since \ k \in \ker(\propto), b \in A]$

So $\propto^{-1} \left(\propto (A)\right) \subseteq A + \ker(\propto)$ ... (1)

Let $x \in A + \ker(\propto)$, $then \ \exists b \in B$, $k \in \ker(\propto) \ni x = b + k$

$\to \propto (x) = \propto (b + k) \to \propto (x) = \propto (b) + \propto (k)$

$\to \propto (x) = \propto (b)[since \ k \in \ker(\propto)] \to x \in \propto^{-1} (\propto (A))$

So $A + \ker(\propto) \subseteq \propto^{-1} \left(\propto (A)\right)$ ... (2)

So from (1) ,(2) we get $\propto^{-1} \left(\propto (A)\right) = A + \ker(\propto)$.

Definition 1.10 [4] Let $A \leq M$ then $B \leq M$ is called addition complement of A in M (briefly adco) iff :

(1)A+B=M

(2)$B \leq M$ minimal in A+B=M , i.e $\forall B \leq M$ with A+B=M , i.e $\forall U \leq M$ with A+U=M and $U \leq B$ imply U=B

$D \leq M$ is called intersection complement of A in M (beieflyinco) iff

(1) $A \cap D = 0$

(2) D is a maximal in $A \cap D = 0$

i.e. $\forall C \leq M$ with $A \cap C = 0 \ \wedge D \leq C$ implies C=D.

Corollary 1.11 Let $A \leq M \ \ and \ B \leq M \ \ \ then \ \ A \oplus B = M \Leftrightarrow$ B is adco and inco of A in M.

Proof. $\Longrightarrow$) Suppose that B is adco and inco of A

Then A+B=M resp. $A \cap B = 0 \ \Longrightarrow M = A \oplus B$

$\Longleftarrow$) Suppose that $A \oplus B = M$, hence A+B=M and $A \cap B = 0$

Let $C \leq M$ with A+C=M and $C \leq B, (A + C) \cap B = M \cap B \Longrightarrow (A + C) \cap B = B \rightarrow (A \cap B) = C = B \Longrightarrow C = B[A \cap B = 0]$

So B is adco of A in M

Let $C \leq M$ with $A \cap C = 0 \ \ and \ B \leq C$

Since A+B=M⟹A+C=M [since $A + B \subseteq A + C$]

$\rightarrow A \oplus C = M \Longrightarrow A \oplus C = A \oplus B$ [$A \oplus B = M$by assumption]

$\frac{A \oplus C}{A} = \frac{A \oplus B}{A} \Longrightarrow C = B \longrightarrow$ so B is inco of A in M.

Lemma 1.12 [3] Let M=A+B , then we have B is adco of A in M $\Leftrightarrow A \cap B \ll B$.

Proof. $\Longrightarrow$) $let \ U \leq B \ (A \cap B) + U = B$

Then $M = A + (A \cap B) + U \Longrightarrow A + U = M$ $[since \ A \cap B \subseteq A]$

But B is so $A \cap B \ll B$

$\Longleftarrow$) We have by assumption M=A+B , let $U \leq M$ with A+U=M and $U \leq B$

$\rightarrow (A+U) \cap B = M \cap B \rightarrow (A+U) \cap B = B$ $[B \leq M]$ $\rightarrow (A+B) \cap U = B$ [by modular law]

But $A \cap B \ll B$, hence U=B , thus B is adco to A in M.

CHAPTER TWO

1. Essential Extensions

Definition 2.1.1 Consider a submodule A of a module B . We say that B is an essential extension of A if every nonzero submodule of B has nonzero inter-section with A. We also say that A is an essential submodule (or a large submodule) of B , and we writ $A \leqq_e B$ to denote this situation. In order to test for this condition, we need only check whether all nonzero cyclic submodules of B have nonzero intersection with A, whicg is equivalent to the condition that every nonzero element of B has a nonzero multiple in A. Note that A always has at least one essential extension, since $A \leqq_e A$. *Also, note that* $0 \leqq_e A \ iff \ A = 0$.

Proposition 2.1.2

(i) If $A \leqq B \leqq C, then\ A \leqq_e C\ if\ and\ only\ if\ A \leqq_e B \leqq_e C$.

(ii) If $A \leqq_e B \leqq C\ and\ \grave{A} \leqq_e \grave{B} \leqq C_,\ then\ A \cap \grave{A} \leqq_e\ B \cap \grave{B}$.

