And Scientific Research

Al-Qadisiyah University

College of Education

Department of Mathematics

S- Essential and e-small submodules

A research

Submitted to the College of Education, University of Al-Qadisiya in Partial Fulfilment of the Requirements for the Degree of Bachelor of Science in mathematics.

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بِسْمِ اللَّهِ الرَّحْمَنِ الرَّحِيمِ

قَالَ الَّذِي عِنْدَهُ عِلْمٌ مِنَ الْكِتَابِ أَنَا آتِيكَ بِهِ قَبْلَ أَنْ يَرْتَدَّ إِلَيْكَ طَرْفُكَ فَلَمَّا رَآهُ مُسْتَقِرًّا عِنْدَهُ قَالَ هَذَا مِنْ فَضْلِ رَبِّي لِيَبْلُوَنِي أَأَشْكُرُ أَمْ أَكْفُرُ وَمَنْ شَكَرَ فَإِنَّمَا يَشْكُرُ لِنَفْسِهِ وَمَنْ كَفَرَ فَإِنَّ رَبِّي غَنِيٌّ كَرِيمٌ

صدق الله العلي العظيم

سورة النمل (40)

الشكر والاهداء

إلى كل من علمني علما نافعا ولو حرفا, إلى كل من أنار لي الطريق إلى النجاح إلى من ارشدني وعلمني أتقدم بالشكر والعرفان الجزيل, للأستاذ /فرحان داخل الذي افادنا من علمه مما ساعدنا في اعداد هذا المشروع واخراجه بهذه الصوره التي اجتهدنا ان تكون بافضل صورة قدر المستطاع ......

والشكر ايضا الى كل من يقرأ هذا البحث بغرض الاطالع والاستفادة منه ومن ثم المقدرة على التحديث والتطوير والوصول الى االفضل بإذن الله والشكر الجزيل والامتنان الكبير الى الأب الغالي والأم الغالية فهما اعز النعم التي انعم الله بها علينا فما كان لنا سندا وعونا لإعداد هذا البحث من خلال توفير الجو الملائم للدراسة والاستذكار .

ولابد لنا ونحن نخطو خطواتنا الاخيرة في الحياة الجامعية من وقفه نعود إلى الاعوام قضيناها في رحاب الجامعة مع أساتذتنا الكرام الذين قدموا لنا الكثير باذلين بذالك جهودا كبيرة في بناء جيل الغد لتبعث ألامة من جديد ....وقبل أن نمضي تقدم أسمى آيات الشكر و الامتنان والتقدير والمحبةإلى الذين حملوا أقدس رسالة في الحياة ...الى جميع أساتذتنا الافاضل.. الذين مهدوا لنا طريق العلم والمعرفة.

الاهداء

إلى اليد الطاهرة التي أزالت من أمامنا أشواك الطريق

ورسمت المستقبل بخطوط من الامل والثقة

إلى الذي لا تفيه الكلمات والشكر والعرفان بالجميل أبي الحبيب

إلى من ركع العطاء أمام قدميها

وأعطتنا من دمها وروحها وعمرها حبا وتصميما

ودفعا لغٍد أجمل

إلى الغالية التي التي نرى الامل من عينيها أمي الحبيبة

إلى أزهار النرجس التي تفيض حباً وطفولةً ونقا ًء

وعطراً

إلى من أخذ بيدي ... ورسم الاملفي كل خطوة مشيتها

إلى أصدقائي الذين تسكن صورهم وأصواتهم الاجمل في

شكري الجزيل وامتناني

CONTANTES

Introduction

Let M be an R-module. Asubmodule A of an R-module M is said to be essential in M (denoted by $A \leq_{ess} M$) if $A \cap W \neq (0)$ for every non-zero submodule W of M equivalenty $A \leq_{ess} M$ if whenever $A \cap W = 0$, $W \leq M$ then W= 0. Asubmodule A of an R-module M is said to be small in M if Whenever N+W=M ,W submodule of M implies W=M.

The socal of an R-module M is denoted by Soc(M) and defined as the sum of the simple submodules of M. If M has no simple submodule then we set Soc(M) = 0 . Let M be a right R-module The Jocobson radical of M denoted by J(M)

And defined as the intersection of all maximal Submodules of M . If M has no maximal Submodules then we set J(M)=M . let M and N be modules over ring R. A function

$f: M \longrightarrow N$ is an R-module homomorphism if f (m+ n) = f(m) + f(n) and f (rm) = rf (m) for all $m,n \in M$ and $r \in R$.

