University of Qadisiya

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Department of Mathematics

On the Subnormality of the Composition Operator C_σ

A search

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Abstract

Lat U denote the unit boll in the complix plane, the Hordy space H^2 is the set of functions $f(z)=\sum_{n=0}^{\infty} f^{\wedge}(n)z^n$ holomarphic on U such that $\sum_{n=0}^{\infty} |f^{\wedge}(n)|^2 < \infty$ with $f^{\wedge}(n)$ denotes then the Taylor coeffecient of f .

Let ϕ be a holomarphic self-map of U, the composition operator C_ϕ induced by ϕ is defined on H^2 by the equation

$$C_\phi f = f \circ \phi \quad (f \in H^2)$$

We have studied the subnormelity of the composetion operator induced by the bijective map ψ and descussed the adjoint of the composetion of the bijective map ψ We have look also at some known properties on composetion operators and tried to see the analogue properties in arder to show how the resultes are changed by changing the function ϕ in U.

In arder to make the work accessible to the reader, we have included some known results with the details of the proofs for some cases and proofs for the properties .

المستخلص

ليكن U يرمز إلى كرة الوحدة في المستوى العقدي، إن فضاء هاردي H^2 هو مجموعة كل

الدوال $f(z)=\sum_{n=0}^{\infty} f^{\wedge}(n)\, z^n$ التحليلية على U بحيث أن $\sum_{n=0}^{\infty} \left|f^{\wedge}(n)\right|^2 < \infty$, $f^{\wedge}(n)$ يرمز إلى معاملات تيلر

لتكن $\phi : U \to U$ دالة تحليلية على U ، المؤثر التركيبي المتولد من ϕ يعرف على فضاء هاردي

H^2 بواسطة :

$$C_{\phi} f = f \circ \phi \quad (f \in H^2).$$

درسنا في هذا البحث الطبيعية الجزئية للمؤثر التركيبي المتولد من الدالة المتقابلة ψ حيث ناقشنا المؤثر المرافق للمؤثر التركيبي المتولد من الدالة المتقابلة ψ . بالإضافة إلى ذلك نظرنا إلى بعض النتائج المعروفة وحاولنا الحصول على نتائج مناظرة لنتمكن من ملاحظة كيفية تغير النتائج عندما تتغير الدالة التحليلية ϕ .

ومن أجل جعل مهمة القارئ أكثر سهولة ، عرضنا بعض النتائج المعروفة عن المؤثرات التركيبية وعرضنا براهين مفصلة وكذلك برهنا بعض النتائج .

Contents

المقدمة

هذا البحث يشمل فصلين . في الفصل الأول , سوف نتناول الدالة المتقابلة ψ وخواصها ، ونناقش النقاط الصامتة الداخلية والخارجية للدالة ψ أيضا وكذلك نناقش أيضا الدوران المحوري حول الأصل للدالة ψ وكذلك نناقش أيضا هل الدالة ψ قطع ناقص , وكذلك نناقش أيضا هل الدالة ψ تحويل كسوري خطيّ .

في الفصل الثاني ، سوف نتناول المؤثر التركيبي C_ψ المتولد بالدالة ψ وخواصه ، وكذلك نناقش أيضا المرافق للمؤثر التركيبي C_ψ المتولد بالدالة σ , وكذلك نناقش أيضا هل المؤثر التركيبي C_ψ مؤثر تركيبي وحدوي وكذلك نناقش أيضا هل المؤثر التركيبي C_ψ مؤثر تركيبي طبيعي جزئي .

Chapter one

Properties of the Map ψ

Introduction

This search consists of two chaptars . In chaptar one ,we are going to study the bijective map ψ and proporties of ψ, and alsa discuss the interior and exterior fixed points of ψ and also discuss ψ is a rototion around the origen and ψ is elliptic and ψ is a linear fractional trancformation .

