Republic of Iraq

Ministry of Higher Education and Scientific Research

Al-Qadisiyah University

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Additivity of Jordan Triple Product Homomorphisms on Generalized Matrix Algebras

A Research Submitted by

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To the Council of the Department of Mathematics / College of Education, University of Al-Qadisiyah as a partial Fulfillment of the Requirements for the Bachelor Degree in Mathematics.

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A.D. 2019

بِسْمِ اللَّـهِ الرَّحْمَـٰنِ الرَّحِيمِ

﴿بَلْ هُوَ آيَاتٌ بَيِّنَاتٌ فِي صُدُورِ الَّذِينَ أُوتُوا الْعِلْمَ ۚ وَمَا يَجْحَدُ بِآيَاتِنَا إِلَّا الظَّالِمُونَ﴾

صدق الله العظيم

سورة العنكبوت\جزء من الآية 49

شكر و تقدير

الحمد لله رب العالمين والسلام على نبيه وسيد المرسلين (صلى الله عليه وآله وسلم) وبعد الحمد والثناء للبارئ عز وجل ورسوله لا يسعني إلا أن أتقدم بخالص الشكر والتقدير الى الدكتورة رجاء جفات شاهين (المشرفة على البحث) والتي كانت خير عون لي في اتمام هذا البحث فجزاها الله عني أفضل الجزاء. كما أتقدم بخالص شكري وأمتناني لجميع أساتذتي في قسم الرياضيات. كما أقف احتراما لأقدم جميع كلمات الشكر والتقدير والبر لوالدي ووالدتي لما قدماه لي من عطاء ليومنا هذا.

Abstract

In this article, it is proved that under some conditions every bijective Jordan triple product homomorphism from generalized matrix algebras onto rings is additive. As a corollary, we obtain that every bijective Jordan triple product homomorphism from M_n (A) (A is not necessarily a prime algebra) onto an arbitrary ring Ŕ is additive.

Introduction

Let R be a commutative ring with identity, A and B be two associative algebras over R. Let M be an (A , B)-bimodule and N a (B, A)-bimodule. Assume that there are two bimodule homomorphosms $\varphi : M \otimes_B N \rightarrow A$ and $\psi : N \otimes_A M \rightarrow B$ satisfying the associativity conditions: $(MN)\,M' = M\,(N\,M')$ and $(NM)\,N' = N\,(MN')$ for all $M, M' \in M$ and $N, N' \in N$ where we put $MN = \varphi\,(M \otimes N)$ and $NM = \psi\,(N \otimes M)$. A generalized matrix algebra Mat (A , M , N , B) is an associative algebra of the form.

$$\text{Mat}\,(A, M, N, B) = \left\{ \begin{bmatrix} A & M \\ N & B \end{bmatrix} : A \in A\,,\ M \in M\,,\ N \in N\,,\ B \in B \right\}$$

Under the usual matrix – like multiplication, where at least one of the two bi – modules M and N is distinct from zero. In the above definition of generalized matrix algebras, if M is faithful as a left A-module and also as a right B-module and N ={0}, then the associative R-algebra

$$\text{Tri Mat}(A, M, B) = \left\{ \begin{bmatrix} A & M \\ O & B \end{bmatrix} : A \in A, M \in M, B \in B \right\}$$

is usually called a triangular algebra.

The most important examples of triangular algebras are upper triangular matrix algebras, block upper triangular matrix algebras and nest algebras. Obviously, the triangular algebras and M_n (A).

In studying preserves on algebras or rings, one usually assumes additivity in advance. Recently, however, a growing number of papers began investigating preservers that are not necessarily additive. Characterizing the interrelation between the multiplicative and additive

structures of a ring or algebra is an interesting topic. This question was first studied by Martindale (18) who showed the surprising result that every bijective multiplicative map from a prime ring containing a nontrivial idempotent onto an arbitrary ring is necessarily additive. For operator algebras, the same problem was treated in (1, 14, 21). In the papers (2, 3, 7, 12,13, 15, 19), the additivity of maps on operator algebras which are multiplicative with respect to other products, such as the Jordan product, the Jordan triple product or Jordan triple product homomorphisms were investigated. Also, the papers (6, 9, 20) studied the similar questions for elementary maps and Jordan elementary maps on rings or operator algebras.