(iii) If $F: B \to C\ and\ A\ \leqq_e C_,\ than\ f^{-1}\ A \leqq_e B$

(iv) If $\{A\alpha\}$ is an independent family of submodules of C, and if $A_\alpha \leqq_e B_\alpha \leqq C$

$for\ each\ a_,\ than\ \{B\alpha\}\ is\ an\ independent\ family\ and\ \oplus A_\propto\ \leqq_e \oplus B_\alpha$.

Proof: (i) First let $A \leqq_e B\ \leqq_e C\ and\ consider\ any\ nonzero\ M \leqq C$.

Since $B \leqq_e C\ we\ have\ M \cup B \neq 0$, and then since $A \leqq_e B$ we obtain $(M \cap B) \cap A \neq 0$, that is, $M \cap A \neq 0$. Thus $A \leqq_e C$.

Now assume that $A \leqq_e C$. Since any nonzero submodule of C has nonzero intersection with A, the same can be said for nonzero submodule of C. Hence $A \leqq_e B$. Also ,since any nonzero submodule M of C satisfies $M \cap C\ satisfies\ M \cap A \neq 0, it\ mus\ satfy\ M \cap B \neq 0; thus\ B \leqq_e C$.

(v) If M is any nonzero submodule of B$\cap B'$, $than\ since\ A \leqq_e B\ we\ have\ M \cap A \neq 0. Since\ \grave{A} \leqq_e\ \grave{B}\ as\ well, we\ obtain\ (M \cap A) \cap \grave{A} \neq 0, and\ thus\ A \cap \grave{A} \leqq_e\ B \cap \grave{B}$

(vi) If not, than B has a nonzero submodule of M such that $M \cap f^{-1}A\ = 0$.

In particular, M$\cap$ $(kerf) = 0; hence\ fmaps\ m\ isomrhically\ onto\ FM, so\ that\ fM\ so\ a\ nonzero$

Sub module of C. However, since M$\cap$ $f^{-1}\ A = 0\ we\ obtain\ fM \cap A = 0, which\ is\ impossible.$

(vii) First consider the case when the index set consists of exactly two elements, say $\{1, 2\}. According\ to\ (ii)\ 0 \leqq_e B_1 \cap= B_2; hence\ B_1 \cap B_2 = 0\ and\ so\{B_1, B_2\}$ is independent. Applying (iii) to the projection maps $B_1 \oplus B_{2\to} B_1$ and $B_1 \oplus B_{2\to} B_2$, we obtain $A_1 \oplus B_2 \leqq_e B_1 \oplus B_2$ and $B_1 \oplus A_2 \leqq_e B_1 \oplus B_2$; hence it follows from (ii) that $A_1 \oplus A_2 \leqq_e B_1 \oplus B_2$ Thus (iv) hold for index sets with two elements. Now consider the case when the index set consists of $\{1,2,\dots,n\}$, and assume that (iv) holds for index sets with $n-1$ elements. Then $\{B_1 \cdots, B_{n-1}\}$ is independent, and $A_1 \oplus \cdots \oplus A_{n-1} \leqq_e B_1 \oplus \cdots \oplus B_{n-1}$.Using the case above, we see that $(A_1 \oplus \cdots \oplus A_{n-1}) \oplus A_n \leqq_e (B_1 \oplus \cdots \oplus B_{n-1}) \oplus B_n$

Therefore (iv) holds for all finite index sets, and we are ready to prove the general case. Given distinct indices $\propto(0), \propto(1), \cdots, \propto(n)$, we know that $\{B_{\propto(0)}, \cdots, B_{\propto(n)}\}$ is independent, whence $B_{\propto(0)} \cap (B_{\propto(1)} + \ldots + B_{\propto(n)}) = 0$. Thus $\{B_\propto\}$ is independent. Now any nonzero sub module$M \leqq \oplus B_\propto$ contains a nonzero element, which must belong to $B_{\propto(1)} \oplus \ldots \oplus B_{\propto n}$ for some $\propto(i)$ As a result, $M \cap (B_{\propto(1)} \oplus \ldots \oplus B_{\propto(nn)} \neq 0$, from which we obtain

$$M \cap (B_{\propto(1)} \oplus \ldots \oplus B_{\propto(n)}) \cap (A_{\propto(1)} \oplus \ldots \oplus A_{\propto(n)}) \neq 0$$

And consequently $M \cap (\oplus A_\propto) \neq 0$. Therefore $\oplus A_\propto \leqq_e \oplus B_\propto$.

