University Of AL - Qadisiya

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ON FUZZY TOPOLOGICAL GAME $G^*(K, X)$

A research

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بسم الله الرحمن الرحيم

﴿وليعلم الذين أوتوا العلم أنه الحق
من ربك فيؤمنوا به فتخبت له قلوبهم
وإن الله لهاد الذين آمنوا إلى
صراط مستقيم﴾

صدق الله العلي العظيم

سورة الحج الآية (٥٤)

الإهـــــداء

إلى من بلغ الرسالة وأدى الأمانة.. ونصح الامة الى نبي الرحمة ونور العالمين

سيدنا محمد صل الله عليه واله وسلم

إلى من كلله الله بالهيبة والوقار الى من علمني العطاء بدون انتظار الى من أحمل أسمه بكل افتخار

والدي العزيز

إلى معنى الحب

وإلى معنى الحنان والتفاني

إلى بسمة الحياة وسر الوجود

إلى من كان دعائها سر نجاحي وحنانها بلسم جراحي

أمـــي الحبيـبـة

إلى من بهم اكبر وعليهم أعتمد

التي تنير ظلمة حياتي

إلى الشموع المتقدة

إلى من بوجودهم أكتسب قوة ومحبة لا حدود لها

إلى من عرفت معهم معنى الحياة

إخـوتي وأخـواتي

شكر وتقدير

الحمد لله يقول الله في محكم كتابه { لئن شكرتم لأزيدنكم } والصلاة والسلام على اشرف خلق الله سيدنا محمد (صلى الله عليه واله وسلم) القائل: من لم يشكر المخلوق لم يشكر الخالق.

بداية اشكر الله عز وجل الذي ساعدني على اتمام بحثي وتفضل علينا بإتمام هذا العمل.. وبعد

شكرا وتقديرا لحضرة الاستاذ الفاضل فراس جواد على ما بذله من سعة صدر وكرم طبعه ورحابة خاطره وارشاد وتوجيه وتسديد لأفكاري

فجزاه الله خير جزاء المحسنين

الباحث

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Abstract:

In this paper we generalize the concept of topological games in to a fuzzy topological space and some results related to them are obtained. Just like in the case of $G\ (K, X)$, the fuzzy topological game $G^*(K, X)$ has plenty of applications in fuzzy topology especially in fuzzy metacompactness etc.

Introduction:

The concept of a fuzzy set, which was introduced in [1], provides a natural framework for generalizing many of the concepts of general topology to what might be called fuzzy topological spaces. In the interest of brevity, we shall confine our attention in this note to the more basic concepts such as open set, closed set, neighborhood, interior set, continuity and compactness, following closely the definitions, theorems and proofs given in Kelly [2]. Our notation and terminology for fuzzy sets follow that of Zadeh [1].

In chapter one section one we introduce definition of basic of topological space, some examples and remark.

In chapter one section two preview some definitions and properties of topological space, continuity, homomorphism and locally compact. Also some examples and theorem.

In chapter one section three we introduce some definitions and examples of fuzzy and fuzzy topological space.

In chapter two section one we preview definitions, examples and theorems of fuzzy topological game $G^*(K,X)$. In chapter two section two we introduce some proposition, definition, examples and theorems of finite and countable union.

Finally In chapter two section three we introduce theorems of game and mapping. A pursuit evasion game $G\ (K,X)$ in which the pursuer and the evader choose certain subsets of a topological space in a certain way is defined and ystudied b Telgarsky [T_2].

CHAPTER ONE

SOME PROPERTIES OF TOPOLOGICAL SPACE

1.1 Basic of Topological space.

1.1.1 Definition :

Let X be a set. A topology on X is a collection $T \subseteq P(X)$ of subsets of X satisfying

1. T contains $\emptyset$ and X;
2. T is closed under arbitrary unions, i.e. if $U_i \in T$ for $i \in I$ then $\bigcup_{i \in I} U_i \in T$;
3. T is closed under finite intersections, i.e. if $U_1, U_2 \in T$ then $U_1 \cap U_2 \in T$.

1.1.2 Definition :

A topological space (X, T) is a set X together with a topology T on it. The elements of T are called open subsets of X. A subset $F \subseteq X$ is called closed if its complement $X \setminus F$ is open. A subset N containing a point $x \in X$ is called a neighborhoods of x if there exists U open with $x \in U \subseteq N$. Thus an open neighbourhood of x is simply an open subset containg x.

Normally we denote the topological space by X instead of (X, T).

1.1.3 Definition :

Let $A \subseteq X$ be a subset of a topological space X. The interior of A is the biggest open subset contained in A. One has $A^{\circ} = \bigcup_{A \supseteq U}$ open U. Dually the closure of A is the smallest closed subset containing A. One has $\overline{A} = \bigcap_{A \subseteq F}$ closed F.

1.1.4 Example:

Consider the following set consisting of 3 points; $X = \{a, b, c\}$ and determine if the set $T = \{\emptyset, X, \{a\}, \{b\}\}$ satisfies the requirements for a topology.

This is, in fact, not a topology because the union of the two sets $\{a\}$ and $\{b\}$ is the set $\{a, b\}$, which is not in the set τ

1.1.5 Example:

Find all possible topologies on $X = \{a, b\}$

1. $\emptyset, \{a, b\}$
2. $\emptyset, \{a\}, \{a, b\}$
3. $\emptyset, \{b\}, \{a, b\}$
4. $\emptyset, \{a\},\{b\},\{a,b\}$

1.1.6 Example:

When X is a set and τ is a topology on X, we say that the sets in τ are open. Therefore, if X does have a metric (a notion of distance), then ^{T} ={all open sets as defined with the ball above} is indeed a topology. We call this topology the Euclidean topology. It is also referred to as the usual or ordinary topology.

