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D-Essential Submodules

A research

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By

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بِسْمِ اللَّهِ الرَّحْمَنِ الرَّحِيمِ

أمن هو قانتٌ آناءَ الليلِ ساجداً وقائماً يحذرُ الاخرةَ ويرجو رحمةَ

ربهِ قُل هل يستوي الذين يَعلمون وَالذين لَا يعلمونَ

انما يتذكرُ اولُو الألبابِ ۞

صدقَ اللهُ العليُ العظم

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Certification

I certify that this paper was prepared under my supervision at the university of AL-Qadisiyah, college of Education, Dep. of Mathematics, as a partial fulfillment for the degree of B.C. of science in Mathematics.

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In view of the available recommendations, I forward this paper for debate by the examining committee.

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Dedication

I dedicate this humble to cry resounding silence in the sky to the martyrs of Iraq wounded. Also, I dedicate my father treasured, also I dedicate to my supervisor ***Dr. Tha'ar Younis Ghawi****. Finally, to everyone who seek knowledge, I dedicate this humble work.*

Acknowledgment

And later the first to thank in this regard is the Almighty God for the blessing that countless, including the writing of this research modest. Then thanks and gratitude and deepest gratitude to my teacher **Dr. Tha'ar Younis Ghawi**, which prefer to oversee the research and guidance in spite of concern, and not to ask him reordering Almighty to help him I also wish to express my thanks to the staff of the department of mathematics.

Contents

INTRODUCTION

We introduce and study the concept of D-essential submodules, where D is arbitrary submodules of a module M. This notion is a proper generalization of the notion of essential submodule. Indeed, essential submodule is respectively M-essential submodule. As a special case, we will show that s-essential submodule introduced by Zhou et al. are exactly Rad(M)-essential submodules. In ring and module theory, a submodule K of a module M over a ring R is said to be essential in M, written as $K \trianglelefteq M$, if for every $L \leq M, K \cap L = 0$ implies L=0. Recently, various generalizations of this notion were proposed by many authors [2,5]. For example the following notions of s-essential submodule were introduced and studied by Zhou and Zhang in [1]. A submodule of a module M over a ring R is said to be s-essential in M, written as $K \trianglelefteq_s M$, if for every $L \ll M, K \cap L = 0$ implies L=0. In this paper, we introduce a new generalization of essential submodule. Let D and F be submodule of a module M over a ring R. A submodule K of M is said to be D-essential in M, written as $K \lhd_D M$, if for every $L \leq D$, $K \cap L = 0$ implies L=0. In this work, we investigate and characterize the notion of D-essential submodules. Throughout this article, R will denote an associative ring with identity element and all considered modules will be unital left modules over R.

CHAPTER ONE

Background of Modules

Definition 1.1 [4] A module M is said to be semisimple if $\forall N \leq M \exists K \leq M \ni N \oplus K = M$.

Definition 1.2 Let M be an R module A subset X of M is called basis of M iff :

(1) X is generated M , i.e. $M = \langle X \rangle$.

(2) X is linearly independent , that is for every finite subset$\langle x_1, x_2, \dots, x_n \rangle$ of X with $\sum_{i=1}^{n} X_i \propto_i = 0$, $\forall \propto_i \in R$ $then$ $\propto_i = 0$, $\forall 1 \leq i \leq n$.

Definition 1.3 An R-module M is said to be free if satisfy the following condition :

(1) M has basis.

(2) $M = \oplus_{\forall i \in I} A_i \wedge \forall i \in I \; [A_i \equiv R_R]$.

Example 1.4 Z as Z-module is a free module.

Example 1.5 Z as Z-module is free since ⟨1⟩=Z

$\langle 1 \rangle = \{1.a | a \in Z\} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$

And $\forall \propto \in Z, \propto .1 = 0 \Longrightarrow \propto = 0$.

Zoren's lemma 1.6 If A is non-empty partial order set such that every chain in A has an upper bound in A, then A has maximal element.

Modular law 1.7 [3] If $A, B, C \leq M \wedge B \leq C$, $then$ $(A + B) \cap C = (A \cap C) + (B \cap C) = (A \cap C) + B$.

Theorem 1.8 If $\propto: M \to N$, $\beta N \to K$ modular homomorphism on R-ring then $\ker(\beta \propto) = \propto^{-1} (\ker(\beta))$.