In chaptar two, we are going to study the Composetion Operator C_ψ induced by the map σ and proporties of C_ψ, and also discuss the adjoint of Composetion Operator C_ψ induced by the map σ and alsa discuss C_ψ is an unitary operator and discuss C_ψ is a normil operator and discuss C_ψ is a normility operator and discuss C_ψ is an subnormil operator.

Definition(1.1) :

Lat U = $\{z \in C: |z| < 1\}$ is a unit boll in complix plane C and

∂U= $\{z \in C: |z| = 1\}$ is a boundary of U.

Definition(1.2):

Lat $\psi : U \to U$ holomarphic on U and define $\psi(z) = \frac{-3z}{3-3\overline{\beta}z}$ $(z, \beta \in U)$

Proposition (1.3):

ψ is bijective .

Proof:

Since $\psi(z) = \frac{-3z}{3-3\overline{\beta}z}$ $(z, \beta \in z)$

Suppoise $\psi(z_1) = \psi(z_2)$ that is $\frac{-3z_1}{3-3\overline{\beta}z_1} = \frac{-3z_2}{3-3\overline{\beta}z_2}$, thus

$-9z_1 + 9\overline{\beta}z_1z_2 = -9z_2 + \overline{\beta}z_1z_2$, hence $z_1 = z_2$. Thus ψ is injective .

Let $y = \psi(z)$, that is $y = \frac{-3z}{3-3\overline{\beta}z}$, therefare, then $3y - 3\overline{\beta}yz = -3z$, hence

$z = \frac{-3y}{3-3\overline{\beta}y}$, $\psi(z) = \sigma\left(\frac{-3y}{3-3\overline{\beta}y}\right) = \frac{\frac{9y}{3-3\overline{\beta}y}}{3-3\overline{\beta}(\frac{-3y}{3-3\overline{\beta}y})} = \frac{\frac{9y}{3-3\overline{\beta}y}}{\frac{9-9\overline{\beta}y+9\overline{\beta}y}{3-3\overline{\beta}y}} = \frac{9y}{9} = y$, for every

$y \in U$, there exists $z \in U$ such that $\psi(z) = y$ .Thus ψ is surjective . Hance ψ is bijective.

Definition(1.4) :

A point $p \in C$ is a fixid point for the map $\emptyset$, if $\emptyset(z) = z$.

Proposition (1.5) :

0 , $\frac{2}{\bar{\beta}}$ are fixid points for ψ.

Proof :

Lat $\psi(z) = z$ that is $\frac{-3z}{3-3\bar{\beta}z} = z$, therefare$6z - 3\bar{\beta}z^2 = 0$.Hance ψ has two fixid points $z_1 = 0$, $z_2 = \frac{2}{\bar{\beta}}$

Definition(1.6):

Lat $\emptyset: U \to U$ be holomarphic map on U with a fixid point r, than:

1) r as interior fixid point for $\emptyset$ if $r \in U$
2) r as exterior fixid point for $\emptyset$ if $r \notin U$
3) r as boundary fixid point for $\emptyset$ if $r \in \partial U$

Proposition (1.7):

0 is interior fixid point and $\frac{2}{\bar{\beta}}$ is exterior fixid points for σ .

Proof :

Since ψ has two fixid points $z_1 = 0$, $z_2 = \frac{2}{\bar{\beta}}$

, $|z_1| = |0| = 0 < 1$.Thus z_1 as interior fixid point for ψ.

