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On the Norm of the composition operator C_ϕ

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Abstract

Let U denate the unit ball in the complix plane, the Hardy space H^2 is the set of functions $f(z)=\sum_{n=0}^{\infty} f^{\wedge}(n)z^n$ holomarphic on U such that $\sum_{n=0}^{\infty} \left|f^{\wedge}(n)\right|^2 < \infty$ with $f^{\wedge}(n)$ denates then the Taylor coefficient of f .

Let ψ be a holomarphic self-map of U, the composetion operater C_ψ induced by ψ is defined on H^2 by the equation

$$C_\psi f = f \circ \psi \qquad (f \in H^2)$$

We have studied the eignvalue of the composition operator induced by the bijective map σ and discussed the adjoint of the composition operator induced by the bijective map σ .We have look also at some known properties on composition operators and tried to see the analogue properties in order to show how the results are changed by changing the function ψ in U.

In order to make the work accessible to the reader, we have included some known results with the details of the proofs for some cases and proofs for the properties .

المستخلص

ليكن U يرمز إلى كرة الوحدة في المستوى العقدي، إن فضاء هاردي H^2 هو مجموعة كل الدوال

$f(z) = \sum_{n=0}^{\infty} f^{\wedge}(n)\ z^n$ التحليلية على U بحيث أن $\sum_{n=0}^{\infty} \left|f^{\wedge}(n)\right|^2 < \infty$ ، $f^{\wedge}(n)$ يرمز إلى معاملات تيلر

لتكن $\psi: U \to U$ دالة تحليلية على U ، المؤثر التركيبي المتولد من ψ يعرف على فضاء هاردي H^2 بواسطة:

$$C_\psi f = f \circ \psi \quad (f \in H^2).$$

درسنا في هذا البحث القيم الذاتية للمؤثر التركيبي المتولد من الدالة المتقابلة σ حيث ناقشنا المؤثر المرافق للمؤثر التركيبي المتولد من الدالة المتقابلة σ . بالإضافة إلى ذلك نظرنا إلى بعض النتائج المعروفة وحاولنا الحصول على نتائج مناظرة لنتمكن من ملاحظة كيفية تغير النتائج عندما تتغير الدالة التحليلية ψ .

ومن أجل جعل مهمة القارئ أكثر سهولة، عرضنا بعض النتائج المعروفة عن المؤثرات التركيبية وعرضنا براهين مفصلة وكذلك برهنا بعض النتائج .

Contents

المقدمة

هذا البحث يشمل فصلين . في الفصل الأول سوف نتناول الدالة المتقابلة σ وخواصها ، وكذلك نناقش أيضا الدوران المحوري حول الأصل للدالة σ وكذلك نناقش أيضا هل الدالة σ تحويل كسوري خطيّ .

في الفصل الثاني، سوف نتناول المؤثر التركيبي C_σ المتولد بالدالة σ وخواصه ، وكذلك نناقش أيضا المرافق للمؤثر التركيبي C_σ المتولد بالدالة σ وكذلك نناقش أيضا هل المؤثر التركيبي C_σ مؤثر تركيبي قابل للعكس وكذلك نناقش المعيار للمؤثر التركيبي C_σ

Chapter one
Properties of the Map σ

Definition(1.1) :

Let U = $\{ z \in C : |z| < 1 \}$ is a unit ball in complix plane C and ∂U = $\{ z \in C : |z| = 1 \}$ is a boundary of U

Definition(1.2):

For α ∈U, define $\sigma : U \to U$ and $\sigma(z) = \dfrac{2z - 2\alpha}{2\bar{\alpha}z - 2}$ $(\alpha, z \in U)$.

Proposition (1.3):

σ is bijective .