Proof. Let $x \in \ker(\beta \propto) \to \beta \propto (x) = 0' \to \left(\propto (x)\right) = 0' \to \propto (x) \in \ker(\beta) \to x \in \propto^{-1} (\ker(\beta))$. So $\ker(\beta \propto) \subseteq \propto^{-1} (\ker(\beta))$ … (1)

Let $x \in \propto^{-1} (\ker(\beta)) \to \propto (x) \in \ker(\beta) \to \left(\propto (x)\right) = 0' \to \beta \propto (x) = 0' \to x \in \ker(\beta \propto)$. So $\propto^{-1} (\ker(\beta)) \subseteq \ker(\beta \propto)$ … (2)

Form (1),(2)$\to \ker(\beta \propto) = \propto^{-1} (\ker(\beta))$.

Theorem 1.9 [2] If $\propto: M \to N$, $\beta: N \to K$ modular homomorphism on R-ring then if $A \leq M$ *then* $\propto^{-1} \left(\propto (A)\right) = A + \ker(\propto)$.

Proof. Let $x \in \propto^{-1} (\propto (A)) \to \propto (x) \in \propto (A)$.

Then $\exists b \in A \ \ni \propto (x) = \propto (b)$

$\to \propto (x - b) = 0' \to x - b \in \ker(\propto)$, *then* $\exists k \in \ker(\propto) \ni x - b = k$

$\to x = b + k \to x \in A + \ker(\propto)$ [*since* $k \in \ker(\propto), b \in A$]

So $\propto^{-1} \left(\propto (A)\right) \subseteq A + \ker(\propto)$ ... (1)

Let $x \in A + \ker(\propto)$, *then* $\exists b \in B$, $k \in \ker(\propto) \ni x = b + k$

$\to \propto (x) = \propto (b + k) \to \propto (x) = \propto (b) + \propto (k)$

$\to \propto (x) = \propto (b)$[*since* $k \in \ker(\propto)$] $\to x \in \propto^{-1} (\propto (A))$

So $A + \ker(\propto) \subseteq \propto^{-1} \left(\propto (A)\right)$ ... (2)

So from (1) ,(2) we get $\propto^{-1} \left(\propto (A)\right) = A + \ker(\propto)$.

Definition 1.10 [2] Let $A \leq M$ then $B \leq M$ is called addition complement of A in M (briefly adco) iff :

(1)A+B=M

(2)$B \leq M$ minimal in A+B=M , i.e $\forall B \leq M$ with A+B=M , i.e $\forall U \leq M$ with A+U=M and $U \leq B$ imply U=B

$D \leq M$ is called intersection complement of A in M (beieflyinco) iff

(1) $A \cap D = 0$

(2) D is a maximal in $A \cap D = 0$

i.e. $\forall C \leq M$ with $A \cap C = 0 \ \wedge D \leq C$ implies C=D.

Corollary 1.11 Let $A \leq M \ \ and \ B \leq M \ \ \ then \ \ A \oplus B = M \Leftrightarrow$ B is adco and inco of A in M.

Proof. $\Longrightarrow$) Suppose that B is adco and inco of A

Then A+B=M resp. $A \cap B = 0 \ \Longrightarrow M = A \oplus B$

$\Longleftarrow$) Suppose that $A \oplus B = M$, hence A+B=M and $A \cap B = 0$

Let $C \leq M$ with A+C=M and $C \leq B, (A + C) \cap B = M \cap B \Longrightarrow (A + C) \cap B = B \rightarrow (A \cap B) = C = B \Longrightarrow C = B[A \cap B = 0]$

So B is adco of A in M

Let $C \leq M$ with $A \cap C = 0 \ \ and \ B \leq C$

Since A+B=M$\Longrightarrow$A+C=M [since $A + B \subseteq A + C$]

$\rightarrow A \oplus C = M \Longrightarrow A \oplus C = A \oplus B$ [$A \oplus B = M$by assumption]

$\frac{A \oplus C}{A} = \frac{A \oplus B}{A} \Longrightarrow C = B \longrightarrow$ so B is inco of A in M.

Lemma 1.12 [3] Let M=A+B , then we have B is adco of A in M $\Leftrightarrow A \cap B \ll B$.

Proof. $\Longrightarrow$) $let \ U \leq B \ (A \cap B) + U = B$

Then $M = A + (A \cap B) + U \Longrightarrow A + U = M \ [since \ A \cap B \subseteq A]$

But B is so $A \cap B \ll B$.