1.1.7 Example:

If $Y \subseteq X$ and τx is a topology on X, one can define the Induced topology as $\tau y = \{O \cap Y \mid O \in \tau x\}$.

This last example gives one reason why we must only take finitely many intersections when defining a topology.

1.1.8 Remark:

As promised, we can now generalize our definition for a closed set to one in terms of open sets alone which removes the need for limit points and metrics

1.1.9 Definition:

A set C is closed if $X - C$ is open.

Now that we have a new definition of a closed set, we can prove what used to be definition 1.3.3 as a theorem: *A* set *C* is a closed set if and only if it contains all of its limit points.

Proof: Suppose a set *A* is closed. If it has no limit points, there is nothing to check as it trivially contains its limit points. Now suppose *z* is a limit point of *A*. Then if $z \in A$, it contains this limit point. So suppose for the sake of contradiction that *z* is a limit point and *z* is not in *A*. Now we have assumed *A* was closed, so its complement is open. Since *z* is not in *A*, it is in the complement of *A*, which is open; which means there is an open set *U* containing *z* contained in the complement of *A*. This contradicts that z is a limit point because a limit point is, by definition, a point such that every open set about z meets *A*

Conversely: if *A* contains all its limit points, then its complement is open. Suppose *x* is in the complement of *A*. Then it can not be a limit point (by the assumption that *A* contains all of its limit points). So *x* is not a limit point which means we can find some open set around *x* that doesn't meet *A*. This proves the complement is open, i.e. every point in the complement has an open set around it that avoids *A* .

1.1.10 Remark:

Since we know the empty set is open, *X* must be closed.

1.1.11 Remark:

Since we know that *X* is open, the empty set must be closed.

Therefore, both the empty set and *X* and open and closed.

1.1.12 Example :

When *X* is a set and τ is a topology on *X*, we say that the sets in *τ* are open. Therefore, if *X* does have a metric (a notion of distance), then *τ* ={all open sets as defined with the ball above} is indeed a topology. We call this topology the Euclidean topology. It is also referred to as the usual or ordinary topology.

1.1.13 Definition:

A subset S of topological space (x, T) is said clopen if it is both open and closed subset of X.

1.2 Some properties of Topological space.

1.2.1 Continuity

In topology a continuous function is often called a function. There are 2 different ideas we can use on the idea of continuous functions.

Calculus Style

1.2.2 Definition:

$f : R n \to R m$ is continuous if for every $\in > 0$ there exists $\delta > 0$ such that when $|x - x0| < \delta$ then $|f(x) - f(x0)| < \in$.

The map is continuos if for any small distance in the pre-image an equally small distance is apart in the image. That is to say the image does not jump

Topology Style. In topology it is necessary to generalize down the definition of continuity, because the notion of distance does not always exist or is different than our intuitive idea of distance.

1.2.3 Definition :

A function $f : X \to Y$ is continuous if and only if the pre-image of any open set in Y is open in X. If for whatever reason you prefer closed sets to open sets, you can use the following equivalent definition:

1.2.4 Definition :

A function $f : X \to Y$ is continuous if and only if the pre-image of any closed set in Y is closed in X.

1.2.5 Definition :

Given a point x of X, we call a subset N of X a neighborhood of X if we can find an open set O such that $x \in O \subseteq N$.

1. A function $f : X \to Y$ is continuous if for any neighborhood V of Y there is a neighborhood U of X such that $f(U) \subseteq V$.

2. A composition of 2 continuous functions is continuous

1.2.6 Definition :

A function $f{:}X \to Y$ between two topological spaces is called continuous if every $U \subseteq Y$ open in Y the inverse image $f^{-1}(U)$ is open in X.

1.2.7 Proposition :

The identity function is continuous. A composition of two continuous maps is continuous. Thus topological spaces and continuous maps between them form a category, the category of topological spaces.

1.2.8 Definition :(Homeomorphisms)

A homeomorphism is a function $f : X \to Y$ between two topological spaces X and Y that

1.is a continuous bijection; and
2.has a continuous inverse function f^{-1} .

Another equivalent definition of homeomorphism is as follows.

1.2.9 Definition :

Two topological spaces X and Y are said to be homeomorphic if there are continuous function $f : X \to Y$ and $g : Y \to X$ such that $f \circ g = I_Y$ and $g \circ f = I_X$.

Moreover, the functions f and g are homeomorphisms and are inverses of each other, so we may write f^{-1} in place of g and g^{-1} in place of f.

Here, I_X and I_Y denote the identity maps .

1.2.10 Definition:

Let $\mathfrak{I}$ and $\mathfrak{I}^*$ be two topologies on a given set X. If $\mathfrak{I}^* \supseteq \mathfrak{I}$ then $\mathfrak{I}$ is coarser than $\mathfrak{I}^*$.

1.2.11 Definition :

a topological space *(X,T)* is said to be completely regular space iff every closed subset *F* of X and every point $x \in X-F$ there exist a continuous function $f:X \rightarrow [0,1]$ such that $f(x)=0$, $f(F)=\{1\}$

1.2.12 Definition :(tychonoff)

a tychonoff space or space is completely regular $T_{1\text{-space}}$

1.2.13 Definition :

Say that a family of sets A is linked if for every $A, B \in A$, $A \cap B = \emptyset$.

1.2.14 Definition :(pathwise)

Let X be a topological space, and $x, y \in X$. A continuous function $p : I \rightarrow X$ such that $p(0) = x$ and $p(1) = y$ is called a path from x to y. X is called pathwise.

1.2.15 Definition :

A collection U of open subsets of a topological space X is called an (open) cover if its union is the whole of X, *i.e.* $\bigcup_{i \in I} U = \bigcup_{U_0 \in U} U = X$. A subcollection $U_0 \subseteq U$ is called a sub-cover if it is itself a cover.