Ling and Lu (11) studied Jordan maps of nest algebras, a kind of triangular algebras coming from operator theory. They showed that every Jordan bijective map on a standard subalgebra of a nest algebra is additive. This result was extended by Ji (4) to Jordan surjective map pair of triangular algebras. Cheng and Jing (1) proved that every multiplicative bijective map, Jordan bijective map, Jordan triple bijective map and elementary surjective map on triangular algebras is additive. Recently, Li and Xiao (10) extended the results of Ji (4) to generalized matrix algebras. In fact, they proved that every multiplicative bijective map, Jordan bijective map, Jordan triple bijective map on a generalized matrix algebra is additive under some conditions. For the Jordan triple product homomorphisms, Li and Jing (8) showed that if R is a 2 – torsion free prime ring containing a nontrivial idempotent and Ŕ is an arbitrary ring, then every bijective Jordan triple product homomorphism is additive. Kuzma (7) described the forms of Jordan triple product homomorphisms on matrix algebras and Moinar (19) obtained the exact forms of Jordan

triple product homomorphisms between standard operator algebras. There is a connection between Jordan triple product homomorphisms and Lie triple product homomorphisms, the R – linear maps $\phi : A \to B$ such that for every A,B, C, ϵ A, $\phi([[A, B], C]) = [[\phi(A), \phi(B)], \phi(C)]$, where $[A,B] = AB - BA$ is the Lie multiplication.

Chapter One

Preliminaries

Definition 1.1:- Module / let (R ,+,.) be a ring and let (M , +) be abelian group, Then (M , +) is called left R – modules (left module over The ring R)

If there is a mapping . :$R \times M \rightarrow M$ satisfying

1- $r.(m_1 + m_2) = r.m_1 + rm_2$

2- $(r_1 + r_2).m = r_1.m + r_2.m$

3- $(r_1 . r_2).m = r_1.(r_2.m)$

Definition 1.2 : - Abi -module is an abelian group that is both a left and right module such that the left and right multiplication compatible.

Definition 1.3 : A generalized matrix algebra

$$\text{Mat}(A, M, N, B) \left\{ \begin{bmatrix} A & M \\ N & B \end{bmatrix} = : A \in A, M \in M, N \in N, B \in B \right\}$$

Under the usual matrix – like multiplication, where at least one of the two bi-modules M and N is distinct from zero.

Definition 1.4: if M is faithful as a left A-module and also as a right B-module and N ={0}, then the associative R-algebra

$$\text{Tri}\ (A, M, B) = \left\{ \begin{bmatrix} A & M \\ O & B \end{bmatrix} : A \in A, M \in M, B \in B \right\}$$

is usually called a triangular algebra.

Definition 1.5 : Let ϕ be a map from A to B and A , B , C be arbitrary elements of A

1) ϕ is said to be multiplicative if

   $\phi (A , B) = \phi (A) \phi (B)$

2) ϕ is called a Jordan map if

   $\phi (AB + BA) = \phi (A) \phi (B) + \phi (B) \phi (A)$

3) ϕ is called a Jordan triple map if

$$\phi(ABC + CBA) = \phi(A)\,\phi(B)\,\phi(C) + \phi(C)\,\phi(B)\,\phi(A)$$

4) ϕ is said to be a Jordan triple product homomorphism if

$$\phi(ABA) = \phi(A)\,\phi(B)\,\phi(A)$$

Remark 1.6:-

$$G11 = \left\{ \begin{bmatrix} A & O \\ O & O \end{bmatrix} : A \in \mathcal{A} \right\}$$

$$G12 = \left\{ \begin{bmatrix} O & M \\ O & O \end{bmatrix} : M \in \mathcal{M} \right\}$$

$$G21 = \left\{ \begin{bmatrix} O & O \\ N & O \end{bmatrix} : N \in \mathcal{N} \right\}$$

$$G_{22} = \left\{ \begin{bmatrix} O & O \\ O & B \end{bmatrix} : B \in \mathcal{B} \right\}$$

Then every element $A \in G$ can be written as $A = A_{11} + A_{12} + A_{21} + A_{22}$, where $A_{ij} \in G_{ij}$ for $1 \leq i , j \leq 2$.

Chapter Two

Main Results

Lemma 2.1

ϕ (0) = 0.

Proof:-Since ϕ is surjective there exists an element A ϵ G such that ϕ (A) = 0.

Then ϕ (0) = ϕ (0 A 0) = ϕ (0) ϕ (A) ϕ (0) = 0

Lemma 2.2.