$\Longleftarrow$) We have by assumption M=A+B , let $U \leq M$ with A+U=M and $U \leq B$

$\rightarrow (A+U) \cap B = M \cap B \rightarrow (A+U) \cap B = B$ $[B \leq M]$ $\rightarrow (A+B) \cap U = B$ [by modular law]

But $A \cap B \ll B$, hence U=B , thus B is adco to A in M.

CHAPTER TWO

1. D-Essential Submodules

Definition 2.1.1 Let M be a module and D anon zero submodule of M. A submodule K of M is said to be D-essential, written as $K \lhd_D M$, if for every submodule L of D, $K \cap L = 0$ implies that $L = 0$

Remarks 2.1.2

(i) By above definition, it is clear that essential submodules are M-essential submodule.

(ii) A submodule containing D is D-essential ; in particular D is D-essential .

(iii) It is clear that if submodule K of M is essential in M, then K is D-essential in M for an arbitrary submodule D of M. However, the converses are not true in general. Let R=$\mathbb{Z}_{24}$ and M= $\mathbb{Z}_{24}$,D=6 $\mathbb{Z}_{24}$ then 3 $\mathbb{Z}_{24}$ is D- essential because D $\subseteq 3\mathbb{Z}_{24}$. But 8 $\mathbb{Z}_{24} \cap 3$ $\mathbb{Z}_{24}$ =0, thus 3 $\mathbb{Z}_{24}$ is not essential in M.

Proposition 2.1.3 Let D and K be submodules of a module M. If D$\trianglelefteq M$, then K$\lhd_D$ M $\Leftrightarrow$ K$\trianglelefteq M$.

Proof. Assume K $\lhd_D$ M and D $\trianglelefteq M$. Let L $\leq M$ such that $K \cap L = 0$. Then $K \cap (D \cap L) = 0$. Since $D \cap L \leq D$ and $K \lhd_D M$, then $D \cap L = 0$.By hypothesis $D \trianglelefteq M$, thus $L = 0$, which means that $K \trianglelefteq M$. The converse is obvious.

Definition 2.1.4 Let M and N are two modules with $D \leq M$. A monomorphism f : $K \to M$ is said to be D-essential, whenever $Imf \lhd_D M$.

Proposition 2.1.5 Let D be a submodule of a module M. For a submodule K of M, the following statements are equivalent:

(1) $K \lhd_D \ M;$

(2) The inclusion $map\ \ i : K \to M$ is a D-essential monomorphism;

(3) For every module N and for each h$\in Hom(m, N) whit\ kerh \leq D, (kerh) \cap K = 0$ implies ker $h = 0$.

Definition 2.1.6 A homomorphism $f: M_1 \longrightarrow M_2$ is said to be monic if for some homomorphisim $g_1: N \longrightarrow M_1$ and $g_2: N \longrightarrow M_2$ with $fg_1 = fg_2$ implies $g_1 = g_2$, where N is an R-module.

Corollary 2.1.7 Let L, M be modules. A monomorphism $f: L \to M$ is D-essential if and only if , for all homomorphisms (equivalently, epimorphism) $h\ \ with\ \ f\ kerh \leq D,$ if hof is monic, then h is monic.

Proposition 2.1.8. Let D and K be submodules of a module M. Then the following statements are equivalent:

(1) $K \lhd_D \ M;$

(2) $K \cap D \trianglelefteq D;$

(3) For each $0 \neq x \in D,$ there exists an element $r \in R$ such that $0 \neq rx \in K$.

Proof.(1)⇒(2). Assume that $K \lhd_D \ M$ and let L be a submodule of D such that $K \cap D \cap L = 0$. Then ,0= $K \cap D \cap L = K \cap L$. Since $K \lhd_D \ M,$ then $L = 0$. Thus $K \cap D \trianglelefteq D$.
(2)⟹(3). Let K be a submodule of M such that $K \cap D \trianglelefteq D$. Then for each $0 \neq x \in D,$ we have $0 \neq Rx$ and $K \cap D \cap Rx \neq 0$. Then there exists an element $r \in R$ such that $0 \neq rx \in K$.
(3)⟹(1). Assume that $L \leq D$ and $0 \neq x \in L$. Then there exists an element $r \in R$ such that $0 \neq rx \in K \cap L$. Thus $K \lhd_D \ M$.