1.2.16 Definition :

A topological space X is called compact if every open cover admits a finite sub-cover

1.2.17 Definition :(locally compact)

A topological space is locally compact if every point $x \in X$ has a compact neighborhood.

1.2.18 Example 1.2. Any compact space is locally compact

1.2.19 Definition :

Product topology Given two topological spaces *(X, T)* and *(Y, T ')*, we define the product topology on X × Y as the collection of all unions $\bigcup_i U_i \times V_i$, where each Ui is open in X and each Vi is open in Y .

1.2.20 Theorem.

Projection maps are continuous Let *(X, T)* and *(Y, T')* be topological spaces. If $X \times Y$ is equipped with the product topology, then the projection map $p_{1=} : X \times Y \to X$ defined by $p_1(x, y) = x$ is continuous. Moreover, the same is true for the projection map $p_2 : X \times Y \to Y$ defined by $p_2(x, y)= y$ □

1.3 Fuzzy Topological space

1.3.1 Definition

Let A and B be fuzzy sets in a space $X = \{x\}$, with the grades of membership of x in A and B denoted by $\mu_A(X)$ and $\mu_B(X)$, respectively. Then

$$A = B \Leftrightarrow \mu_A(x) = \mu_B(x) \qquad for\ all\ x \in X.$$

$$A \subset B \Leftrightarrow \mu_A(x) \leq \mu_B(x) \qquad for\ all\ x \in X.$$

$$C = A \cup B \Leftrightarrow \mu_C(x) = Max[\mu_A(x), \mu_B(x)] \qquad for\ all\ x \in X.$$

$$D = A \cap B \Leftrightarrow \mu_D(x) = Min[\mu_A(x), \mu_B(x)] \qquad for\ all\ x \in X.$$

$$E = A' \Leftrightarrow \mu_E(x) = 1 - \mu_A(x)\ for\ all\ x \in X$$

More generally, for a family of fuzzy sets, $A = \{A_i;\ i \in I\}$, the union, $C = \bigcup_i A_i$, and the intersection, $D = \bigcap_i A_i$, are defined by

$$\mu_C(x) = sup_I\{\mu_{A_i}(x)\}, x \in X$$

$$\mu_D(x) = inf_I\{\mu_{A_i}(x)\}, x \in X$$

The symbol Φ will be used to denote an empty fuzzy set ($\mu_\Phi(x) = 0$ for all $x\ in\ X$). For X, we have by definition $\mu_x(x) = 1$ for all $x\ in\ X$.

We are now ready to define a fuzzy topological space.

1.3.2 Definition

A fuzzy topology is a family T of fuzzy sets in X which satisfies the following conditions:

(a) $\Phi, X \in T$,

(b) $If\ A, B \in T, then\ A \cap B \in T$,

(c) $If\ A_i \in T\ for\ each\ i \in I, then\ U_I\ A_i \in T$.

T is called a fuzzy topology for X, and the pair (X, T) is a fuzzy topological space, or fts for short. Every member of T is called a T-open fuzzy set. A fuzzy set is T-closed if and only if its complement is T-open. In the sequel, when no confusion is likely to arise, we shall call a T-open (T-closed) fuzzy set simply an open (closed) set. As (ordinary) topologies, the indiscrete fuzzy topology contains only $\Phi\ and\ X$, while the discrete fuzzy topology contains all fuzzy sets. A fuzzy topology U is said to be coarser than a fuzzy topology T if and only if $U \subset T$.

1.3.3 Definition

A fuzzy set U in a fts (X, T) is a neighborhood, or nbhd for short, of a fuzzy set A if and only if there exists an open fuzzy set 0 such that $A \subset 0 \subset U$.

The above definition differs somewhat from the ordinary one in that we consider here a nbhd of a fuzzy set instead of a nbhd of a point.

1.4. SEQUENCES OF FUZZY SETS

1.4.1 Definition

A sequence of fuzzy sets, say $\{A_n, n = 1, 2, \ldots\}$, is eventually contained in a fuzzy set A iff there is an integer m such that, if $n \geq m$, then $A_n \subset A$. The sequence is frequently contained in A iff for each integer m there is an integer n such that $n \geq m$ and $A_n \subset A$. If the sequence is in a fts (X, T), then we say that the sequence converges to a fuzzy set A iff it is eventually contained in each nbdh of A.

1.4.2 Definition

Let N be a map from the set of non-negative integers to the set of non-negative integers. Then the sequence $\{B_i, i = 1, 2, \ldots\}$ is a subsequence of a sequence $\{A_n, n = 1, 2, \ldots\}$ iff there is a map N such that $B_i = A_{n\,(i)}$ and for each integer m there is an integer n such that $N(i) \geq m$ whenever $i \geq n$.

1.4.3 Definition

Suppose $\mathbb{F}$ is a family of fuzzy sets in X. which satisfies the following axioms:

$(T1) O, 1 \in \mathbb{F}$.

$(T2) if\ A, B \in \mathbb{F}. then\ A \cap B \in \mathbb{F},$

$(T3) if\ A_i; \in \mathbb{F}, i \in \wedge. then\ \cup_{i \in \wedge}\ A_i \in \mathbb{F},$

then $\mathbb{F}$ is called a fuzzy topology for X and the pair $(X, \mathbb{F})$ is a fuzzy topological space.

Every member of $\mathbb{F}$ is called an $\mathbb{F}$ -open fuzzy set (or simply open fuzzy set) and its complement is an $\mathbb{F}$ -closed fuzzy set (or closed fuzzy set).