Proof:

Since $\sigma(z) = \dfrac{2z - 2\alpha}{2\bar{\alpha}z - 2}$ $(\alpha, z \in U)$

Suppose $\sigma(z_1) = \sigma(z_2)$, that is $\dfrac{2z_1 - 2\alpha}{2\bar{\alpha}z_1 - 2} = \dfrac{2z_2 - 2\alpha}{2\bar{\alpha}z_2 - 2}$, tharefore

$4\bar{\alpha}z_1z_2 - 4z_2 - 4|\alpha|^2 z_1 + 4\alpha = 4\bar{\alpha}z_1z_2 - 4z_1 - 4|\alpha|^2 z_2 + 4\alpha$,therefare

$4z_1 - 4|\alpha|^2 z_1 = 4z_2 - 4|\alpha|^2 z_2$, hance $z_1 = z_2$. Thas σ is injective .

Let $y = \sigma(z)$, that is $y = \dfrac{2z - 2\alpha}{2\bar{\alpha}z - 2}$, therefare $2\bar{\alpha}zy - 2y = 2z - 2\alpha$, than $2y - 2\alpha = 2\bar{\alpha}zy - 2z$, hance

$$z = \frac{2y - 2\alpha}{2\bar{\alpha}y - 2}, \mu(z) = \mu\left(\frac{2y - 2\alpha}{2\bar{\alpha}y - 2}\right) = \frac{2\left(\dfrac{2y - 2\alpha}{2\bar{\alpha}y - 2}\right) - 2\alpha}{2\bar{\alpha}\left(\dfrac{2y - 2\alpha}{2\bar{\alpha}y - 2}\right) - 2} = \frac{\dfrac{4y - 4\alpha - 4|\alpha|^2 y + 4\alpha}{2\bar{\alpha}y - 2}}{\dfrac{4\bar{\alpha}y - 4|\alpha|^2 - 4\bar{\alpha}y + 4}{2\bar{\alpha}y - 2}} = \frac{(4 - 4|\alpha|^2)y}{(4 - 4|\alpha|^2)} = y$$

, for every $y \in U$, there exists $z \in U$ such that $\sigma(z) = y$ .Thus σ is surjective .

Hence σ is bijective.

Definition(1.4) :

$$1-|\sigma(z)|^2 = \frac{4(1-|\alpha|^2)(1-|z|^2)}{|2\bar{\alpha}z-2|^2}$$

Proof:

$$1-|\sigma(z)|^2=1-\left|\frac{2z-2\alpha}{2\bar{\alpha}z-2}\right|^2 = 1-\frac{|2z-2\alpha|^2}{|2\bar{\alpha}z-2|^2} = \frac{|2\bar{\alpha}z-2|^2-|2z-2\alpha|^2}{|2\bar{\alpha}z-2|^2} =$$

$$\frac{(2\bar{\alpha}z-2)(2\alpha\bar{z}-2)-(2z-2\alpha)(2\bar{z}-2\bar{\alpha})}{|2\bar{\alpha}z-2|^2} = \frac{4|\alpha|^2|z|^2-4\bar{\alpha}\,z-4\alpha\bar{z}+4-[4|z|^2-4\bar{\alpha}z-4\alpha\bar{z}+4|\alpha|^2]}{|2\bar{\alpha}z-2|^2}$$

$$= \frac{4|\alpha|^2|z|^2-4\bar{\alpha}\,z-4\alpha\bar{z}+4-4|z|^2+4\bar{\alpha}z+4\alpha\bar{z}-4|\alpha|^2}{|2\bar{\alpha}z-2|^2} = \frac{4|\alpha|^2|z|^2+4-4|z|^2-4|\alpha|^2}{|2\bar{\alpha}z-2|^2} =$$

$$\frac{-4|z|^2(1-|\alpha|^2)+4(1-|\alpha|^2)}{|2\bar{\alpha}z-2|^2} = \frac{4(1-|\alpha|^2)(1-|z|^2)}{|2\bar{\alpha}z-2|^2}$$

Proposition (1.5) :

σ take ∂U inta ∂U

Proof :

Let $z \in \partial U$, that is $|z|=1$, therefare $|z|^2=1$, hance $1-|z|^2=0$, than $1-|\sigma(z)|^2=0$, hance $|\sigma(z)|^2=1$, hance $|\sigma(z)|=1$, hance σ take ∂U inta ∂U

Definition(1.6):

Let $\psi: U \to U$ be holomorphic map on U, ψ is called an inner map if $|\psi(z)|=1$.