This work consists of two chapters . In chapter one we deal with certain know result which is useful throught this work .In chapter two we study e- small and s- Essential submodules . Let N be a submodule of a module M . N is said to be e- samall in M if N+L=M, When $L \ll_{ess}$ implies L=M. And Let N be a submodule of a module M. N is said to be e- small in M if $N \cap L=0$, When $L \ll_{ess}$ implies L=0. and some properties about it .Let N be a submodule of a module M . Also who show that $N \ll_{ess} M$ if and only if X+N = M, then X is a direct summand of M with M/X a semisimple module.As well as explain us who close tha notion of s- Essential submodules at to then of Essential submodule in addition. We use the concepts of e-small and s-essential submodules to characterize some properties of homomorphisms.

Let $0 \neq K \leq M$ be a module. Then $K \trianglelefteq_s M$ if and only if for each $0 \neq x \in M$, if $Rx \ll M$, then there is an element $r \in R$ such that $0 \neq rx \in K$. Let N be a module and N,K,L are submodules of M with $K \subseteq N$. If $N \ll_{ess} M$, then $K \ll_{ess} M$ and $N/K \ll_{ess} M/K$., $N + L \ll_e M$ if and only if $N \ll_{ess} M$ and $L \ll_{ess} M$.,If $K \ll_{ess} M$ and $f : M \to N$ is a homomorphism, then $f(K) \ll_{ess} N$.

if $K \ll_{ess} M \subseteq N$, then $K \ll_{ess} N$.

Assume that $K_1 \subseteq M_1 \subseteq M$, $K_2 \subseteq M_2 \subseteq M$ and $M = M_1 \oplus M_2$, then $K_1 \oplus K_2 \ll_{ess} M_1 \oplus M_2$ if and only if $K_1 \ll_{ess} M_1$ and $K_2 \ll_{ess} M_2$.

Let M be a module. Then $Rad_e(M) = P\ \{ N \subseteq M \mid N \ll_{ess} M\}$. $Soc_s(M) = \cap\{ L \leq M \mid L \trianglelefteq_s M\}$.

Chapter

One

Preliminarie

Essential and Small submodules :-

Definition (1.1)[5] :- A submodul A of an R-module M is said to be essential in M (denoted by $A \leq_{ess} M$), if $A \cap W \neq (0)$ for every non-zero submodul W of M.Equivalently $A \leq_{ess} M$ if whenever $A \cap W = 0, W \leq M$ then $W = 0.''$

Example (1.2):- Find an essential in Z_{12}

Solution/

$$M=Z_{12}$$

<0> ={0}

<2> ={0,2,4,6,8,10}

<3> ={ 0,3,6,9}

<4> ={ 0,4,8}

<6> ={ 0,6}

<12> ={ 0,12,2,3,4,5,6,7,8,9,10,11}

<0> ∩<2> =0 , <0> ∩ <3> =0 , <0>∩<4> =0 , <0> ∩<6> =0

<0> ={0} is $\nleq$ ess

<2> ∩<0> =0

<2> ∩<3> ={0,6}

<2> ∩<4> ={0,4,8}

<2> ∩<6> ={0,6}

< 2> ∩< 12> = { 0,2,4,6,8,10}

$<2>$ is $\leq_{ess}$

$<3> \cap <0> = 0$

$<3> \cap <2> = \{0,6\}$

$<3> \cap <4> = \{0\}$

$<3>$ is $\not\leq_{ess}$

$<4> \cap <0> = 0$ $\quad$ $<4> \cap <3> = \{0\}$ $\quad$ $<4> \cap <2> = \{0,4,8\}$

$<4>$ is $\not\leq_{ess}$

$<6> \cap <0> = 0$ $\quad$ $<4> \cap <3> = \{0,6\}$ $\quad$ $<6> \cap <2> = \{0,6\}$ $\quad$ $<6> \cap <4> = \{0\}$

$<6>$ is $\not\leq_{ess}$

$<12> \cap <0> = 0$ $\quad$ $<12> \cap <2> = \{0,2,4,6,8,10\}$ $\quad$ $<12> \cap <3> = \{0,3,6,9\}$

$<12> \cap <4> = \{0,4,8\}$ $\quad$ $<12> \cap <6> = \{0,6\}$ $\quad$ $<12> \cap <12> = Z_{12}$

$<12>$ is $\not\leq_{ess}$

Definition (1.3) [5] :- Let N be a submodule of module M

N is said small in M if whenever $M = N + W$, W submodul of M implies $W = M$.

For example ,$<2>$ is a small in M, where $M = Z_4$.

Definition (1.4)[5]:- Let A is submodule of module M A is direct summand of M and denoted by A⊕B=M if $M = A + B$ and A∩B=0.