Let A be a fuzzy set in fts $(X, \mathbb{F})$. The closure $\overline{A}$ and interior A^0 of A are defined. respectively, by

$$\overline{A} = \cap \{B: B \supset A, B' \in \mathbb{F}\}$$

and

$A^0 = \cup \{B: B \subset A, B \in \mathbb{F}\}$.

1.4.4 Definition: (Fuzzy set)

Let x be anon-empty set. Then u is said Fuzzy set if $u: X \longrightarrow I$ Such that $I = [0,1]$,

$u = \{(X, u(x)) \quad : x \in X \quad , u(x) \in I\}$.

1.4.5 Example:

Let $u: X \longrightarrow I$ and let $= \{a, b, c\}$,

$u(a) = \frac{1}{2}$, $u(b) = \frac{1}{3}$, $u(c) = \frac{1}{4}$

Is u Fuzzy set ?

Solution.

u is Fuzzy set

Since all numbers are enclosed between zero and one.

CHAPTER TWO

THE FUZZY TOPOLOGICAL GAME $G^*(K, X)$

THE FUZZY TOPOLOGICAL GAME $G^*(K, X)$

2.1 The Fuzzy Topological Game

2.1.1 Notation

By K we denote a non-empty family of fuzzy topological spaces, where all spaces are assumed to be T_1.That is all fuzzy singletons are fuzzy closed. I^x Denote the family of all fuzzy closed subsets of X Also $X \in K$ implies $I^x \subseteq \boldsymbol{K}.\boldsymbol{DK}$ $(\boldsymbol{FK})$ denote the class of all fuzzy topological spaces which have a discrete (finite) fuzzy closed α-shading by members of $\boldsymbol{K}$.

2.1.2 Definition

Let K be a class of fuzzy topological spaces and let $X \in \boldsymbol{K}$. Then the fuzzy topological game $G^*(K, X)$ is defined as follows. There are two players Prayer I and player II. They alternatively choose consecutive terms of the sequence $(E_1, F_1, E_2, F_2, \dots)$ of fuzzy subsets of X.When each player chooses his term he knows K, X and their previous choices. A sequence $(E_1, F_1, E_2, F_2, \dots)$ is a play for $G^*(K, X)$ if it satisfies the following conditions for each $h \geq 1$.

(I) E_n is a choice of Player I

(2) F_n is a choice of Player II

(3) $E_n \in I^x \cap K$

(4) $F_n \in I^x$

(5) $E_n \vee F_n < F_{n-1}$ Where F_0=X

(6) $E_n \wedge F_n = 0$

Player I wins the Play if $Inf\ F_n = 0.$ Otherwise Player II wins the Game.

2.1.3 Definition

A "finite sequence $(E_1, F_1, E_2, F_2, \dots, E_m, F_m)$ is admissible if it satisfies conditions (1) $--$ (6) for each $n \leq m$.

2.1.4 Definition

Let S' be a crisp function defined as follows

$$S:\cup\,(I^x)^n \xrightarrow{into} I^x \cap K$$

Let $S_1 = \{X\}$

$S_2 = \{F \in I^x : (S'(X), F)$ Is admissible for $G^*(K, X)\}$, Continuing like this inductively we get

$S_n = \{(F_1, F_2, F_3, \ldots F_n) : (E_1, F_1, E_2, F_2, \ldots, E_n, F_n)$ is admissible for $G^*(K, X)$ where $F_0 = X$ and

$E_i = S'(E_1, F_1, E_2, F_2, \ldots, F_{n-1})$ for each $i \leq n$ }. Then the restriction S of S' to

$\cup_{n\geq 1} S_n$ Is called a fuzzy strategy for player I in $G^*(K, X)$.

2.1.5 Definition

If Player I wins every play $(E_1, F_1, E_2, F_2, \ldots, E_n, F_n$ ....) such that

$E_n = S(F_1, F_2, \ldots \ldots, F_{n-1})$, then we say that S is a fuzzy winning strategy.

2.1.6 Definition

$S:\cup\,(I^x)^n \xrightarrow{into} I^x \cap K$ is called a fuzzy stationery strategy for

Prayer I in $G^*(K,X)$ if $S\ (F) < F$ for each $F \in I^x$. we say that S is a fuzzy stationary winning strategy if he wins every play $(S(X), F_1, S(F_1), F_2, \dots)$.

From definitions above, we get

2.1.7 Result

A function $S{:}\cup\ (I^x)^n \xrightarrow{into} I^x \cap K$ is a fuzzy stationary winning strategy if and only if it satisfies

(I) For each $F \in I^x$, $S(F) < F$

(ii) If $\{F_n : n \geq 1\}$ satisfies $S(X) \wedge\ F_1 =\ 0$ and $S(F_n)\ \wedge\ F_{n+1} =\ 0$ for each n $\geq$ 1 then$Inf\ F_n = 0$.

2.1.8 Theorem

Player I have a fuzzy winning strategy in $G^*(K,X)$ if and only if he has a fuzzy winning strategy in it.

Proof is similar to that of Yajima $[Y_1]$ and for completeness we are including it.

Proof.

Sufficiency part follows clearly. Conversely let S be a fuzzy winning strategy of, Plyer I for$G^*(K,X)$. Well order $I^x \backslash \{o\}$ by <. Let H be any non-empty closed fuzzy subset of X.