Since

$\beta \in U$, then $|\beta| < 1$, since $|\bar{\beta}| = |\beta|$ therefore $|\bar{\beta}| < 1$ and $1 < 2$ hence $|\bar{\beta}| < 2$ hence $\frac{2}{|\beta|} > 1$, since $\frac{2}{|\beta|} = \left|\frac{2}{\bar{\beta}}\right|$ hence $\left|\frac{2}{\bar{\beta}}\right| > 1$ hence $|z_2| = \left|\frac{2}{\bar{\beta}}\right| > 1$
then z_2 is exterior fixid point for ψ

Proposition (1.8) :

$$\psi^{-1}(z) = \frac{-3z}{3-3\bar{\beta}z} = \psi(z) .$$

Proof :

Let $y = \psi^{-1}(z)$, than $z = \psi(y)$, hance $z = \frac{-3y}{3-3\bar{\beta}y}$, thus $3z - 3\bar{\beta}yz = -3y$, therefare $-3z = 3y - 3\bar{\beta}yz$.Thus$-3z = y(3 - 3\bar{\beta}z)$, hance $y = \frac{-3z}{3-3\bar{\beta}z}$, then $\psi^{-1}(z) = \frac{-3z}{3-3\bar{\beta}z} = \psi(z)$.

Remark(1.9) :

If $\beta \in U$, then $\psi'(0) = -1$, $\psi'(\beta) = \frac{-1}{(1-|\beta|^2)^2}$.

Definition(1.10) :

Let $\emptyset: U \to U$ be holomarphic map on U. We say that ϕ is a rototion round the origin if there exists $r \in \partial U$ such that $\emptyset(z) = rz$ $(z \in U)$

Proposition (1.11):

If $\beta = 0$, then ψ as a rototion a round the origin

Proof:

Since $\psi(z) = \frac{-3z}{3-3\overline{\beta}z}$, since $\beta = 0$, hance

$\psi(z) = \frac{-3z}{3-3\overline{\beta}z}$,$\beta = 0, \psi(z) = -z = rz, r = -1, |r| = |-1| = 1, r \in \partial U$ than by

(1.10) ψ is a rototion a round the origen.

Definition(1.12):

Let $\emptyset: U \to U$ be holomarphic map on U. We say that ϕ is an elliptic if $\emptyset$

has interior fixid point and bijective.

Proposition (1.13):

ψ as an elliptic

Proof:

Since ψ has interior fixid point by(1.7) and ψ is bijective by (1.3) hance by

(1.12) ψ as an elliptic

Definition(1.14):

A linear fractional trancformation is a mapping of the form $\tau(z) = \frac{az+b}{cz+d}$ where a , b , c , and d are complix numbers , and we sometame denote it by $\tau_A(z)$ where A is the non-sangular 2×2 complix matrix A=$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Proposition (1.15) :

If $\beta \in U$, then ψ as a linear fractional trancformation .

Proof :

Since $\psi(z) = \frac{-3z}{3-3\overline{\beta}z}$ such that a = -3 , b = 0 , c = - 3$\overline{\beta}$, d = 3 and a , b , c , and d are complix numbers and A=$\begin{bmatrix} -3 & 0 \\ -3\overline{\beta} & 3 \end{bmatrix}$, hance by (1.14) ψ is a linear fractional trancformation .

Chapter two

Subnormality of the C_ψ

Definition(2.1):

Lat U denote the unit boll in the complex plane, the Hordy space H^2 is the set of functions $f(z) = \sum_{n=0}^{\infty} f^\wedge(n)\, z^n$ holomarphic on U such that $\sum_{n=0}^{\infty} |f^\wedge(n)|^2 < \infty$ with $f^\wedge(n)$ denotes then the Taylor coeffecient of f and $H^2 : U \to C$.

Remark (2.2) :

We can define an inner praduct of the Hordy space functions as follows: $f(z) = \sum_{n=0}^{\infty} f^\wedge(n)\, z^n$, $g(z) = \sum_{n=0}^{\infty} g^\wedge(n)\, z^n$, then the inner praduct of f and g is define $\langle f, g\rangle = \sum_{n=0}^{\infty} f^\wedge(n)\, \overline{g^\wedge(n)}$

Definition (2.3) :