Definition (1.5) [5]: - Let M be an R-module N , A is called Semisimple if A≤M then A⊆ ⊕ M

Definition (1.6) [5] :- for R-modules N and A. N is said to be A- projective, if every submodule X of A, any homomorphism $\emptyset: N \longmapsto \frac{A}{X}$ can be lifted to a homorphism, $\psi: N \longmapsto A$, that is if $\pi: A \longmapsto \frac{A}{X}$, be the-natural epiomorphism, then there exists a homorphism $\psi: N \longmapsto A$

$$\begin{tikzcd} & N \arrow[dl, dashed, "\psi"'] \arrow[dr, "\emptyset"] & \\ A \arrow[rr, "\pi"'] & & A/X \end{tikzcd}$$

such that $\pi \circ \psi = \emptyset$.

M is called projective if M is N-projective for every R-module N. If M is M-projective, M is called self-projective".

For examples:

(1) Z as Z-module is projective.
(2) Z_2 as Z-module is self-projective.

Z_{p^∞} as Z- module is Z-projective.

Definition (1.7) [5] : The socal of a an R-module M is denoted by Soc(M) and defined as the sum of the simple submodules of M. If M has no simple submodule then we set Soc(M)=0.

For examples :

(1) $\mathrm{Soc}(Z_Z) = 0$;

(2) $\mathrm{Soc}(Z_6)= Z_6$;

(3) $\mathrm{Soc}(Z_4) =< 2 >$.

Definition (1.8) [5] :- Let M be a right R-module . The Jocobson radical of M denoted by $\mathrm{J}(M)$ and defined as the intersection of all maximal submodules of M. If M has no maximal submodule, then we set $\mathrm{J}(M) = M$.

For examples :

(1) $\mathrm{J}(Z_Z) = 0$;

(2) $\mathrm{J}(Z_6) = 0$;

(3) $\mathrm{J}(Q_Z) = Q_Z$;

(4) $\mathrm{J}(Z_4) =< 2 >$.

Definition (1.9) [5] :- Let M and N be modules over ring R. A function $f: M \longmapsto N$ is an R-module homomorphism if $f(m+n) = f(m) + f(n)$ and $f(rm) = rf(m)$ for all $m, n \in M$ and $r \in R$.

Chapter

Two

e-smallands-essential submodul

e-Small and s-Essential Submodules:-

All result of this chapter from [14]

Definition (2.1) :- Let N be a submodule of a module M.

N is said to be e- samall in M **if N+L=M**, When **L$\ll_{ess}$** implies **L=M**

Example(2.2) :- find an **e**-small in Z_{24}

N={ 0,4,8,12,16,20} **L**= { 0,2,4,6,8,10,12,14,16,18,20,22}

Solution/

N is **e**-small of **M**

Since **L** essential of **M** **L+N=M**

Definition (2.3) :- Let N be a submodule of a module M.

N is said to be **e**- small in **M if N ∩ L=0**, When **L$\ll_{ess}$** implies **L=0**

Example (2.4) :- find a **s**- essential **in Z_6**

L= { 0,2,4} N={ 0,3}

N is **e**-small of **M**

K∩N =0 ,K≪N

Proposition 2.5 -: Let N be a submodule of a module M. The following are equivalent.

(1) N $\ll_{ess}$M;

(2) if X+N = M, then X is a direct summand of M with M/X a semisimple module.

Proof :

(1) ⇒ (2). Let **Y** be a complement of **X** in **M**, then $\mathbf{X} \oplus \mathbf{Y} \leq_{ess} \mathbf{M}$. Since

$\mathbf{X + Y + N = M}$ and $\mathbf{N} \ll_{ess} \mathbf{M}$, it follows that $\mathbf{X} \oplus \mathbf{Y} = \mathbf{M}$. To see that $\mathbf{M/X} \cong \mathbf{Y}$ is semisimple, let A be a submodule of Y . Then X + A + N = M. Arguing as above with X + A replacing **X**, we have that $\mathbf{X + A = X} \oplus A$ is a direct summand of **M**, implying that **A** is a direct summand of **Y** , so **M/X** is semisimple.

(2) ⇒ (1). Let $\mathbf{K} \leq_{ess} \mathbf{M}$ and $\mathbf{K + N = M}$, then K is a direct summand of **M**, so $\mathbf{K = M}$. We have $\mathbf{N} \ll_{ess} \mathbf{M}$. **In particular if M** is a projective module, then every e-small submodule N of M

The next proposition, which will be used frequently, explains how close the notion of s-essential submodules is to that of essential submodules.