Claim-(1) now we will prove that there is some

$F(H) = (F_1, F_2, F_3, \dots F_m) \in (I^x)^m$

Satisfying

(I) $S(F_0, F_1, \dots F_i) \wedge H = 0$ For $0 \le i \le m - 1$

(ii) $S(F_0, F_1, \dots F_m) \wedge H \neq 0$

(Iii) F_{i+1} =Min$\{F \in I^x : \ H \ \le F \ \le F_i \ \ and \ F \wedge S(F_1, F_2, \dots F_i) = O\}$ for $O \le \ i \ \le \ m - 1 \ where$

$F_0 = X$ And $F(H) = 0$ may occur

To prove the above claim assumes the contrary. Then we can inductively choose

Some $(F_1, F_2, \dots) \in (I^x)^\omega$ such that $S(F_1, F_2, \dots F_k) \wedge H = O$ and

$F_k = Min(F \in I^x : H \le F \le F_{k-1}$ And

$S(F_1, F_2, \dots F_{k-1}) \wedge H = O\} for \ each \ k \ge 1$.

Now $(E_1, F_1, E_2, F_2, \dots.)$ where $E_k = \ S(F_1, F_2, \dots F_{k-1})$ is a play for each $k \ge 1$ for $G^*(K, X)$ and by definition of fuzzy strategy, we have$Inf_{k \ge 1} \ F_k = \ O$. Also $H \le F_k$ for all $k \ge 1$.There fore

$H \le Inf_{k \ge 1} \ F_k = O$. This is a contradiction to $H \ \neq O$. Thus claim-(1) holds.

Take $S^*(O) \ = 0$ and $S^*(H) \ = \ S(F_1, F_2, \dots F_m) \wedge H$ where $F(H)(F_1, F_2, \dots F_m)$.

For each $H \in I^x \backslash \{o\}$ .Then S^* is a function from I^x into $I^x \cap K$ such that $S^*(H) \le H$ for each $H \in I^x$. We will prove that S^* is a fuzzy stationary winning strategy for Player I in $G^*(K, X)$.

Let $(E_1, H_1, E_2, H_2, ...)$ be a play such that $E_1 = S^*(X)$ and

$E_1 = S^*(H_{n-1})$ For $n \geq 2$ we show that $Inf_{k\geq 1} H_n = 0$. For $n \leq m$, take $F(H)/n = (F_1, F_2, ... F_n)$ and $|F(H)| = m$

Claim-(2)

We will show that there are some $(F_1, F_2, ...) \in (I^x)^\omega$ and a sequence

$k(1) < k(2) < \ldots$.Such that $k > k(n)$ implies $(F_1, F_2, ... F_n) = F(H_n)/n$ for each $n \geq 1$

Take $F_0 = X$ and assume that $(F_1, F_2, ... F_n) \in (I^x)^n$ and{ $k(i): i \leq n$} has been

Already chosen, first we will prove that $|F(H_k)| > n$ for each $k > k(n)$. $Let\ k > k(n)$, then by induction we have $F(H_k)/n = F(H_{k(n)})/n = (F_1, F_2, ... F_n)$.

If $S(F_0, F_2, ... F_n) \wedge H_{k(n)} = 0$, then from $H_k < H_{k(n)}$ it follows that $S(F_0, F_2, ... F_n) \wedge H_k = 0$. Otherwise if $S(F_0, F_2, ... F_n) \wedge H_k \neq 0$ by (ii) of Claim-(1) above we have $F(H_{k(n)}) = (F_1, F_2, ... F_n)$. So that $S^*\left(H_{k(n)}\right) = S(F_0, F_2, ... F_n) \wedge H_{k(n)}$.

Hence $S(F_0, F_2, ... F_n) \wedge H_k = S^*\left(H_{k(n)}\right) \wedge H_k$

$$< E_{k(n)+1} \wedge H_{k(n)+1}$$

$$= 0$$

Thus in both cases $S(F_0, F_2, ... F_n)$ is disjoint from H_k.By the choice of $F(H_k)$ this means $|F(H_k)| > n$.

Let $F_{n+1}(k)$ be the $(n+1)^{st}$ term of $F(H_k)$ for $k > k(n)$. This exists since we have already proved that $|F(H_k)| > n$. Now take F_{n+1} =Min { $F_{n+1}(k): k\ >\ k(n)$}.

Choose some $k(n+ 1 » k(n)$ such that $F_{n+1} = F_{n+1}(k(n+1))$.
Let $k > k(n+1)$.

Clearly $F_{n+1} \leq F_{n+1}(k), also\ \ F(H_k)/n = F(H_{k(n)})/n$

$$= (F_1, F_2, \dots F_n)\ and\ H_k < H_{k(n+1)}$$

By (ii) of claim-(1) above we obtain $F_{n+1}(k) \leq\ F_{n+1}\big(k(n+1)\big) = F_{n+1}$. Hence $F_{n+1} = F_{n+1}(k)$ whenever $k\ \geq\ k(n+1)$ this means $(F_1, F_2, \dots F_n)\ =\ F(H_k)/n+1$ for each $k\ >\ k(n+1)$. Thus

Claim-(2) holds.

Now consider $(E_1, F_1, E_2, F_2, \dots, E_n, F_n)$ such that
$E_i = S(F_0, F_1, F_2, \dots F_{i-1})$ for

$1 \leq\ i\ \leq\ n$ And $F_0 := X$ this is an admissible sequence in $G^*(K, X)$. By the definition of fuzzy winning strategy we have $Inf F_n\ =\ 0$ Also by claim-(2). Each F_n is in terms of some $F(H_k)$.

Then from (ii) of claim-(1), it follows that $H_k\ < F_n$ for each F_n Therefore we have $Inf_{n\geq 1} H_n \leq Inf_{n\geq 1} F_n$ But $Inf_{n\geq 1} F_n\ =\ O$. Therefore it follows that $Inf_{n\geq 1} H_n\ =\ O$. Thus S^* is a fuzzy stationary winning strategy for Player I in $G^*(K, X)$.