ϕ (A11 + A12 + A21 + A22) = ϕ (A11) + ϕ (A12) + ϕ (A21) + ϕ (A22)

for every Aij ϵ Gij

Proof / Let SϵG be an element such that

ϕ (S) = ϕ (A11) + ϕ (A12) + ϕ (A21) + ϕ (A22)

Then we have for every Xij , $1\leq i \leq j \leq 2$

ϕ (xij s xij) = ϕ (xij) ϕ (s) ϕ (xij)

$$= \phi\ (xij) \sum_{l,t} \phi\ (Alt)\ \phi\ (xij)$$

$= \phi(A_{11} + A_{12}) + \phi(A_{11} B_{12} C_{22})$

$= \phi(A_{11}) + \phi(A_{12}) + \phi(A_{11} B_{12})$

$= \phi(A_{11}) + \phi(A_{12}) + \phi(B_{12})$

From which it follows that

$$\phi(A_{12} + B_{12}) = \phi(A_{12}) + \phi(B_{12})$$

Lemma 2.4.

ϕ is additive on G_{21}

<u>**Proof :-**</u> **Let S be an element of G such that $\phi(S) = \phi(A_{21}) + \phi(B_{21})$.**

Then for every Xij , we have

(2.1) $\phi(Xij\ S\ Xij) = \phi(Xij)(\phi(A_{21}) + \phi(B_{21}))\phi(Xij)$

$= \phi(Xij\ A_{21}\ Xij) + \phi(Xij\ B_{21}\ Xij)$

Then by lemma 2.3. , we have

$$\phi(X_{12}\ S\ X_{12}) = \phi(X_{12} A_{21} X_{12}) + \phi(X_{12} B_{21} X_{12})$$

$$= \phi(X_{12} A_{21} X_{12} + X_{12} B_{21} X_{12})$$

Hence we have

$$X_{12}\ S\ X_{12} = X_{12} A_{21} X_{12} + X_{12} B_{21} X_{12}$$

Then by condition (2), we get

$S_{21} = A_{21} + B_{21}$. By (2.1) we have:

$$\phi (X_{11} S X_{11}) = \phi (X_{22} S X_{22}) = \phi (X_{21} S X_{21}) = o$$

Then it follows that $S_{11} = S_{22} = o$ and $S_{21} = o$ by condition (1),and hence

$$\phi (A_{21} + B_{21}) = \phi (S_{21}) = \phi (S) = \phi (A_{21}) + \phi (B_{21})$$

Lemma 2.5:

$\phi (A_{11} + B_{11}) = \phi (A_{11}) + \phi (B_{11})$ for every A_{11} , $B_{11} \epsilon G_{11}$

Proof :- We first claim that for every $C_{11} \epsilon G_{11}$ and $D_{12} \epsilon G_{12}$ we have that (2.2)

$$\phi (C_{11} D_{12}) = \phi (P_{11}) \phi (C_{11}) \phi (D_{12}) + \phi (D_{12}) \phi (C_{11}) \phi (P_{11})$$

Where P_{11} is the identity element of A. By Lemma 2.2, we have:

$$\phi (C_{11}) + \phi (C_{11} D_{12}) = \phi (C_{11} + C_{11} D_{12})$$

$$= \phi ((P_{11} + D_{12}) C_{11} (P_{11} + D_{12}))$$

$$= \phi (P_{11} + D_{12}) \phi (C_{11}) \phi (P_{11} + D_{12})$$

$$= (\phi (P_{11}) + \phi (D_{12}) \phi (C_{11}) (\phi (P_{11}) + \phi (D_{12}))$$

$$= \phi (P_{11}) \phi (C_{11}) \phi (P_{11}) + \phi (D_{12}) \phi (C_{11}) \phi (D_{12})$$

$$+ \phi (P_{11}) \phi (C_{11}) \phi (D_{12}) + \phi (D_{12}) \phi (C_{11}) \phi (P_{11})$$

$$= \phi (C_{11}) + \phi (P_{11}) \phi (C_{11}) \phi (D_{12}) + \phi (D_{12}) \phi (C_{11}) \phi (P_{11})$$

From which it follows that (2.2) holds

Now choose $S \epsilon G$ such that

$\phi (S) = \phi (A_{11}) + \phi (B_{11})$ since for

every $Xij \epsilon Gij$

$\phi (Xij\ S\ Xij) = \phi (Xij)\ \phi (S)\ \phi (Xij) =$

$\phi (Xij\ A_{11}\ Xij) + \phi (Xij\ B_{11}\ Xij),$

We have

$\phi (X_{12}\ S\ X_{12}) = \phi (X_{21}\ S\ X_{21}) =$

$\phi (X_{22}\ S\ X_{22}) = o$

Hence we get $S_{12} = S_{21} = S_{22} = o$

by condition (1) and (2) we also have

$\phi (S_{11}\ X_{12}) = \phi (P_{11})\ \phi (S_{11})\ \phi (X_{12}) + \phi (X_{12})\ \phi (S_{11})\ \phi (P_{11})$