Proposition 2.6:- Let $\mathbf{0 \neq K \leq M}$ be a module. Then $\mathbf{K} \trianglelefteq_s \mathbf{M}$ **if and only if**

for each $\mathbf{0 \neq x \in M}$**, if** $\mathbf{Rx \ll M}$, then there is an element $\mathbf{r \in R}$ such that $\mathbf{0 \neq rx \in K}$.

Proof. -:

Let **K** be a submodule of **M** and $\mathbf{K} \trianglelefteq_s$ M. For each $\mathbf{0 \neq x \in M}$, if $\mathbf{Rx \ll M}$, then $\mathbf{Rx \neq 0}$ and $\mathbf{K} \cap \mathbf{Rx} \neq \mathbf{0}$. Thus there is an element $\mathbf{r \in R}$ such that $\mathbf{0 \neq rx \in K}$.

(⇐) **Suppose L** is a small submodule of **M** and $\mathbf{0 \neq x \in L}$. We have $\mathbf{Rx \ll M}$, hence there exists an element $\mathbf{r \in R}$ such that $\mathbf{0 \neq rx \in K \cap L}$. That is, $\mathbf{K} \trianglelefteq_s \mathbf{M}$.

Proposition 2.7. Let **M** be a module.

(1) Assume that **N,K,L** are submodules of **M** with $\mathbf{K \subseteq N}$.

(a) If $\mathbf{N \ll_{ess} M}$, then $\mathbf{K \ll_{ess} M}$ and $\mathbf{N/K \ll_{ess} M/K}$.

(b) $\mathbf{N + L \ll e\ M}$ **if and only if** $\mathbf{N \ll_{ess} M}$ and $\mathbf{L \ll_{ess} M}$.

(2) If $\mathbf{K \ll_{ess} M}$ and $\mathbf{f : M \to N}$ is a homomorphism, then $\mathbf{f(K) \ll_{ess} N}$.

In particular

if $\mathbf{K \ll_{ess} M \subseteq N}$, then $\mathbf{K \ll_{ess} N}$.

(3) Assume that $\mathbf{K_1 \subseteq M_1 \subseteq M}$, $\mathbf{K_2 \subseteq M_2 \subseteq M}$ and $\mathbf{M = M_1 \oplus M_2}$, then

$\mathbf{K_1 \oplus K_2 \ll_{ess} M_1 \oplus M_2}$ **if and only if** $\mathbf{K_1 \ll_{ess} M_1}$ **and** $\mathbf{K_2 \ll_{ess} M_2}$.

Proof:-

(1) (a) Suppose that $\mathbf{L \leq_{ess} M}$ and $\mathbf{L+K = M}$, then $\mathbf{N+L = M}$, thus $\mathbf{L = M}$ for $N \ll_{ess} M$, **so** $\mathbf{K \ll_{ess} M}$

If $\mathbf{L \leq M}$ with $\mathbf{L/K \leq_{ess} M/K}$ and $\mathbf{L/K + N/K = M/K}$, then $\mathbf{N + L = M}$ and $\mathbf{L \leq_{ess} M}$.

Hence $\mathbf{L = M}$ and $\mathbf{L/K = M/K}$. Therefore $\mathbf{N/K \ll_{ess} M/K}$.

(b) The necessity follows immediately from (a). Conversely, suppose $\mathbf{K \leq_{ess} M}$ with $\mathbf{N + L + K = M}$, then $\mathbf{L + K = M}$ since $\mathbf{L + K \leq_{ess} M}$ and $\mathbf{N \ll e\ M}$. Whence $\mathbf{K = M}$ for $\mathbf{K \leq_{ess} M}$ and $\mathbf{L \ll_{ess} M}$.

(2) Suppose that $\mathbf{A \leq_{ess} N}$ and $\mathbf{A+f(K) = N}$. Then $\mathbf{f^{\leftarrow}(A) \leq_{ess} M}$, and $\mathbf{f^{\leftarrow}(A)}+K = M$

Since $\mathbf{K \ll_{ess} M}$, we have $\mathbf{f^{\leftarrow}(A) = M}$. Thus $\mathbf{f(K) \subseteq A}$ and $\mathbf{A = N}$. So $\mathbf{f(K) \ll_{ess} N}$.

(3) Immediate from (1) and (2).

It is proved in [13, Lemma 1.3] that if $K \ll M$ and $N/K \ll M/K$, then $N \ll M$.

The following example shows that the converse of Proposition 2.5 (a) is false.

Example 2.8. Assume that $\mathbf{R = \mathbb{Z}}$, $\mathbf{M} = \mathbb{Z}_{24}$ $\mathbf{K = 6\mathbb{Z}_{24}}$ and $\mathbf{N = 3\mathbb{Z}_{24}}$. Then $\mathbf{K \ll M}$ and

$\mathbf{N/K \ll_{ess} M/K}$. But **N** is not e-small in **M**.