2.1.9 Proposition

Let K_1 and K_2 be two classes of fuzzy topological spaces with $K_1 \subset K_2$ and if Player I has a fuzzy winning strategy in $G^*(\ K_1, X)$, then he has a fuzzy winning strategy in $G^*(K_2, X)$.

Proof.

From Theorem 2.2.8 it follows that Player I has a fuzzy stationary winning strategy in $G^*(K_1, X)$. say S. From theorem 2.2.8 it suffices to prove that Player I has a fuzzy stationary winning strategy in $G^*(K_2, X)$. Now $S: I^x \xrightarrow{into} I^x \cap K_1$, Then by result 2.2.7 we have $S(F) < F$ where $F \in I^x$. Where and if $\{F_n : n \geq N\} \subseteq I^x$ satisfies $S(X) \wedge F_1 = 0$ and $S(F_n) \wedge F_{n+1} = 0$ for all $n \geq 1$, then $Inf_{n\geq1}F_n = O$.

Now define $S^*: I^x \xrightarrow{into} I^x \cap K_1$ by $F \rightarrow S(F) \wedge K_2$ now we will show that S^* is a fuzzy winning strategy for $G^*(K_2, X)$.

Now $S^*(F) = S(F) \wedge K_2$

$$\leq S(F)$$

$$\leq F$$

Therefore S^* is a stationary strategy for Player I in $G^*(K_2, X)$.

Now to prove that S^* is winning, we want to prove that Player I wins every play of the form $(S^*(X), F_1, S^*(F_1), \ldots..)$. For that we want to prove that $Inf_{n\geq1}F_n = O$.

Now we have $S^*(X) \wedge F_1 = [S(X) \wedge K_2] \wedge F_1$

$$= s(X) \wedge K_2 \wedge F_1$$

$= 0$ Since S is a stationary winning strategy of Player I in $G^*(K_1, X)$.

Also $S^*(F_n) \wedge F_{n+1} = S(F_n) \wedge K_2 \wedge F_{n+1}$

By Result 2.2.7 it follows that $Inf_{n\geq1}F_n = O$. Therefore S^* is a fuzzy stationary winning strategy for Player I in $G^*(K_2, X)$.

2.1.10 Proposition

Let Y is a fuzzy closed subspace of a fuzzy topological space X.If Player I has a fuzzy winning strategy in $G^*(K,X)$. Then he has a winning strategy in $G^*(K,Y)$.

Proof.

Let $S^*: I^x \xrightarrow{into} I^x \cap K$ be a fuzzy stationary winning strategy of $G^*(K,X)$.

Now define $S^*: I^Y \xrightarrow{into} I^Y \cap K$ by $F' \longrightarrow S(F) \wedge Y$ where $F' = F \wedge Y$ and $F \in I^x$ now $S^*(F') \; = \; S(F) \wedge Y$

$$< F \wedge Y$$

$$= \; F'$$

Thus S^* is a fuzzy stationary strategy of Player I in $G^*(K,X)$.

Let $\{ F_n{}' : n \geq 1\} \; \subset \; I^Y$ where $F_n{}' = F_n \wedge Y$ Y for some $F_n \in I^x$

Now $S^*(Y) \; \wedge F_1{}' = [S(X)] \wedge \; Y] \; \wedge F_1{}'$

$$= [S(X)] \wedge \; Y] \; \wedge [F_1 \wedge Y]$$

$= S(X) \wedge \; Y \wedge F_1$

$= \;$ 0 since S is winning

Also $S^*(F_n{}') \; \wedge F_{n+1}{}'$ follows clearly. Therefore from Result 2.2.7, it follows that

$Inf_{n\geq1}F_n = O$. Therefore it follows that $Inf_{n\geq1}F_n{}' = O$. Thus proving S^* is a fuzzy stationary winning strategy of Player I in $G^*(K, X)$.

2.2 Finite and Countable Unions

Clearly we have $K \subseteq FK$ and $X \in FK$ implies $I^x \subseteq FK$.

2.2.1. Proposition

If Player I has a fuzzy winning strategy in $G^*(FK, X)$, then he has a' fuzzy winning strategy in $G^*(K, X)$.

Proof.

Let S be a fuzzy winning strategy for Player I in $G^*(FK, X)$. We will try to define a fuzzy strategy t for $G^*(K, X)$, Now take $E_0 = X, E_1 = S(E_0)$ and $F_0 = E_0$ Now $E_1 \in I^x \cap FK$ .Therefore $E_1 = \vee \{ H_{1,m} : m: \leq k_1\}$ where $\{ H_{1,m} : m: \leq k_1\} \subseteq I^x \cap K$. We set $F_1 = H_{1,0}$ and $1(F_0) = F_1$. Also take $F_2 \in I^x$ in such a way that $F_1 \wedge F_2 = 0$ and also set

$F_3 = F_2 \wedge H_{1,1}$ and $t(F_0, F_1, F_2) = F_3$. Continuing like this we get an admissible sequence $(F_0, F_1, \dots, F_{2k1})$ for $G^*(K, X)$. Take $F_{2k1+1} = t(F_0, F_1, \dots, F_{2k1}) = F_{2k1} \wedge H_{1,1k}$. Take $F_{2k1+2} \in I^x$.

With $F_{2k1+2} \leq F_{2k1}$ and $F_{2k1+2} \wedge F_{2k1+1} = O$. Take $E_2 = F_{2k1+2}$, now clearly $E_1 \wedge E_2 = 0$ and set $E_3 = S(E_0, E_1, E_2)$.

Since $E_3 \in I^x \cap FK$, we have $E_3 = \vee \{H_{3,m}, \quad m \leq K_3$, Where each $H_{3,m} \in I^x \cap K$.