$= \phi (P_{11})\ (\phi (A_{11}) + \phi (B_{11}))\ \phi (X_{12}) + \phi (X_{12})\ (\phi (A_{11}) + \phi (B_{11})$

$\phi (P_{11})$

$= \phi (A_{11}\ X_{12}) + \phi (B_{11}\ X_{12})$

$= \phi (A_{11}\ X_{12} + B_{11}\ X_{12}),$

Where the first and third equations hold by (2.2) and the last equation comes by the additivity of ϕ on G_{12}. Then we get $S_{11} = A_{11} + B_{11}$ by condition (3), and hence $\phi (A_{11} + B_{11}) = \phi (A_{11}) + \phi (B_{11})$

<u>Lemma 2.6:-</u>

ϕ is additive on G_{22}

<u>Proof :-</u> Let A_{22} and $S \epsilon G$ be an element such that $\phi (S) = \phi (A_{22}) + \phi (B_{22})$.

Since for every $X_{ij} \epsilon G_{ij}$

(2.3) $\phi (X_{ij} S X_{ij}) = \phi (X_{ij}) \phi (S) \phi (X_{ij}) =$

$$\phi (X_{ij} A_{22} X_{ij}) + \phi (X_{ij} B_{22} X_{ij}),$$

We have

$\phi (X_{11} S X_{11}) = \phi (X_{11}) \phi (A_{22}) \phi (X_{11}) + \phi (X_{11}) \phi (B_{22}) \phi (X_{11}) = o$

Hence $X_{11} S X_{11} = o$ and $X_{11} S_{11} X_{11} = o$. Then it follows that $S_{11} = o$.

Since

$\phi (X_{12} S X_{12}) = o$ by (2.3), we have $X_{12} S X_{12} = o$ and hence $S_{21} = o$

Similarly $S_{12} = o$ Hence

$$\phi (S_{22}) = \phi (A_{22}) + \phi (B_{22})$$

Considering. For every X_{12} , $Y_{12} \epsilon G_{12}$

ϕ (X_{12} S_{22} Y_{21}) = ϕ ((X_{12} + Y_{21}) S_{22} (X_{12} + Y_{21})

= ϕ (X_{12} + Y_{21}) ϕ (S_{22}) ϕ (X_{12} + Y_{21})

= ϕ (X_{12} + Y_{21}) (ϕ (A_{22}) + ϕ (B_{22})) ϕ (X_{12} + Y_{21})

= ϕ ((X_{12} + Y_{21}) A_{22} (X_{12} + Y_{21})) + ϕ ((X_{12} + Y_{21}) B_{22} (X_{12} + Y_{21}))

= ϕ (X_{12} A_{22} Y_{21}) + ϕ (X_{12} B_{22} Y_{21})

= ϕ (X_{12} A_{22} Y_{21} + X_{12} B_{22} Y_{21}),

We have

$$X_{12} S_{22} Y_{21} = X_{12} A_{22} Y_{21} + X_{12} B_{22} Y_{21}$$

Therefore, by condition (4), it follows that $S_{22} = A_{22} + B_{22}$, and hence

$$\phi (A_{22} + B_{22}) = \phi (A_{22}) + \phi (B_{22})$$

We now prove our main result.

<u>Theorem 2.7:-</u> Let R be a commutative ring with identity, A and B be two unital algebras over R. Let G be the generalized matrix algebra Mat (A, M, N, B) which satisfies the following conditions:

1- For M ϵ M , NMN =0 for every N ϵ N implies M = 0

2- For N ϵ N , MNM = 0 for every M ϵ M implies N = 0

3- M is faithful as a left A- module, that is for A ϵ A , AM = 0 for every M ϵ M implies A = 0 ,

4- For B ϵ B , MBN = 0 for every M ϵ M and N ϵ N implies B = 0

Then every bijective Jordan triple product homomorphism ϕ from G onto an arbitrary ring Ŕ is additive.