Dually, we have the following conclusions on s-essential submodules.

Proposition 2.9. Let **M** be a module.

(1) Assume that **N,K,L** are submodules of **M** with $\mathbf{K \subseteq N}$.

(a) If $\mathbf{K \trianglelefteq s\ M}$, then $\mathbf{K \trianglelefteq s\ N}$ and $\mathbf{N \trianglelefteq s\ M}$.

(b) $\mathbf{N \cap L \trianglelefteq s\ M}$ if and only if $\mathbf{N \trianglelefteq s\ M}$ and $\mathbf{L \trianglelefteq s\ M}$.

(2) If $\mathbf{K \trianglelefteq s\ N}$ and $\mathbf{f : M \rightarrow N}$ is a homomorphism, then $\mathbf{f\leftarrow(K) \trianglelefteq s\ M}$.

(3) Assume that $\mathbf{K1 \subseteq M_1 \subseteq M, K_2 \subseteq M_2 \subseteq M}$ and $\mathbf{M = M_1 \oplus M_2}$, then

$\mathbf{K_1 \oplus K_2 \trianglelefteq s\ M_1 \oplus M_2}$ if and only if $\mathbf{K_1 \trianglelefteq s\ M1}$ and $\mathbf{K_2 \trianglelefteq s\ M_2}$.

The converse of Proposition 2.7 (1)(a) is **not true**

Example 2.10. Let $\mathbf{R = \mathbb{Z}, M = \mathbb{Z}_{36}, N = 6\mathbb{Z}_{36}}$ and $\mathbf{K = 18\mathbb{Z}_{36}}$. Then $\mathbf{K \trianglelefteq s\ N, N \trianglelefteq s\ M}$. But **K** is **not** s-essential in **M**.

The socle and radical of a module are important in the study of modules and rings. In [13], the radical of a module M is generalized as follows

$\mathbf{(M)} = \cap\{\mathbf{K \leq M \mid M/K}$ is singular and simple$\}$.

Furthermore, we have

Definition 2.11. Let M be a module. Define

$Rad_e(M) = \cap\{ N \leq_{ess} M \mid N$ is maximal in $M \}$, And

$\mathrm{Soc}_s(\mathbf{M}) = \mathbf{X}\{\ \mathbf{N} \ll \mathbf{M} \mid \mathbf{N}$ is minimal in $\mathbf{M}\ \}$.

Obviously,

$\mathrm{Soc}_s(\mathbf{M}) \subseteq \mathrm{Rad}(\mathbf{M}) \subseteq (\mathbf{M}) \subseteq \mathrm{Rad}_e(\mathbf{M})$

and

$\mathrm{Soc}_s(\mathbf{M}) \subseteq \mathrm{Soc}(\mathbf{M}) \subseteq \mathrm{Rad}_e(\mathbf{M})$**.**

In the following we use e-small submodules and s-essential submodules to characterize $\mathrm{Rad}_e(\mathbf{M})$ and $\mathrm{Soc}_s(\mathbf{M})$**.**

Theorem 2.10:- Let **M** be a module. Then

(1) $\mathrm{Rad}_e(\mathbf{M}) = \mathbf{P}\ \{\ \mathbf{N} \subseteq \mathbf{M} \mid \mathbf{N} \ll_{ess} \mathbf{M}\}$**.**

(2) $\mathrm{Soc}_s(\mathbf{M}) = \cap\{\ \mathbf{L} \leq \mathbf{M} \mid \mathbf{L} \trianglelefteq \mathbf{s}\ \mathbf{M}\}$**.**

Proof. (1). Let $\mathbf{U} = \mathbf{P}\ \{\ \mathbf{N} \subseteq \mathbf{M} \mid \mathbf{N} \ll_{ess} \mathbf{M}\}$. Suppose that $\mathbf{L} \ll_{ess} \mathbf{M}$ and $\mathbf{K} \leq_{ess} \mathbf{M}$ is maximal in **M**, hence $\mathbf{L} \leq \mathbf{K}$. Otherwise, we have $\mathbf{K} + \mathbf{L} = \mathbf{M}$. But $\mathbf{L} \ll_{ess} \mathbf{M}$**,** hence $\mathbf{K} = \mathbf{M}$**, a** contradiction. It follows that $\mathbf{U} \subseteq \mathbf{Rad_e(M)}$**.** On the other hand, for $\mathbf{x} \in \mathbf{Rad_e(M)}$ suppose that **Rx** is not e-small in **M**.