Proposition (1.7):

σ is an inner map .

Proof :

By (1.5) σ take ∂U into ∂U, hence $|\sigma(z)|=1$, then by (1.6) σ is an inner map .

Proposition (1.8) :

$$\sigma^{-1}(z) = \frac{2z - 2\alpha}{2\bar{\alpha}z - 2} = \sigma(z)$$

Proof :

Let y = $\sigma^{-1}(z)$ then $z = \sigma(y)$ therefare $z = \frac{2y - 2\alpha}{2\bar{\alpha}y - 2}$, hance

$2\bar{\alpha}yz - 2z = 2y - 2\alpha$, thas $2\bar{\alpha}yz - 2y = 2z - 2\alpha$ then $y = \frac{2z - 2\alpha}{2\bar{\alpha}z - 2}$. Hance

$$\phi^{-1}(z) = \frac{2z - 2\alpha}{2\bar{\alpha}z - 2} = \phi(z)$$

Remark(1.9) :

$$\sigma'(\alpha) = \frac{-1}{\left(1 - |\alpha|^2\right)} \;,\; \sigma'(0) = -(1 - |\alpha|^2) \,.$$

Definition(1.10) :

Let $\psi : U \to U$ be holomarphic map on U . We say that ψ is a rototion around the origin if there exists $\sigma \in \partial U$ such that $\psi(z) = \sigma z \;\; (z \in U)$

Proposition (1.11):

If $\alpha = 0$, σ is a rototion around the origin

Proof:

Since $\alpha = 0$, $\sigma(z) = \frac{2z - 2\alpha}{2\bar{\alpha}z - 2} = -z = \sigma z, \sigma = -1, |\sigma| = 1 \in \partial U$, then by (1.10) σ is a rototion around the origin .

Definition(1. 12):

A linear fractionil transformation is a mapping of the form $\tau(z)=\frac{az+b}{cz+d}$ where a , b , c and d are complex numbers, and sometime denate it by $\tau_A(z)$ where A is the non–singular 2×2 complex matrix $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

<u>Proposition (1.13)</u> :

σ is a linear fractionil transformation .

<u>Proof</u> :

Since $\sigma(z)=\frac{2z-2\alpha}{2\bar{\alpha}z-2}=\frac{az+b}{cz+d}$ such that $a=2$, $b=2\alpha$, $c=2\bar{\alpha}$, $d=-2$ and a , b , c , and d are complix numbers and $A=\begin{bmatrix} 2 & 2\alpha \\ 2\bar{\alpha} & -2 \end{bmatrix}$, hence by (1.10) σ is a linear fractionil transformation .

Chapter two
TheNorm of the C_σ

Definition(2.1):

Let U denate the unit ball in the complix plane, the Hordy space H^2 is the set of functions $f(z)=\sum_{n=0}^{\infty} f^{\wedge}(n)\, z^n$ holomarphic on U such that $\sum_{n=0}^{\infty} |f^{\wedge}(n)|^2 < \infty$ with $f^{\wedge}(n)$ denotes then the Taylor coefficient of f .