Remark 2.1.3 We note that 1.1(ii) may fail for infinite intersections. For example, nZ $\leqq_e Z$ for all positive integers n, and yet $\cap\,(nZ) = 0$, which is essential in Z. Also, 1.1(ii) may fail if the family $\{A_\propto\}$ is not independent, as the following example shows.

$A + \grave{A} \not\leqq_e B + \grave{B}$.

Proof: Set $R = Z, C = Z\oplus(Z/2Z)$, $A = \grave{A} = (0,2)R$, $B = (1,0)R$, and $\grave{B} = (1,\bar{1})R$. Any nonzero element of $\grave{B}$ has the form $(n,\bar{n})$ for some nonzero $n \in Z$, and $(n,\bar{n})2 = (2n,0)$ is a nonzero element of $\grave{A}$. Thus

$\grave{A} \leqq_e \grave{B}$, and similarly $A \leqq_e B$. Observing that $(0,\bar{1})R \cap A = 0$, we see that $A \not\leqq_e C$,

That is, $A + \grave{A} \not\leqq_e B + \grave{B}$.

Definition 2.1.4 Let A be a sub module of C. A relative complement for A in C is any sub module B of C which is maximal with respect to the property $\boldsymbol{A \cap B = 0}$. Such sub modules B always exist, by virtue of $\boldsymbol{zren's}$ Lemma; in fact, any sub module $\boldsymbol{B_0}$ ***of C satisfying*** $\boldsymbol{A \cap B_0 = 0}$ can be enlarged to a relative com-plement for A. Of course, if A is actually a direct summand of C, say $\boldsymbol{C = A\oplus B}$, than the complementary summand B is a relative complement for A. for example, if F is field, $\boldsymbol{C = F\oplus F}$, ***and*** $\boldsymbol{A =\oplus 0}$, then for any $\boldsymbol{x \in F}$ the subspace $\boldsymbol{(x,1)F}$ is a relative complement for A in C. In case F is infinite, this provides an example in which A has infinitely many distinct relative com-elements in C. (See the importance of relative complements is that they can be used to construct essential submodules, as in the following proposition.

Proposition 2.1.5 Let $A \leqq C$. If B is any relative complement for A in C, then$A\oplus B \leqq_e C$.

Proof: Since $A \cap B = 0$, we have $A + B = A\oplus B$, so that $A\oplus B$ is a submodule if C. Suppose that $M \leqq C$ *with* $M \cap (A\oplus B) = 0$, Then the sum $(A\oplus B) + M$ is direct, that is, $(A\oplus B) + M = A\oplus B\oplus M$, whence $A\cap(B\oplus M) = 0$. By the maximality of B, we obtain $B\oplus M = B$ and thus $M = 0$. Therefore $A\oplus B \leqq_e C$.

Definition 2.1.6 A submodule A of a module C is said to be a closed submodule of C if A has no proper essential extensions inside C, that is, if the only solution of the relation $A \leqq_e B \leqq C \, is \, A = B$. For example, $0 \, and \, C$ and always closed submodules of C. Also, every direct summand of C is a closed submodule.

Proposition 2.1.7 If $B \leqq_e C$ then the following conditions are equivalent:

(i)B is a closed submodule of C.

(ii) B is relative complement for some $A \leqq C$.

(iii) If A is any relative complement for $B \, in \, C$, then B is a relative com plement for $A \, in \, C$.

(iv) $If \; B \leqq K \leqq_e C$, then $K/B \leqq_e C/B$.

Proof: $(i) \Longrightarrow (ii)$: $If \;\; M/B$ is submodule of $C/B \;\; such \, that \; (M/B) \cap (K/B) = 0, then \; M \cap K = B$. Since $K \leqq_e C, we \; have \; M \cap K \leqq_e M \cap C, i.e., B \leqq_e M$. The assumption that B is closed in C gives $B = M$, and thus $M/B = 0$.

$(ii) \Longrightarrow (iii)$: Since $A \cap B = 0$, B can be enlarged to a relative complement $\grave{B} for \; A$. By the modular law, $(A \oplus B) \cap \grave{B} = B + \left(A \cap \grave{B}\right) = B$, whence $[(A \oplus B)/B] \cap \left[\grave{B}/B\right] = 0$. According to 1.3, $A \oplus B \leqq_e C$, and then from(ii) we obtain $(A \oplus B)/B \leqq_e C/B$. Thus$\grave{B}/B = 0$, and so $B = \grave{B}$ is a relative complement for A.