Set

$$\boldsymbol{\Gamma} = \{\ \mathbf{B} \mid \mathbf{B} \neq \mathbf{M},\ \mathbf{B} \leq_{ess} \mathbf{M} \text{ and } \mathbf{R}_x + \mathbf{B} = \mathbf{M}\}.$$

Clearly, Γ is a non-empty subposet of the lattice of submodules of M. By the Maximal Principle, Γ has a maximal element, say B_0. Now we claim that B_0 is maximal in M. Otherwise, there is a submodule C of M such that $B_0 \subsetneqq C \subsetneqq M$, thus

$$\mathbf{R_x + C \supseteq R_x + B_0 = M}$$

and $\mathbf{C} \leq_{ess} \mathbf{M}$, hence $\mathbf{C} \in \mathbf{\Gamma}$, which contradicts the maximality of $\mathbf{B_0}$. So $\mathbf{B_0}$ is maximal in **M** and $\mathbf{B_0} \leq_{ess} \mathbf{M}$. Thus $\mathbf{x} \in \mathrm{Rad}_e(\mathbf{M}) \subseteq \mathbf{B_0}$ **and** $\mathbf{R_x} \subseteq \mathbf{B_0}$.

Since $\mathbf{R_x + B_0 = M}$, it follows that $B_0 = M$, a contradiction. So $\mathbf{R_x} \ll_{ess} \mathbf{M}$, hence $\mathrm{Rad}_e(\mathbf{M}) \subseteq \mathbf{U}$. Therefore

$$\mathrm{Rad}_e(\mathbf{M}) = \sum \{ \mathbf{N} \subseteq \mathbf{M} \mid \mathbf{N} \ll_{ess} \mathbf{M} \}.$$

(2). Let $\mathbf{S} = \cap \{ \mathbf{L} \leq \mathbf{M} \mid \mathbf{L} \trianglelefteq_s \mathbf{M} \}$. Suppose that $\mathbf{L} \trianglelefteq_s \mathbf{M}$ and $\mathbf{K} \ll \mathbf{M}$ is minimal in**M**, then $\mathbf{K} \leq \mathbf{L}$. Otherwise, $\mathbf{K} \cap \mathbf{L} = \mathbf{0}$, hence $\mathbf{K} = \mathbf{0}$, a contradiction. So $\mathrm{Soc}_s(\mathbf{M}) \subseteq \mathbf{S}$. Note that $\mathbf{S} \subseteq \mathbf{Soc(M)}$, thus $\mathrm{Soc}_s(\mathbf{M})$ and S are semisimple

modules. If **S " Soc**$_s$**(M)**, there is a simple module **T** such that $\mathbf{T} \leq \mathbf{S}$ and T is not small in **M**. Let **K** be a proper submodule such that $\mathbf{K + T = M}$.

(a) If $\mathbf{K} \cap \mathbf{T} \neq \mathbf{0}$, then $\mathbf{T} \subseteq \mathbf{K}$, hence $\mathbf{K = M}$, a contradiction.

(b) If $\mathbf{K} \cap \mathbf{T} = \mathbf{0}$, then $\mathbf{M} = \mathbf{K} \oplus \mathbf{T}$. For each $\mathbf{H} \leq \mathbf{M}$, if $\mathbf{H} \ll \mathbf{M}$ and $\mathbf{K} \cap \mathbf{H} = 0$, then $\mathbf{H+K}$ is a proper submodule of **M** and $H \cong \mathbf{(H + K)/K}$ is a submodule of**M/K**, where $\mathbf{M/K} \cong \mathbf{T}$ is a simple module. Thus $\mathbf{H = 0}$. Then $\mathbf{K} \trianglelefteq_s \mathbf{M}$, that is, $\mathbf{T} \subseteq \mathbf{S} \subseteq \mathbf{K}$, a contradiction.

Thus $\mathbf{T} \ll \mathbf{M}$, a contradiction. Therefore $\mathbf{S} = {}_{\mathbf{Socs}}\mathbf{(M)}$.

Corollary 2.12. Let **M** and **N** be modules.

(1) If $\mathbf{f : M \to N}$ is an **R**–homomorphism, then $\mathbf{f(Rad_e(M)) \subseteq Rad_e(N)}$. In particular, Rade**(M)** is a fully invariant submodule of **M**.

(2) If every proper essential submodule of **M** is contained in a maximal submodule of **M**, then **Rad_e(M)** is the unique largest e-small submodule of M.

Proof.

(1) By **Proposition 2.7** and **Proposition 2.12**.