Continuing like this we get the Play $(E_0, E_1, E_2, \dots.)$ of $G^*(FK, X)$and $(F_0, F_1, F_2, \dots.)$ Of $G^*(FK, X)$, Since S is a fuzzy winning strategy for $G^*(FK, X)$, $Inf_{n\geq 1} F_{2n} = O$.

Now $\{E_{2n}: n \in N\} \subseteq \{F_{2n} : n \in N\}$. Therefore it follows that $Inf_{n\geq 1} F_{2n} = O$. Therefore t is a fuzzy winning strategy for Player I in $G^*(K, X)$.

2.2.2 Remark

From $K \subseteq FK$ and Proposition 2.2.9 it follows that if Player I has a fuzzy winning strategy in $G^*(K, X)$, then he has a fuzzy winning strategy in $G^*(FK, X)$.

From Remark 2.3.2 and Proposition 2.3.1 we get

2.2.3 Theorem

Player I has a fuzzy winning strategy in $G^*(K, X)$ if and only if he has the same in $G^*(FK, X)$.

2.2.4 Proposition

If a fuzzy topological space X has a fuzzy closed countable α-shading $\{X_n : n \in N\}$ such that Player I has a fuzzy winning strategy in $G^*(K, X_n)$ for each $n \in N$ then he has a fuzzy winning strategy in $G^*(K, X)$.

Proof.

Let Sn be a fuzzy stationery winning strategy for Player I in $G^*(K, X_n)$for each $n \in N$, now it is enough if we prove that Player I has a fuzzy winning strategy in $G^*(FK, X)$.

Now we take $S(X) = S_1(X)$ and assume that $(E_1, F_1, E_2, F_2, \ldots, E_n, F_n)$ is an admissible sequence in $G^*(FK, X)$ such that $E_i = S(F_1, F_2, F_3, \ldots F_{i-1})$ for each $i \leq n$ where $F_0 = X$ Take $E_{n+1} = S(F_1, F_2, F_3, \ldots F_n) = Sup_{k \leq n+1} S_k(F_n \wedge X_k)$.

Consider the Play $(E_1, F_1, E_2, F_2, \ldots.)$ in $G^*(FK, X)$ such that $E_n = S(F_1, F_2, F_3, \ldots F_{n-1})$ for all $n \geq 1$. Now take an $m \geq 1$. By definition of Play we have $E_{n+1} \wedge F_{n+1} = 0 \cdots\cdots$ (1)

Here $E_{n+1} = Sup_{k \leq n+1} S_k(F_n \wedge X_k)$

$$\geq S_m(F_n \wedge X_m)$$

Also $F_{n+1} \wedge X_m \leq F_{n+1}$, therefore from (1) it follows that

$[S_m(F_n \wedge X_m)] \wedge [F_{n+1} \wedge X_m] = 0$ For each $n \geq m$, now since S_m is a stationary winning strategy for Player I in $G^*(K, X_m)$, we have

$S_m(F_n \wedge X_m) \leq F_n \wedge X_m$ For each $n \geq m$,

Therefore $F_n \wedge X_m] \wedge [F_{n+1} \wedge X_m] = 0$ for each $n \geq m$, thus $\wedge_{n \geq m} [F_{n+1} \wedge X_m] = 0$ we also have $F_{n+1} < F_n$ and hence it follows that $Inf_{n \geq 1} F_n = O$. Thus Player I has a winning strategy in $G^*(FK, X)$ hence the proof is complete by Theorem 2.3.3.

2.2.5 Theorem

Let X be a fuzzy topological space with a fuzzy subset E such that $E \in I^x \cap E$. If Player I has a fuzzy winning strategy in $G^*(K, F)$ for each $F \in I^x$ with $E \wedge F = 0$, then Player I has a fuzzy winning stratey in $G^*(K, X)$.

Proof.

For each $F \in I^x$ with $E \wedge F = 0$, Let S_F is a fuzzy stationary winning strategy for Player I in $G^*(K, F)$. Now we will find out a fuzzy winning strategy S for Player I in $G^*(K, X)$.

Define $S(X) = E$ and $(E_1, F_1, E_2, F_2, \dots, E_n, F_n)$ be an admissible sequence in $G^*(K, X)$ such that $E_1 = S(F_0, F_1, F_2, \dots F_{i-1})$ for each $i \leq n$ where $F_0 = X$. Take $E_{n+1} = S(F_0, F_1, F_2, \dots F_n) = S_{F\,1}(F_n)$. Consider the play $(E_1, F_1, E_2, F_2, \dots)$ now clearly $E_{n+1} \wedge F_{n+1} = O$. That is $S_{F\,1}(F_n) \wedge F_{n+1} = O$. Also $S_{F\,1}(X) \wedge F_1 = E_1 \wedge F_1 = 0$ Since $S_{F\,1}$ is a stationery winning strategy, it follows that $Inf_{n\geq 1}F_n = O$. Thus Player I has a fuzzy winning strategy in $G^*(K, X)$.

2.3 Games and Mappings

2.3.1. Theorem

Let X and Y be two fuzzy topological spaces and $\boldsymbol{K_1}$ and $\boldsymbol{K_2}$ be two classes of fuzzy topological spaces such that $X \in \boldsymbol{K_1}$ and $Y \in \boldsymbol{K_2}$. If I is an F-continous function from X on to Y which maps all $E \in I^x \cap \boldsymbol{K_1}$ to $f(E) \in I^x \cap \boldsymbol{K_2}$ and if player I has a fuzzy winning strategy in $G^*\ (\boldsymbol{K_1}, X)$, then Player I has a fuzzy winning strategy in $G^*\ (\boldsymbol{K_2}, X)$.

Proof.