$(iii) \Longrightarrow (ii)$is automatic.

$(ii) \Longrightarrow (i)$: Suppose that $B \leqq_e \grave{B} \leqq C$. Since $\left(\grave{B} \cap A\right) \cap B = A \cap B = 0, wehave \; \grave{B} \cap A = 0$ and then the maximality of B implies that $\grave{B} = B$. Thus B is closed in C.

Proposition 2.1.8 Let A' be a relative complement for A in B, and let B' be a relative complement for B in C. According to 1.3, $B \oplus \grave{B} \leqq_e C$; hence 1.4 shows that $(B \oplus \grave{B})/B \leqq_e C/B$. We now see from 1.1 that$(B \oplus \grave{B})/A \leqq_e C/A \, , or \; [B/A] \oplus \left[(A \oplus \grave{B})/A\right] \leqq_e C/A$. Using 1.3 and

1.4 again, we obtain $A \oplus \grave{A} \leqq_e B \; and \; then \; (A \oplus \grave{A})/A \leqq_e B/A$. According to 1.1, it follows that
$[(A \oplus \grave{A})/A] \oplus [(A \oplus \grave{B})/A] \leqq_e C/A \; or \; (A \oplus \grave{A} \oplus \grave{B})/A \leqq_e C/A$.

Now suppose that we have $A \leqq_e K \leqq C$. Since $A \cap (\grave{A} \oplus \grave{B}) = 0, 1.1 \; show \; that \; K \cap (\grave{A} \oplus \grave{B}) = 0$. Using the modular law, we find that $K \cap (A \oplus \grave{A} \oplus \grave{B}) = A, whence \; [K/A] \cap [(A \oplus \grave{A} \oplus \grave{B})/A] = 0$. Inasmuch as $(A \oplus \grave{A} \oplus \grave{B})/A \leqq_e C/A \, , we \; obtain \; K/A = 0, so \; that \; K = A$.Therefore A is closed in C.

Example 2.1.9 There exist module $A, B \leqq C$ such that A and B are closed in C, but $A \cap B$ is not closed in A,B or C.

Proof: Set $R = Z, \; C = Z \oplus (Z/2Z), A = (1,0)R, and \; B = (1, \bar{1})R$. Since A is a direct summand of C, it must be closed in C. Observing that $C = B \oplus (0, \bar{1})R$, we see that B is closed in C also. Note that $A \cap B = (2,0)R$. As observed in 1.2,$A \cap B \leqq_e A \; and \; A \cap B \leqq_e B$,whence $A \cap B$ is not closed in A,B or C.

2. On Extending Modules

Remark 2.2.1

(i) Let $\{A_\propto\}$ and $\{B_\propto\}$ be collections of modules such that $A_\propto$ is closed submodule of $B_\propto$ for each $\propto$. Hence $\oplus A_\propto$ must be closed in $\oplus B_\propto$.

(ii) Let $A \leqq B \leqq C$. If B is closed in C, then B/A is closed in C/A.

(iii) If A is closed in and $N \leqq_e C$then $A \cap N$ is closed in N.

Definition 2.2.2 A module M is called extending, or a CS-module, if every closed submodule is a direct summand. Equivalently, M is an extending module if and only if every submodule is essential in a direct summand of M .

Thus notion is the key one in this monograph and in this section we explore some of the basic properties of extending module.

Definition 2.2.3 An R-module M is called semisimple if 0 and M are only direct summand of M.

Example 2.2.4 Z_6 as Z-module semismple but Z is not semismple.

Definition 2.2.5 A nonzero R- module M is called uniform if all it is submodule are essential.

Example 2.2.6 Z as Z-module is uniform but Z_6 is not uniform as Z-module.

Definition 2.2.7 An R-module N is called injective if for Any R-module A, B and for any monomorphism $f: A \longrightarrow B$ and honomorphism $g: A \longrightarrow N$ there exist a homomorphism $h: B \longrightarrow N$ such that $g = h \circ f$.

Definition 2.2.8 A module M is said to be completely extending if every direct summand of M is extending.

Example 2.2.9

(ii) Any uniform module is CS. (so for instance, any subgroup of $\mathbb{Q}$ *is a CS module over* $\mathbb{Z}$).

(iii) Any injective module is CS. (More generally, any ``quasi-injective`` module, to be defined in $6G$, is already CS; see (6.80).)

(iv) A closed submodule N of a CS module M is always CS. In fact, let $C \leqq^c N$. By(6.24)(2)., we have $C \leqq^c M$. Since M is CS, C is a direct summand of M, and hence of N.