Proposition 2.1.9 Let M and N be module and $f: M \ \longrightarrow N$ be an homomorphism. If $K \ \lhd_D \ N$, then$f^{-1}(K) \lhd_{f^{-1}(D)} \ M$.

Proof. Assume that $K \lhd_D N$ and let $0 \neq L \leq f^{-1}(D)$. If $f(L) = 0$, then $L \leq \ker f \leq f^{-1}(K)$. Hence $0 \neq L = L \cap f^{-1}(K)$. If $f(L) \neq 0$, then $f(L) \subseteq D$. Since $K \lhd_D N$, then $0 \neq K \cap f(L)$. Therefore there exist anon-zero $l \in L$ with $f(l) \in K$. Thus $0 \neq l \in L \cap f^{-1}(K)$,i.e. $f^{-1}(K) \lhd_{f^{-1}(D)} M$.

Proposition 2.1.10 Let M and N be modules and $f: M \longrightarrow N$ be an homomorphism. If $K \lhd_{Imf} N$, then $f^{-1}(K) \trianglelefteq M$. In particular, if f is monomorphism, then $K \lhd_{Imf} N$ if and only if $f^{-1}(K) \trianglelefteq M$.

Proof. Since $f^{-1}(Imf) = M$, then from Proposition 2.1.8, $K \lhd_{Imf} N$ implies $f^{-1}(K) \trianglelefteq M$. Now assume that f is monice. $L \leq Imf$ such that $K \cap L = 0$. Then $f^{-1}(K \cap L) = f^{-1}(K) \cap f^{-1}(L) = 0$. Since f is monic, then $L = 0$,that is $K \lhd_{Imf} N$.

Proposition 2.1.11 Let K be a submodule of a module M. If C, D are submodules of M such that $0 \subseteq C \subseteq D$. Then $K \lhd_D M \Longrightarrow K \lhd_C M$

Proposition 2.1.12 Let K,N,D be submodules of a module M such that $K \leq N$. Then $K \lhd_D M \Leftrightarrow K \lhd_{(D \cap N)} N$ and $N \lhd_D M$.

Proof. Necessity. Assume $K \lhd_D M$. Then from Proposition 2.1.11, we have $K \lhd_{(D \cap N)} N$. Let $L \leq D$ with $N \cap L = 0$, then $K \cap L = 0$. Since $K \lhd_D M$, then $L = 0$.Sufficiency. Let $0 \neq x \in D$, then there exists an $r \in R$ such that $0 \neq xr \in N \cap D$. Since $K \lhd_{(D \cap N)} N$, then there exists an $\acute{r} \in R$ such that $0 \neq \acute{r}rx \in K$. So $K \lhd_D M$.

Proposition 2.1.13 Let K,L and D be submodules of a module M. Then $K \cap L \lhd_D M \Leftrightarrow K \lhd_D M$ and $L \lhd_D M$;

Proof. Assume that $K \cap L \lhd_D M$. Since $K \cap L \subseteq K$ and $K \cap L \subseteq L$, Then from Proposition 2.1.11, We have $K \lhd_D M$ and $L \lhd_D M$. Conversely, suppose that $K \lhd_D M$ and $L \lhd_D M$. Then from Proposition 2.1.7, $K \cap D \trianglelefteq D$ and $L \cap D \trianglelefteq D$. Thus, $(K \cap D) \cap (L \cap D) = (K \cap L) \cap D \trianglelefteq D$, i.e. $K \cap L \lhd_D M$.

Proposition 2.1.14 Let M be a module. Suppose that $K_i \leq M_i \leq M$ and $D_i \leq M_i$ for $i = 1,2$. If $D_1 \cap D_2 = 0$, then $K_1 \lhd_{D_1} M_1$ and $K_2 \lhd_{D_2} M_2$ implies $K_1 + K_2 \lhd_{(D_1 \oplus D_2)} M_1 + M_2$.