Remark (2.2) :

We can define an inner product of the Hardy space functions as follows:

f(z) = $\sum_{n=0}^{\infty} f^{\wedge}(n)\, z^n$ and $g(z)=\sum_{n=0}^{\infty} g^{\wedge}(n)\, z^n$, then the inner product of f and g is:

$$\langle f,g\rangle = \sum_{n=0}^{\infty} f^{\wedge}(n)\, \overline{g^{\wedge}(n)}$$

Definition (2.3) :

Let $\alpha \in U$, define $K_{\alpha}(z)=\frac{1}{1-\overline{\alpha}z}(z \in U)$. Since $\alpha \in U$, then $|\alpha|<1$. The geometric series $\sum_{n=0}^{\infty} |\alpha|^{2n}$ is converge . Thus $K_{\alpha} \in H^2$ and $\sum_{n=0}^{\infty} (\overline{\alpha})^n z^n$

Definition(2.4) :

Let $\psi : U \to U$ be holomarphic map on U, the composetion operator C_{ψ} induced by ψ is defined on H^2 as follows $C_{\psi}\, f = f \circ \psi \quad (f \in H^2)$

Definition(2.5) :

Let T be a bounded operater on a Hilbert space H, then the norm of an operater T is defined by $\|T\| = \sup\{\|Tf\| : f \in H, \|f\| = 1\}$.

Theorem (2.6):

If $\psi : U \to U$ is holomarphic map on U, then $f \circ \psi \in H^2$ and

$\|f \circ \psi\| \le \sqrt{\frac{1+|\psi(0)|}{1-|\psi(0)|}}\|f\|$ for every $f \in H^2$.

The goal of this theorem $C_\psi : H^2 \to H^2$.

Definition(2.7) :

The composetion operater C_σ induced by σ is defined on H^2 as follows $C_\sigma f = f \circ \sigma$, for all $f \in H^2$.

Proposition(2.8) :

If $\sigma(z) = \frac{2z-2\alpha}{2\bar{\alpha}z-2}$, then $f \circ \sigma \in H^2$ and $\|f \circ \sigma\| \le \|f\|$ for each $f \in H^2$.

Proof :

Since $\sigma : U \to U$ is holomarphic map on U, then by (2.6)

$f \circ \sigma \in H^2$ and $\|f \circ \sigma\| \le \|f\|$ for each $f \in H^2$, hance $C_\sigma : H^2 \to H^2$

Remark (2.9) :

1) One can easely show that $C_\kappa C_\psi = C_{\psi \circ \kappa}$ and hence $C_\psi^n = C_\psi C_\psi \cdots C_\psi$

$= C_{\psi \circ \psi \circ \cdots \circ \psi} = C_{\psi_n}$

2) C_ψ is the identity operater on H^2 if and only if ψ is identity map from U

into U and holomarphic on U.

3) It is sample to prove that $C_\kappa = C_\psi$ if and only if $\kappa = \psi$.

Definition(2.10):

Let T be an operater on a Hilbert space H , The operater T^* is the adjoint of T if $\langle Tx, y\rangle = \langle x, T^* y\rangle$ for each $x, y \in H$.

Theorem (2.11) :

$\{K_\alpha\}_{\alpha \in U}$ forms a dense subset of H^2.

Theorem (2.12) :

If $\psi : U \to U$ is holomarphic map on U, then for all $\alpha \in U$, then $C_\psi^* K_\alpha = K_{\psi(\alpha)}$

Definition(2.13):

Let H^∞ be the set of all bounded holomarphic map on U .

Definition(2.14):

Let $g \in H^\infty$, the Toeplits operater T_g is the operator on H^2 given by :

$(T_g f)(z) = g(z)\, f(z) \quad (f \in H^2,\ z \in U)$

Theorem (2.15) :

If $\psi : U \to U$ is holomarphic map on U, then $C_\psi T_g = T_{g \circ \psi} C_\psi \quad (g \in H^\infty)$

Remark (2.16) :

For each $f \in H^2$, it is well- know that $T_h^* f = T_{\bar{h}} f$, such that $h \in H^\infty$.