Proof :-Let

$A = A_{11} + A_{12} + A_{21} + A_{22}$ and $B = B_{11} + B_{12} + B_{21} + B_{22}$ be elements of G. Then by lemmas 2.2, 2.3, 2.4, 2.5, and 2.6 we have

$$\begin{aligned}
\phi (A + B) &= \phi (A_{11} + B_{11} + A_{12} + B_{12} + A_{21} + B_{21} + A_{22} + B_{22}) \\
&= \phi (A_{11} + B_{11}) + \phi (A_{12} + B_{12}) + \phi (A_{21} + B_{21}) + \phi (A_{22} + B_{22}) \\
&= \phi (A_{11}) + \phi (B_{11}) + \phi (A_{12}) + \phi (B_{12}) \\
&\quad + \phi (A_{21}) + \phi (B_{21}) + \phi (A_{22}) + \phi (B_{22}) \\
&= \phi (A_{11}) + \phi (A_{12}) + \phi (A_{21}) + \phi (A_{22}) \\
&\quad + \phi (B_{11}) + \phi (B_{12}) + \phi (B_{21}) + \phi (B_{22}) \\
&= \phi (A) + \phi (B).
\end{aligned}$$

It is clear M_n (A) satisfies the conditions of Theorem 2.6 for (not necessarily prime) unital algebras A. Hence we have the following corollary.

Corollary 2.8:- Let A be a unital (not necessarily prime) algebra over a commutative ring R. Then every bijective Jordan triple product homomorphism ϕ from M_n (A) onto an arbitrary ring Ŕ is additive for $n \geq 2$.

References

[1] X. Cheng and W. Jing, Additivity of Maps on Triangular Algebras, Electron. J.Linear Algebra 17 (2008), 597- 615.

[2] J. Hakeda, Additivity of Jordan * - Maps on AW* - algebras, proc. Amer. Math. Soc. 96 (1986). No 3, 413- 420.

[3] J. Hakeda and K. Saito, Additivity of Jordan * - Maps Between Operator Algebras, J. Math. Soc. Japan 38 (1986), no. 3, 403-408.

[4] P. Ji, Jordan Maps on Triangular Algebras, Linear Algebra APPI-426 (2007), no. 1, 190-192.

[5] P. Ji,R. Liu, and Y. Zhao, Nonlinear Lie Triple Derivations of Triangular Algebras, Linear Multilinear Algebra 60 (2012),no. 10, 1155-1164.

[6] W. Jing, Additivity of Jordan Elementary Maps on Rings, ar xiv: 0706. 0488v1 [math. RA] 4 Jun 2007.

[7] B. Kuzma, Jordan Triple Product Homomorphisms, Monatsh. Math. 149 (2006), no. 2, 119-128.

[8] P.Li and W.Jing, Jordan Elementary Maps on Rings, Linear Algebra APPI. 382 (2004), 237-245.

[9] P. Li and F. Lu, Additivity of Jordan Elementary Maps on Nest Algebras, Linear Algebra APPI.400 (2005), 327-338.

[10] Y. Li and Z. Xiao, Additivity of Maps on Generalized Matrix Algebras, Electron. J. Linear Algebra 22 (2011), 743-757.

[11] Z. Ling and F. Lu, Jordan Maps of Nest Algebras, Linear Algebra APPI. 387 (2004), 311-368.

[12] F. Lu, Jordan Triple Maps, Linear Algebra APPI. 375 (2003), 311-317.

[13] _____, Jordan Maps on Associative Algebras, Comm. Algebra 31 (2003), no. 5, 2273-2268.

[14] ______, Multiplicative Mappings of Operator Algebras, Linear, Linear

Algebra APPI. 347 (2002), 283-291.

[15] _______, Additivity of Jordan Maps on Standard Operator Algebras, Linear Algebra APPI. 357 (2002), 123-131.

[16] A. J. C. Martin and C. M. Gonzalez, The Banach – Lie Group of Lie Triple Automorphisms of an H* - Algebras, Acta Math. Sci. ser. B Engl. Ed. 30 (2010), no. 4, 1219-1226.

[17] ___________, A Linear Approach to Lie Triple Automorphisms of H* - Algebras, J.Korean Math. Soc. 48 (2011), no.1, 117-132.

[18] W. S. Martindale III, When are Multiplicative Mappings Additive, Proc. Amer. Math. Soc. 21 (1969), 695-698.

[19] L. Molnar, On Isomorphisms of Standard Operator Algebras, Studia Math. 142 (2000), no. 3, 295-302.

[20] L. Molnar and P. Semri, Elementary Operators on Standard Algebras, Linear Multilinear Algebra 50 (2002), no. 4, 315-319.

[21] P. Semri, Isomorphisms of Standard Operator Algebras, Proc. Amer. Math. Soc. 123 (1995), no. 6, 1851-1885.

[22] Z. Xiao and F. Wei, Commuting Mappings of Generalized Matrix Algebras, Linear Algebra APPI. 433 (2010), no. 11-12, 2178-2197.

[23] __________, Lie Triple Derivations of Triangular Algebras, Linear Algebra APPI. 437 (2012),no. 5, 1234-1249.