(2) For each essential submodule **K** of **M, if K $\neq$ M**, there is a maximal submodule **L** of **M** such that **K $\subseteq$ L**, then **L**$\leq_{ess}$ **M**. By the definition of **Rad_e(M),** Rad_e**(M) $\subseteq$ L**.

So Rad_e**(M) + K $\subseteq$ L $\subsetneq$ M**. Thus Rad_e**(M) $\ll_{ess}$ M**.

Dually, we have

Corollary 2.13. Let **M** and **N** be modules. Then

(1) If f : M $\rightarrow$ N is an R−homomorphism, then $f(Soc_s(M)) \subseteq Soc_s(N)$. Therefore, Socs(M) is a fully invariant submodules of M.

(2) If M = ⨁ni=1Mi, then Soc_s(M) =⨁n i=1Soc_s(Mi).

(3) If every non-zero small submodule of **M** contains a minimal submodule of **M**, then

Soc_s**(M)** is the unique least s-essential submodule of **M**.

Example 2.14. Let $\mathbf{R} = \mathbb{Z}$, $\mathbf{M} = \mathbb{Z}_{24}$ **and** $\mathbf{N} \leq \mathbf{M}$. All submodules of **M** have the following properties.

$N \leq M$	Small	e-small	essential	s-essential
$\mathbb{Z}_{24}$	×	×	√	√
$2\mathbb{Z}_{24}$	×	√	√	√
$3\mathbb{Z}_{24}$	×	×	×	√
$4\mathbb{Z}_{24}$	×	√	√	√
$6\mathbb{Z}_{24}$	√	√	×	√
$8\mathbb{Z}_{24}$	×	√	×	×
$12\mathbb{Z}_{24}$	√	√	×	√
0	√	√	×	×

According to the above chart, we have

(1) $\mathrm{Rad}(\mathbf{M}) = 6\mathbb{Z}_{24}$, $\mathrm{Rad}_e(\mathbf{M}) = 2\mathbb{Z}_{24}$, $\mathrm{Soc}(\mathbf{M}) = 4\mathbb{Z}_{24}$ **and** $\mathrm{Soc}_s(\mathbf{M}) = 12\mathbb{Z}_{24}$

(2) $\mathrm{Soc}_s(\mathbf{M}) \subsetneq \mathrm{Rad}(M) \subsetneq \mathrm{Rad}_e(\mathbf{M})$ and $\mathrm{Soc}_s(\mathbf{M}) \subsetneq \mathrm{Soc}(M) \subsetneq \mathrm{Rad}_e(\mathbf{M}$

E-small and s-essential homomorphisms

Definition 2.15. Let M and N be modules.

(1) An epimorphism $g : M \to N$ is e-small in case $\mathrm{Ker}g \ll_{\mathbf{ess}} M$.

(2) A monomorphism $f : M \to N$ is s-essential in case $\mathrm{Im}f \trianglelefteq s\ M$.

In the following, we give a useful characterization of e-small homomorphisms and s-essential homomorphisms.

Proposition 2.16. Let M and N be modules.

(1) An epimorphism $g : M \to N$ is e-small if and only if for each essential monomorphism h, if gh is epic, then h is epic.

(2) A monomorphism $f : M \to N$ is s-essential if and only if for each small epimorphism h, if hf is monic, then h is monic.

Proof.

(1) Let $g : M \to N$ be an epimorphism and $K = \mathrm{Ker}g$. Then there is a unique isomorphism $v : M/K \to N$, such that $v\pi = g$ where $\pi: M \to M/K$.

Thus it follows that for each homomorphism h, $v\pi h = gh$ is epic if and only if πh is epic.

(⇒) If g is e-small, then $K \ll_{\mathbf{ess}} M$. Since πh is epic, we have $\mathrm{Im}\ h + K = M$.

Note that h is an essential monomorphism, hence Im$h \leq_{ess}$ M, thus Imh = M. So

h is epic.

(⇐) Let L be an essential submodule of M. Let iL : L → M be the inclusion.

Then i L is essential. If K + L = M, then πiL is epic. By hypothesis, i L is epic,

that is, L = M. So K $\ll_{ess}$ M, hence g is e-small.

(2) Dual to (1).

Proposition 2.17. Suppose that the following diagram of modules and homomorphisms

is commutative and has exact rows.

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[r, "g"] \arrow[d, "\beta"'] & C \arrow[r] \arrow[d, "\gamma"'] & 0 \\
0 \arrow[r] & A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' \arrow[r] & 0
\end{tikzcd}$$

(1) If α is epic and g is e-small, then g′ is e-small.

(2) If γ is monic and f′ is s-essential, then f is s-essenti

Proof.