Proof. Assume that $K_1 \lhd_{D_1} M_1$, $K_2 \lhd_{D_2} M_2$ and $D_1 \cap D_2 = 0$. Let $0 \neq x_1 + x_2 \in M$ with $0 \neq x_1 \in D_1$ and $0 \neq x_2 \in D_2$. Then by Proposition 2.1.7, there is an $r_1 \in R$ such that $0 \neq r_1 x_1 \in K_1$. If $r_1 x_2 \in K_2$, then by independence $0 \neq r_1 x_1 + r_1 x_2 \in (K_1 \cap D_1) \oplus (K_1 \cap D_2) \subseteq K_1 + K_2$. If $r_1 x_2 \notin K_2$ then again by Proposition 2.1.7, there is an $r_2 \in R$ such that $0 \neq r_2 r_1 x_2 \in K_2$ and we have $0 \neq r_2 r_1 x_1 + r_2 r_1 x_2 \in K_1 \bigcap D_1 \bigoplus K_2 \cap D_2 \subseteq K_1 + K_2$. Thus $K_1 + K_2 \lhd_{(D_1 \oplus D_2)} M_1 + M_2$.

Proposition 2.1.15 Let M be a module. Suppose that $K_i \leq M_i \leq M$ and $D_i \leq M_i$ for $i = 1,2$. If $(K_1 + D_1) \cap (K_2 + D_2) = 0$, then $K_1 \bigoplus K_2 \lhd_{(D_1 \oplus D_2)} M_1 + M_2$ if and only if $K_1 \lhd_{D_1} M_1$ and $K_2 \lhd_{D_2} M_2$.

Proof. Necessity. Assume that K_1 is not D_1-essential in M_1, i.e, there exists a nonzero submodule $L_1 \leq D_1$ such that $K_1 \cap L_1 = 0$. Then we will proof that $(K_1 + K_2) \cap L_1 = 0$. Let $l_1 = k_1 + k_2$ with $l_1 \in L_1$, $k_1 \in K_1$ and $k_2 \in K_2$. Then $k_2 = l_1 - k_1 \in (K_1 + D_1) \cap (K_2 + D_2) = 0$. Thus $l_1 = k_1 \in K_1 \cap L_1 = 0$. Hence $(K_1 + K_2) \cap L = 0$

sufficiency. This follows from proposition 2.1.11.

Corollary 2.1.16 Let M be a module. Suppose that $K_i \leq M_i \leq M$ and $D_i \leq M_i$ for $i = 1,2$. If $M_1 \bigoplus M_2 = M$, then $K_1 \bigoplus K_2 \lhd_{(D_1 \oplus D_2)} M_1 \bigoplus M_2$ if and only if $K_1 \lhd_{D_1} M_1$ and $K_2 \lhd_{D_2} M_1$.

Proof. This follows from proposition 2.1.14.

Let M be a module and $K \leq M$. We recall that $K \cap D$ has always a complement $\acute{K}$ in D such that $(K \cap D) \bigoplus \acute{K} \trianglelefteq D$. This means that $K \cap \acute{K} = K \cap \left(D \cap \acute{K}\right) = (K \cap D) \cap \acute{K} = 0$ and $\acute{K}$ as a submodules of D is maximal with respect to this relation. Moreover, $\acute{K} = 0$ if and only if $K \cap D$ is essential in D, in other words, $K \lhd_D M$. In this case we say that, $\acute{K}$ is a D-complement of K which means that $\acute{K}$ is a complement of $K \cap D$. The link between D-complements and D-essential extensions is given in the next result.

Proposition 2.1.17 Let K and D be submodules of a module M and $\acute{K}$ be a D-complements of K. Then

(1) $K \oplus \acute{K} \lhd_D M$;

(2) $(K \oplus \acute{K}) / \acute{K} \lhd_{D/\acute{K}} M / \acute{K}$.

Proof. (1) Let $0 \neq L \leq D$ such that $\left(K \oplus \acute{K}\right) \cap L = 0$, then is follows that $K \cap \left(\acute{K} + L\right) = 0$, contrary to the maximality of $\acute{K}$.
(2)Assume that $L \leq D$ with $\acute{K} \leq L$ and $L/\acute{K} \cap \left(K \oplus \acute{K}\right)/\acute{K} = 0$. Then by modularity, we get $\left(K \oplus \acute{K}\right) \cap L = (L \cap K) \oplus \acute{K} \subseteq \acute{K}$. Hence $L \cap K = 0$ and by maximality of $\acute{K}$, $L = \acute{K}$.

Recall that a module M is called uniform if every nonzero submodule of M is essential and M is called hollow if every proper submodule of M is small.