Let $\alpha \in U$, define $K_\alpha = \frac{1}{1-\bar{\alpha} z}$. Since $\alpha \in U$ than hance the geometric series $\sum_{n=0}^{\infty} |\alpha|^{2n}$ is convergent and $K_\alpha = \sum_{n=0}^{\infty} (\bar{\alpha})^n z^n$ thus $K_\alpha \in H^2$

Definition(2.4) :

Lat $\phi : U \to U$ be holomarphic map on U, the composetion operator C_ϕ induced by ϕ is defined on H^2 is follows $C_\phi\, f = f \circ \phi \quad \left(f \in H^2\right)$

Definition(2.5) :

Let T be a bounded operator on a Hilbart space H, then the norm of an operoter T is defined by $\|T\| = \sup\{\|Tf\| : f \in H, \|f\| = 1\}$.

Littlewood's Subordination Principle (2.6) :

If $\phi : U \to U$ is holomorphic map on U with $\phi(0) = 0$, then $f \circ \phi \in H^2$ and $\|f \circ \phi\| \le \|f\|$ for each $f \in H^2$.

The goal of this theorem $C_\phi : H^2 \to H^2$.

Definition(2.7) :

The composetion operator C_ψ induced by ψ is defined on H^2 is follows

$C_\psi f = f \circ \psi$, $(f \in H^2)$

Proposition(2.8) :

If $\psi(z) = \frac{-3z}{3 - 3\bar{\beta}z}$, than $f \circ \psi \in H^2$ and $\|f \circ \psi\| \le \|f\|$ far each $f \in H^2$.

Proof :

Since $\psi : U \to U$ is holomarphic map on U by (2.6) $f \in H^2, f \circ \psi \in H^2$ and $\|f \circ \psi\| \le \|f\|$

hance $C_\psi : H^2 \to H^2$

Remark (2.9) :

1) One can easaly show that $C_\kappa C_\phi = C_{\phi\circ\kappa}$ and hance $C_\phi^n = C_\phi C_\phi \cdots C_\phi$

$= C_{\phi\circ\phi\circ\cdots\circ\phi} = C_{\phi_n}$

2) C_ϕ is the idintity operator on H^2 if end only if ϕ is idintity map from U into U and holomorphic on U.

3) It is semple to prove that $C_\kappa = C_\phi$ if end only if $\kappa = \phi$.

Definition(2.10):

Let T be an operator on a Hilbart space H , The operator T^* as the adjoint of T if $\langle Tx, y\rangle = \langle x, T^* y\rangle$ for each $x, y \in H$.

Theorem (2.11) :

$\{K_\alpha\}_{\alpha \in U}$ forms a danse subset of H^2.

Theorem (2.12) :

If $\phi : U \to U$ as holomarphic map on U, then for all $\alpha \in U$

$C_\phi^* K_\alpha = K_{\phi(\alpha)}$

Definition(2.13):

Let H^{∞} be the set of oll bounded holomarphic map on U .

Definition(2.14):

Lat $g \in H^{\infty}$, the Toiplits operator T_g is the operator on H^2 given by :

$$(T_g f)(z) = g(z)\ f(z) \quad (f \in H^2,\ z \in U)$$

Theorem (2.15) :

If $\phi : U \to U$ as holomarphic map on U, then $C_\phi T_g = T_{g \circ \phi} C_\phi \quad (g \in H^{\infty})$

Remark (2.16) :

Far each $f \in H^2$, it is will- know that $T_h^* \ f = T_{\bar{h}} \ f$, such that $h \in H^{\infty}$.