Let S be a fuzzy stationary winning strategy for Player I in $G^*\ (\boldsymbol{K_1}, X)$.Thus player I wins every play of the form $(S(X)\ , F_1, S(F_1), \ldots\ldots)$. Now we will define a stationary winning strategy t for Player I in $G^*(\boldsymbol{K_2}, Y)$. Now consider the play $(t(Y), P_1, t(P_1), P_2 \ldots\ldots)$ where $P_n = t(F_n)\ and\ t : I^Y \xrightarrow{into} I^Y \cap \boldsymbol{K_2}$ is defined by $t(P_n) = f[S(F_n)]$. Now t is a stationary winning strategy for $G^*(K2, Y)$.

For, $t(F_n) = f[S(F_n)]$

$< f(F_n)$

$= P_n$. There fore t is a fuzzy stationery strategy.

Now $t(P_n) \wedge P_{n+1} = f[S(F_n)] \wedge f(F_{n+1})$

$= f[S(F_n) \wedge F_{n+1}]$

$= f(O)$

$= 0$

Also $t(Y) \wedge P_1 = f[S(X)] \wedge P_1$

$= f[S(X)] \wedge f(F_1)$

$= f[S(X) \wedge F_1]$

$= f(O)$

$= 0$

Therefore it follows from Result 2.2.7 that $Inf_{n\geq 1} F_n = O$ and hence t is a stationary winning strategy for Player I in $G^*(\boldsymbol{K_2}, Y)$.

2.3.2 Theorem

Let $f: X \xrightarrow{into} Y$ be an F-continuous F-closed mapping such that $f^{-1}(E) \in I^x \cap \boldsymbol{K_1}$ whenever $E \in I^x \cap \boldsymbol{K_2}$. Then if Player I has a fuzzy winning strategy in $G^*(\boldsymbol{K_2}, Y)$, then Player I has a fuzzy winning strategy in $G^*(\boldsymbol{K_1}, X)$.

Proof.

Let S be a fuzzy stationery winning strategy for Player I in $G^*(\boldsymbol{K_2}, Y)$.Therefore Player I wins every play of the form $(S(Y), F_1, S(F_1), \ldots\ldots)$. Now we will define a function.

$t: I^x \xrightarrow{into} I^x \cap \boldsymbol{K_1}$ As follows, Now $f: X \xrightarrow{into} Y$ is F-closed and hence we take

$P_n = f^{-1}(F_n)$ Where $P_n \in I^x$ and $t(P_n) = f^{-1}[S(F_n)]$ for an $P_n \in I^x$

Now $t(P_n) = f^{-1}[S(F_n)]$

$< f^{-1}(F_n)$

$= P_n$. Thus t is a fuzzy stationary strategy.

Now consider the play $(t(X),\ P_1,\ t(P_1), \ldots.)$

$t(P_n) \wedge P_{n+1} = f^{-1}[S(F_n)] \wedge P_n$

$= f^{-1}[S(F_n)] \wedge f^{-1}(F_{n+1})$

$= f^{-1}[S(F_n) \wedge F_{n+1}]$

$= f^{-1}(0)$

$= 0.$

Also $t(X) \wedge P_1 = f^{-1}[S(X)] \wedge P_1$

$= f^{-1}[S(X)] \wedge f^{-1}(F_1)$

$= f^{-1}[S(X) \wedge F_1]$

$= f^{-1}(0)$

$$= (0).$$

Therefore from Result 2.2.7 it follows that $Inf P_n = O$ and hence t is a winning strategy also. Thus t is a fuzzy winning strategy for Player I in $G^*(\boldsymbol{K}_1, X)$. This completes the proof.

As an immediate consequence of Theorem 2.4,1 and Theorem 2.4.2 we get the following two Theorems.

2.2.3 Theorem

Let X and Y are two fuzzy topological spaces and $f: X \xrightarrow{into} Y$ be an F-continuous function and $f^{-1}(E) \in I^x \cap \boldsymbol{K}_1$.

Whenever $E \in I^x \cap \boldsymbol{K}_2$. If Player II has a fuzzy winning strategy in $G^*(\boldsymbol{K}_1, X)$.Then Player II has a fuzzy winning strategy in $G^*(\boldsymbol{K}_2, Y)$.

2.3.4 Theorem

Let $f: X \xrightarrow{into} Y$ be an F-continuous F-closed mapping such that $f^{-1}(E) \in I^Y \cap \boldsymbol{K}_2$ whenever $E \in I^x \cap \boldsymbol{K}_1$. If Player II has a fuzzy winning strategy in $G^*(\boldsymbol{K}_2, Y)$ then Player II has a fuzzy winning strategy in $G^*(\boldsymbol{K}_1, X)$.

2.3.5 Definition

$[M; B_2]$ Let $O \leq a < 1 (resp. O < a \leq 1)$.

An F-closed F-continuous function f from a fuzzy topological space X to a fuzzy topological space Y is said to be α-perfect ($resp. \alpha^*$-perfect) ifand only $f^{-1}(y)$ is α^*-compact ($resp. \alpha^*$-compact) for each $y \in Y$.

2.3.6. Definition

A class $\boldsymbol{K}$ of fuzzy topological spaces is said to be α-perfect if $X \in K$ IS equivalent to $Y \in K$, provided that there exists an α-perfect map from X onto Y.

From Theorems 2.4.1, 2.4.2, 2.4.3 and 2.4.4 next theorems follow immediately.

2.3.7 Theorem

Let $\boldsymbol{K}$ be an α-perfect class .of fuzzy topological spaces and if there is an α-perfect map from X on to Y, then

(i) If Player I has a fuzzy winning strategy in $G^*(K, X)$.then he has the same in $G^*(\boldsymbol{K_2}, Y)$.

(ii) If Player II has a fuzzy winning strategy in $G^*(K, X)$.then he has the same in $G^*(K, Y)$.

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