Proposition(2.17) :

$$C_\sigma^* = T_g\, C_\sigma\, T_h^* \quad \text{where } h(z)=2\bar{\alpha}z-2\,,\ g(z)=\frac{1}{2\bar{\alpha}z-2}\,,\ \sigma(z)=\frac{2z-2\alpha}{2\bar{\alpha}z-2}$$

Proof :

By (2.16), $T_h^* f = T_{\bar{h}} f$ for each $f \in H^2$. Hance for all $\beta \in U$,

$$\langle T_h^* f, K_\beta\rangle = \langle T_{\bar{h}} f, K_\beta\rangle = \langle f, T_{\bar{h}}^* K_\beta\rangle \cdots\cdots (2-1)$$

On the other hand ,

$$\langle T_h^* f,\ K_\beta\rangle = \langle f, T_h K_\beta\rangle = \langle f, h(\alpha) K_\beta\rangle \cdots\cdots (2-2)$$

From (2-1)and (2-2) one can see that $T_{\bar{h}}^* K_\beta = h(\alpha)\, K_\beta$. Hance $T_h^* K_\beta = \overline{h(\alpha)}\, K_\beta$.

Calculation give:

$$C_\sigma^* K_\beta(z) = K_{\sigma(\beta)}(z)$$

$$= \frac{1}{1-\overline{\sigma(\beta)}\, z} = \frac{1}{1-\dfrac{(2\bar{\beta}-2\bar{\alpha})z}{(2\alpha\bar{\beta}-2)}}$$

$$= \frac{1}{\dfrac{(2\alpha\bar{\beta}-2)-(2\bar{\beta}z-2\bar{\alpha}z)}{(2\alpha\bar{\beta}-2)}} =$$

$$= \frac{(2\alpha\bar{\beta}-2)}{(-2+2\bar{\alpha}z)-\bar{\beta}(2z-2\alpha)} = \frac{\left(\overline{2\bar{\alpha}\beta-2}\right)}{(2\bar{\alpha}z-2)-\bar{\beta}(2z-2\alpha)}$$

$$= \left(\overline{2\bar{\alpha}\beta-2}\right) \cdot \left(\frac{1}{2\bar{\alpha}z-2}\right) \cdot \frac{1}{1-\bar{\beta}\left(\dfrac{2z-2\alpha}{2\bar{\alpha}z-2}\right)}$$

$$= \overline{h(\beta)} \cdot T_g\, K_\beta(\sigma(z)) = T_g\, \overline{h(\beta)}\, K_\beta(\sigma(z))$$

$$= T_g\, \overline{h(\beta)}\, C_\sigma\, K_\beta(z) = T_g\, C_\sigma\, \overline{h(\alpha)}\, K_\beta(z)$$

$$= T_g\, C_\sigma\, T_h^*\, K_\beta(z)$$, therefare

$$C_\sigma^* K_\beta(z) = T_g C_\sigma T_h^* K_\beta(z) .$$

But $\overline{\{K_\beta\}_{\beta \in U}} = H^2$, than $C_\sigma^* = T_g C_\sigma T_h^*$

Definition (2.18) :

Let T be a bounded operater on a Hilbert space H, Then the norm of an operater T is defined by $\|T\| = \sup\{\|T f\|: f \in H, \| f \| = 1\}$. If $\|T\| \le 1$, then T is said to be a contraction on H.

Theorem (2.19):

Let ψ be a holomarphic self-map of U, then C_ψ is a bounded operater on H^2 and $\|C_\psi\| \le \sqrt{\frac{1+|\psi(0)|}{1-|\psi(0)|}}$.

Theorem (2.20):

If ψ is inner, then $\|C_\psi\| = \sqrt{\frac{1+|\psi(0)|}{1-|\psi(0)|}}$

Proposition(2.21) :

$$\|C_\sigma\| = \sqrt{\frac{1+|\alpha|}{1-|\alpha|}}$$

Proof

Since σ is inner by(1.7) ,hence by(2.20) $\|C_\sigma\| = \sqrt{\frac{1+|\alpha|}{1-|\alpha|}}$

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