(1) Assume that g is e-small, then $\mathrm{Ker}g \ll e\ B$ and $\beta(\mathrm{Ker}g) \ll_{ess} B'$. It suffices to show $\mathrm{Ker}g' \leq \beta(\mathrm{Ker}g)$. Let $b' \in \mathrm{Ker}g'$. Since the bottom row is exact, there is an element $a \in A$ with $\alpha(a) = b'$. Since the diagram commutes

and the top row is exact, $b' = f'\alpha(a) = \beta f(a)$ and $gf(a) = 0$. Thus there is a $f(a) \in \mathrm{Ker}g$ such that $\beta(f(a)) = b'$. So $b' \in \beta(\mathrm{Ker}g)$, hence $\mathrm{Ker}g' \ll_{ess} B'$.

(2)Dual to (1).

Corollary 2.19. Consider the following diagram

$$\begin{tikzcd}
A \arrow[r, "\alpha_1"] \arrow[d, "\alpha_2"'] & B \arrow[d, "\beta_2"] \\
C \arrow[r, "\beta_1"'] & D
\end{tikzcd}$$

(1) Assume that the diagram is a pullback of β_1 and β_2. If β_1 is a s-essential monomorphism, so is α_1.

(2) Assume that the diagram is a pushout of $\alpha_1$and α_2. If $\alpha_1$is an e-small epimorphism, so is β_1.

Proof.

(1) Assume that the diagram is a pullback of β_1 and β_2 with β_1 a s-essential monomorphism. Then we have a full commutative diagram with exact rows by [7, Proposition 5.1].

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\alpha_1"] \arrow[d, "\alpha_2"'] & B \arrow[r, "\pi_1"] \arrow[d, "\beta_2"] & L \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & C \arrow[r, "\beta_1"'] & D \arrow[r, "\pi_2"] & L \arrow[r] & 0
\end{tikzcd}$$

By Proposition 3.3, α_1 is a s-essential monomorphism.

(2) Dual to (1).

Let R and S be two rings, if F : Mod−R → Mod−S define a Morita equivalence,by Proposition 3.2 we note that f : M → N is e-small (resp., s-essential)

if and only if F(f) : F(M) → F(N) is e-small (resp., s-essential).

For two rings R and S, a bimodule SUR is said to define a Morita duality,

if ${}_SU_R$ is a faithfully balanced bimodule such that SU and UR are injective

cogenerators. A presentation of Morita duality can be found in [2, §23, §24] and

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[11]. If M is a right R-module (left S-module), we let M^* = SHomR(M,U) (=

HomS(M,U)R), andM is said to be U-reflexive if the evaluation homomorphism

eM : M → M^{**} is an isomorphism. According to [2], let $R_R[U]$ and $S_R[U]$

denote the class of all U-reflexive right R-modules and that of all U-reflexive left

S-modules, respectively.

Theorem 2.20. Assume that ${}_SU_R$defines a Morita duality and f : M → N. If M,N are U-reflexive, then

(1) f is an e-small epimorphism if and only if f^* : N^* → M^* is a s-essential monomorphism.

(2) f is a s-essential monomorphism if and only if f^* : N^* → M^* is an e-small epimorphism.

Proof. (1) Let f : M → N be an e-small epimorphism, then f^* : N^* → M^* is monic by [2, Corollary 24.2]. We claim that f∗ is a s-essential monomorphism. Suppose that h : M^* → H is such that hf∗ is a monomorphism and h is a small epimorphism, then $(hf^*)^* = f^{**}h^*$ is an epimorphism and h^* is an essential monomorphism. Since MR and NR are U-reflexive, the evaluation homomorphisms σ_M : M → M^{**} and σ_N : N → N^{**} are isomorphisms, that is, the following diagram commutes:

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\sigma_M"'] & N \arrow[d, "\sigma_N"] \\ M^{**} \arrow[r, "f^{**}"] & N^{**} \end{tikzcd}$$

Since f is an e-small epimorphism, f^{**} is an e-small epimorphism. By Proposition 3.2, h^* is epic. By [2, Corollary 24.2] h is monic. Therefore f^* is a s-essential monomorphism by Proposition 3.2.

Conversely, let f^* : N^* → M^* be a s-essential monomorphism. By [2, Corollary

24.2], f : M → N is an epimorphism. We shall prove that f is e-small.

Suppose that h : H → M is an essential monomorphism such that fh is epimorphic, then $(fh)^* = h^*f^*$ is monic and h^* is a small epimorphism. By Proposition 3.2, h^* is monomorphism. By [2, Corollary 24.2], h is an epimorphism. So f is an e-small epimorphism by Proposition 3.2.

Dually, (2) can be proved.

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