Proposition 2.1.18 Let M a module and D a nonzero submodule of M. Then M is uniform if and only if every nonzero K of M is D-essential.

Proof. Let K and N be nonzero Submodules of M. Since N is D-essential, then $N \cap D \neq 0$. Since K is D-essential, then $K \cap ND \neq 0$. Thus $K \cap N \neq 0$. Hence M is uniform. The converse is obvious.

2. Generalizations of Socle

In this section, we generalize the socle of a module and we will give some of their characteristics.

Definition 2.2.1 Let D be a submodule of a module M. We define $Soc_D(M)$ by $Soc_D(M) = \sum\{L \leq D : L\ is\ minimal\ in\ M\} = Soc(D)$.

Theorem 2.2.2 Let D be a submodule of a module M. Then $\bigcap\{K \leq M : K \lhd_D M\} = Soc_D(M)$.

Proof. Denote $S = \bigcap\{K \cap M : K \lhd_D M\}$. Assume that $0 \neq L$ is a minimal submodule of D and K a D-essential submodule of M. Then, $K \cap L \neq 0$. Since L is minimal, we conclude that $L \subseteq K$. So $Soc(D) \subseteq S$. Conversely, we have the following inclusion :$\{K \leq D: K \trianglelefteq D\} = \{K \leq D : K \lhd_D M\} \subseteq \{K \leq M : K \lhd_D M\}$. Thus $S = \bigcap\{K \leq M : K \lhd_D M\} \subseteq \bigcap\{K \leq D : K \trianglelefteq D\} = Soc(D)$.

Example 2.2.3 In this example we reconsider the previous example and set $D_1 = 2\mathbb{Z}_{36}$, $D_2 = 6\mathbb{Z}_{36}$, $D_3 = 9\mathbb{Z}_{36}$, $D_4 = 12\mathbb{Z}_{36}$.

$N \leq M$	essential	D_1-essential	D_2-essential	D_3-essential	D_4-essential
$\mathbb{Z}_{36}$	√	√	√	√	√
$2\mathbb{Z}_{36}$	√	√	√	√	√
$3\mathbb{Z}_{36}$	√	√	√	√	√
$4\mathbb{Z}_{36}$	×	×	×	×	√
$6\mathbb{Z}_{36}$	√	√	√	√	√
$9\mathbb{Z}_{36}$	×	×	×	√	×
$12\mathbb{Z}_{36}$	×	×	×	×	√
$18\mathbb{Z}_{36}$	×	×	×	√	×
0	×	×	×	×	×

$D_i \leq M$	M	D_1	D_2	D_3	D_4
$Soc_{D_i}(M)$	$6\mathbb{Z}_{36}$	$6\mathbb{Z}_{36}$	$6\mathbb{Z}_{36}$	$18\mathbb{Z}_{36}$	$12\mathbb{Z}_{36}$

(1) D_1 and D_2 are essential in M that is why D_1-essential submodules, and essential Submodules are the same ; see Proposition 2.3.

(2) We also have for $i = 1,2$, $Soc_{D_i}(M) = Soc(M) \cap D_i = Soc(M) = 6\mathbb{Z}_{36}$ because D_1 and D_2 are essential. The submodule D_4 is simple and then we have $Soc_{D_4}(M) = Soc(M) \cap D_4 = D_4 = 12\mathbb{Z}_{36}$. For submodule D_3, we have $Soc_{D_3}(M) = Soc(M) \cap D_3 = 9\mathbb{Z}_{36} \cap 6\mathbb{Z}_{36} = 18\mathbb{Z}_{36}$.

References

[1] C. Abdioglu, M.T. Kosan, S. Sahinkaya, on modules for which all Submodules are projection invariant and the lifting condition, southeast Asian Bull. Math.34(2010) 807-818.

[2] M.A. Ahmed, Weak essential submodule, Um-salama science journal 6(1) (2009)214-220.

[3] F.W. Anderson, K.R. Fuller, Rings and Categories of modules, Springer-Verlag, New York, 1974.

[4] A.S. Mijbass, N.K. Abdullah, Semi-essential Submodules and semi-uniform modules, J. Kirkuk University-Scientific studies 4 (1) (2009) 48-58.

[5] A.C. Ozcan, On μ-essential and μ-M-singular modules, In: proceedings of the fifth China-Japan-Korea Conference, Tokyo, Japan, 2007.