(vi)By (iv) and (i) above, $\mathbb{Z}^n$ *and* $\mathbb{Z}/p^m\mathbb{Z}$ (for any prime p) are CS module over $\mathbb{Z}$ .However, the $\mathbb{Z} - module\ M = \mathbb{Z} - module\ \mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ in (6.17) (iv) turns out to be not CS. (In the notation of (6.17)(5), the submodule $\grave{C}$ is easily seen to be closed in M, but it is not a direct summand) Thus, the direct sum of two CS modules may not be CS. For a complete determination of the f.g. CS module over $\mathbb{Z}$.

Lemma 2.2.10 Any direct summand of an extending module is extending.

Proposition 2.2.12 For any ring R, $\pi - injective$ R-module are extending.

Conversely, suppose that for every submodule $N\ of\ M\ with\ N \cap M_1 = 0\ there\ exists\ a\ submodule\ M'\ of\ M\ such\ that\ M = M_1 \oplus \grave{M}\ and\ N \subset \grave{M}. Let\ L\ be\ a\ submodule\ of\ M_2\ and\ g: L \longrightarrow M_1\ bea\ homomorphism. Put\ H = \{-(x)g + x | x \in L\}. Then\ H\ is\ a\ submodule\ of\ M\ and\ H \cap M_1 = 0. There\ exists\ a\ submodule\ \grave{H}\ of\ M\ such\ that\ M = M_1 \oplus \grave{H}\ and\ H \subset \grave{H}\ . Let\ \pi: MM_1\ denote\ the\ projection\ with\ kernel\ \grave{H}.\ Then\ \pi/M_2 : M_1 \longrightarrow M_2\ and\ for\ any\ x\ in\ L, (x)\pi = \big((x)g + (-(x)g + x)\big)\pi = (x)g. It\ follows\ that\ M_1\ is\ M_2 - injective.$

Lemma 2.2.13 Let $M = M_1 \oplus M_2$ $where$ M_1 and M_2 are both extending modules. Then M is extending if and only if every closed $K \subset M$ $with$ $K \cap M_1 = 0$ or $K \cap M_2 = 0$ is a direct summand of M.

Proof : The necessity is. Conversely, suppose that every closed $K \subset M$ $with$ $K \cap M_1 = 0$ or $K \cap M_2 = 0$ is a direct summand. Let $L \subset M$ be closed. There exists a complement H in L such that $L \cap M_2$ is essential in H. By 1.10(4), H is closed in M. Clearly $H \cap M_1 = 0$. By hypothesis, $M = H \oplus \grave{H}$ for some submodule $\grave{H}$ of M. Now $L = H \oplus (L \cap \grave{H})$. By1.10(4) again, $L \cap \grave{H}$ is closed in M. Also, clearly, $(L \cap \grave{H}) \cap M_2 = 0$. Hypothesis, $L \cap \grave{H}$ is a direct summand of M, and hence also of $\grave{H}$. It follows that L is a direct summand of M. Thus M is extending.

Proposition 2.2.14 Let $M = M_1 \oplus \cdot \cdot \cdot \oplus M_n$ be a finite direct summand sum of relatively incentive modules M_i. The M is extending if and only if all M_i are extending.

Proof : The necessity is clear by 7.3. Conversely, suppose that all M_i are extending. By induction on n, it is sufficient to prove that M is extending when $n = 2$. Let $K \subset M$ be a closed and $K \cap M_1 = 0$. By 7.5 there exists a submodule $\grave{M}$ of M such that $M = M_1 \oplus \grave{M}$ and $K \subset \grave{M}$. Clearly $\grave{M} \simeq M_2$, and hence $\grave{M}$ is extending. Clearly K is closed in $\grave{M}$ and hence K is a of $\grave{M}$, whence a direct summand of M. Similarly any direct summand closed $H \subset M$ $with$ $H \cap M_2 = 0$ is a direct summand. By 7.9, is an extending module.

Lemma 2.2.15 For any (right) R-module M, the following are equivalent:

(i)Every complement (i.e., closed submodule) in M is a direct summand.

(ii)For every submodule $A \leqq M$, there exists a direct summand C of such that $A \leqq_e C$.

Proof: $(2) \Longrightarrow (1)$ is trivial. $(1) \Longrightarrow (2)$ follows by taking C to be an essential closure of A in M.

If M_R satisfies (1),(2) above, M is an extending module.

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