Proposition(2.17) :

If $\beta \in U$, than $C_\psi^* \ = \ T_g \ C_\gamma \ T_h^*$ where $h(z) = 1 - \bar{\beta}\ z$, $g(z) = 1$, $\gamma(z) = \beta + z$

Proof :

By (2.16), $T_h^* \ f = T_{\bar{h}} \ f$ for each $f \in H^2$. Hance for all $\alpha \in U$,

$$\left\langle T_h^* \ f, K_\alpha \right\rangle = \left\langle T_{\bar{h}} f, K_\alpha \right\rangle = \left\langle f, T_{\bar{h}}^* \ K_\alpha \right\rangle \cdots\cdots (2-1)$$

On the other hand ,

$$\left\langle T_h^* f,\ K_\alpha \right\rangle = \langle f, T_h K_\alpha \rangle = \langle f, h(\alpha) K_\alpha \rangle \cdots\cdots (2-2)$$

From (2-1)and (2-2) one can se that $T_{\bar{h}}^* K_\alpha = h(\alpha) K_\alpha$. Hance $T_h^* K_\alpha = \overline{h(\alpha)} K_\alpha$.

Calculotion give:

$$C_\psi^* K_\alpha(z) = K_{\psi(\alpha)}(z)$$

$$= \frac{1}{1-\overline{\psi(\alpha)}\ z} = \frac{1}{1-\frac{\overline{(-3\alpha)}z}{3-3\beta\bar{\alpha}}}$$

$$= \frac{1}{\frac{3-3\beta\bar{\alpha}-3\bar{\alpha}z}{3-3\beta\bar{\alpha}}} = \frac{3-3\beta\bar{\alpha}}{3-3\bar{\alpha}(\beta+z)} = \frac{\overline{(1-\bar{\beta}\alpha)}}{1-\bar{\alpha}(\beta+z)}$$

$$= \overline{(1-\bar{\beta}\alpha)} \cdot (1) \cdot \frac{1}{1-\bar{\alpha}(\beta+z)}$$

$$= \overline{h(\alpha)} \cdot g(z).\ K_\alpha(\gamma(z)) = \overline{h(\alpha)}\ g(z)\ (K_\alpha \circ \gamma)(z)$$

$$= \overline{h(\alpha)} \cdot (T_g\ K_\alpha \circ \gamma)(z) = \overline{h(\alpha)}\ T_g C_\gamma K_\alpha(z)$$

$$= T_g\ \overline{h(\alpha)}\ C_\gamma\ K_\alpha(z) = T_g\ C_\gamma\ \overline{h(\alpha)}\ K_\alpha(z)$$

$= T_g\ C_\gamma\ T_h^*\ K_\alpha(z)$, therefare

$$C_\psi^*\ K_\alpha(z) = T_g\ C_\gamma\ T_h^*\ K_\alpha(z)\ .$$

But $\overline{\{K_\alpha\}_{\alpha \in U}} = H^2$, than $C_\psi^* = T_g\ C_\gamma\ T_h^*$

Definition (2.18) :

Lat T be an operator on a Hilbart space H , T as called normil operator if $T\,T^{*}=T^{*}\,T$, and T as called unitary operator if $T\,T^{*}=T^{*}\,T=I$, and T as called hyponormil operator if $T\,T^{*}\leq T^{*}\,T$

Theorem (2.19) :

If $\phi:U\to U$ is holomarphic map on U, then C_{ϕ} as normil if end only if $\phi(z)=\alpha\,z$ for some $\alpha,|\alpha|\leq 1$

Theorem (2.20) :

If $\phi:U\to U$ be holomarphic map on U, then C_{ϕ} as unitary if end only if $\phi(z)=\alpha\,z$ for some α , $|\alpha|=1$

Proof :

Suppose C_{ϕ} as unitary, hence by (2.18) $C_{\phi}\,C_{\phi}^{*}=C_{\phi}^{*}\,C_{\phi}=I$, hance $C_{\phi}\,C_{\phi}^{*}=C_{\phi}^{*}\,C_{\phi}$, hence C_{ϕ} is normil operator, hance by (2.19) $\phi(z)=\alpha\,z$ for some α , $|\alpha|\leq 1$. It is enough to show that $|\alpha|=1$

$$C_{\phi}^{*}\,C_{\phi}\,K_{\mu}(z)=C_{\phi}^{*}\,K_{\mu}\,(\phi(z))=K_{\phi(\mu)}(\phi(z)).$$

$$=\frac{1}{1-\overline{\phi(\mu)}\,\phi(z)}=\frac{1}{1-\overline{\alpha}\,\overline{\mu}\,\alpha\,z}=\frac{1}{1-|\alpha|^{2}\,\overline{\mu}\,z}$$

On the other hand $C_{\phi}^{*}\,C_{\phi}\,K_{\mu}(z)=K_{\mu}(z)$, hence $\frac{1}{1-|\alpha|^{2}\,\overline{\mu}\,z}=K_{\beta}(z)=\frac{1}{1-\overline{\mu}\,z}$.

Thus $|\alpha|^2 \, \overline{\mu} = \overline{\mu}$, then $|\alpha| = 1$.

Conversely, Suppose $\phi(z) = \alpha\, z$ for some α ,$|\alpha| = 1$.For $\beta \in U$, for every $z \in U$

$$C_\phi^{\,*}\, C_\phi\, K_\mu(z) = C_\phi^{*}\, K_\mu(\phi(z)) = K_{\phi\ (\mu)}(\phi(z))$$

$$= \frac{1}{1 - \overline{\phi(\mu)}\,\phi(z)} = \frac{1}{1 - \overline{\alpha}\ \overline{\mu}\ \alpha\ z} = \frac{1}{1 - |\alpha|^2\ \overline{\mu}\ z}$$

$$= \frac{1}{1 - \overline{\mu}\ z} = K_\mu(z)$$

Moreaver , for every $z \in U$

$$C_\phi\, C_\phi^{\,*}\, K_\mu(z) = C_\phi\, K_{\phi(\mu)}\,(z) = K_{\phi\ (\mu)}(\phi(z))$$

$$= \frac{1}{1 - \overline{\phi(\mu)}\,\phi(z)} = \frac{1}{1 - \overline{\alpha}\ \overline{\mu}\ \alpha\ z} = \frac{1}{1 - |\alpha|^2\ \overline{\mu}\ z}$$

$$= \frac{1}{1 - \overline{\mu}\ z} = K_\mu(z)$$

Hance $C_\phi\, C_\phi^{\,*} = C_\phi^{\,*}\, C_\phi = I$ on the family $\{K_\alpha\}_{\alpha \in U}$. But by (2.11) $\{K_\alpha\}_{\alpha\ \in\ U}$ forms a dense subset of H^2, hance $C_\phi\, C_\phi^{\,*} = C_\phi^{\,*}\, C_\phi = I$ on H^2.Therefare C_ϕ is unitary composetion operator in H^2.

<u>Proposition(2.21)</u> :

If $\beta = 0$, then C_ψ is an unitary composetion operator .

<u>Proof :</u>

Since $\psi(z)=\frac{-3z}{3-3\bar{\beta}z}$, since $\beta=0$, $\psi(z)=\frac{-3z}{3-3\bar{\beta}z}=z=\alpha z$, $\alpha=1$, $|\alpha|=1$ hance by

(2.20) C_ψ is unitary composetion operator .

Remark(2.22) :

From Definition (2.18), we note every unitary composetion operator as a normil composetion operator .

Definition (2.23):

Let T be an operater on a Hilbert space H is subnormil if there exists a normil operater S on a Hilbert space K such that H is a subspace of K, the subspace H is invariant under the operater S and the restriction of S to H coincides with T (M is called an invariant subspace under T if TM $\subseteq$ M). It is well-known that every subnormal operator is normaloid and every normal operater is subnormel operater.

Proposition(2.24) :

If $\beta=0$, then C_ψ as a Subnormil composetion operator .

Proof:

Since $\beta=0$, then C_ψ as an unitary composition operator by (2.21) and

by(2.23) C_ψ as a Subnormil composetion operator .

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