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Coefficients Estimate for a Certain Subclass of m-Fold Symmetric Bi-Univalent Functions

A Research Submitted by

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بسم الله الرحمن الرحيم

اقْرَأْ بِاسْمِ رَبِّكَ الَّذِي خَلَقَ (1) خَلَقَ الْإِنْسَانَ مِنْ عَلَقٍ
(2) اقْرَأْ وَرَبُّكَ الْأَكْرَمُ (3) الَّذِي عَلَّمَ بِالْقَلَمِ (4) عَلَّمَ الْإِنْسَانَ مَا
لَمْ يَعْلَمْ (5).

صدق الله العظيم

(العلق: 1–5)

الإهداء

لابد لنا ونحن نخطو خطواتنا الأخيرة في الحياة الجامعية التي قضيناها مع أساتذتنا الكرام الذين قدموا لنا من علم ومعرفة.

نتقدم باسمى ايات الشكر والامتنان والمحبه

الى شهدائنا الابرار

الى ابي ... وامي ... وعائلتي

الى من رافقوني خلال الدراسه ..... الزملاء والاصدقاء

اهدي لهم هذا الجهد البسيط

Abstract

In this work , we introduce and investigate general subclasses $R_{\Sigma_m}^{q,s}(\tau, \lambda, \eta; \alpha)$ and $R_{\Sigma_m}^{q,s}(\tau, \lambda, \eta; \beta)$ of Σ_m consisting of analytic and m-fold symmetric bi-univalent functions in the open unit disk U which are associated with the linear operator . We obtain estimates on the Taylor – Maclaurin coefficient $|a_{m+1}|$ and $|a_{2m+1}|$. Also, we obtain special cases for our results .

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Chapter One

Basic Definitions

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Basic Definitions

Introduction:

In this chapter, we have mentioned all of the required definitions, some example, some applications of conformaly map and standard results of analytic functions, univalent, multivalent and bi-univalent functions which are needed. The detailed proofs and further discussions may be found in standard texts such as Duren [5], Goodman [9], Miller and Mocanu [12] and other references.

Definition (1.1)[5]: A function f of the complex variable is analytic at a point z_0 if its derivative exists not only at z_0 but each point z in some neighborhoods of z_0. It is analytic in region $\mathbb{U}$ if it is analytic at every point in $\mathbb{U}$.

Definition (1.2)[5]: A function f is said to be univalent (schilcht) if it does not take the same value twice i.e. $f(z_1) \neq f(z_2)$ for all pairs of distinct points z_1 , $z_2 \in U$. In other words, f is one – to – one (or injective) mapping of U onto another domain.

If f assumes the same value more than one, then f is said to be multivalent (p-valent) in U. We also deal with the functions which are meromorphic univalent in the punctured unit disk $U^* = \{z \in \mathbb{C}: 0 < |z| < 1\}$. f is said to be meromorphic if it is analytic at every point in U.

As examples, the function $f(z) = z$ is univalent in U while $f(z) = z^2$ is not univalent in U. Also, $f(z) = z + \frac{z^n}{n}$ is univalent in U for each positive integer n.

Example (1.1) [12]: The function $f(z) = (1+z)^2$ is univalent in U.

Let $z_1, z_2 \in U$ and suppose $f(z_1) = f(z_2)$. Then

$$(1+z_1)^2 = (1+z_2)^2$$

$$\Rightarrow 1 + 2z_1 + z_1^2 = 1 + 2z_2 + z_2^2$$

$$\Rightarrow z_1^2 - z_2^2 + 2(z_1 - z_2) = 0$$

$$\Rightarrow (z_1 - z_2)(z_1 + z_2 + 2) = 0.$$

Since $|z_1|, |z_2| < 1$, we know that $(z_1 + z_2 + 2) \neq 0$. Hence $z_1 - z_2 = 0$ or $z_1 = z_2$.

But the function $f(z) = (1+z)^4$ is not univalent in U.

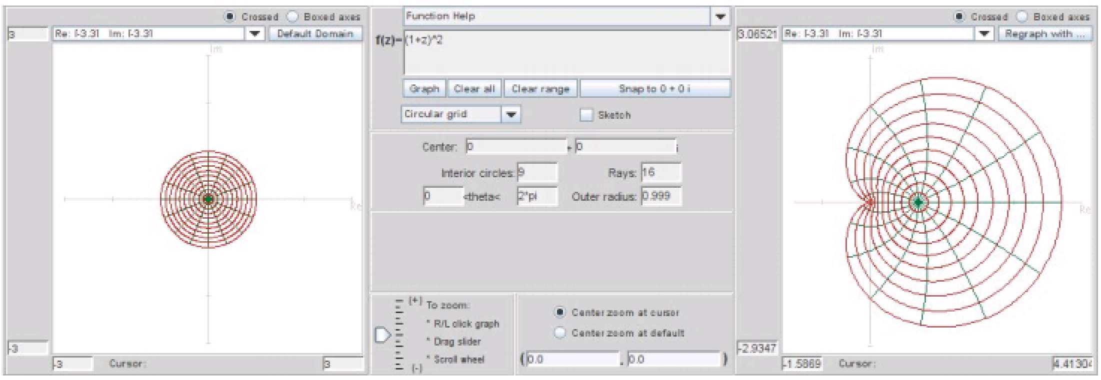

Figure (1.1): The image of the open unit disk under the $f(z) = (1+z)^4$

Definition (1.3) [5]: A function f is said to be locally univalent at a point $z_0 \in \mathbb{C}$ if it is univalent in some neighborhood of z_0. For analytic function f, the condition $f'(z_0) \neq 0$ is equivalent to local univalent at z_0.

Example (1.2)[9]: Consider the domain

$$D = \left\{ z \in \mathbb{C}: 1 < |z| < 2, \;\; 0 < arg z < \frac{3\pi}{2} \right\},$$

and the function $f: D \to \mathbb{C}$ given by $f(z) = z^2$. It is clear that f is analytic on D and local univalent at every point $z_0 \in D$, since $f^{'}(z_0) \neq 2z_0 \neq 0$ for all $z_0 \in D$.

However, f is not univalent on D, since

$$f\left(\frac{3}{2\sqrt{2}} + i\frac{3}{2\sqrt{2}}\right) = f\left(-\frac{3}{2\sqrt{2}} - i\frac{3}{2\sqrt{2}}\right) = \frac{9}{4}i.$$

Definition (1.4)[5]: Let $\mathcal{A}$ denotes the class of functions f of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad n \in \mathbb{N} \tag{1.1}$$

which are analytic and univalent in the open unit disk U.

Definition (1.5)[5]: We say that $f \in \mathcal{A}$ is normalized if f satisfies the conditions $f(0) = 0$ and $f'(0) = 1$.

Definition (1.6)[5]: A set $E \subseteq \mathbb{C}$ is said to be starlike with respect to $w_0 \in E$ if the linear segment joining w_0 to every other point $w \in E$ lies entirely in E. In a more picturesque language, the requirement is that every point of E is visible from w_0. The set E is said to be convex if it is starlike with respect to each of its points, that is , if the linear segment joining any two points of E lies entirely in E.

Definition (1.7)[5]: A function$f$ is said to be conformal at a point z_0 if it preserves the angle between oriented curves passing through z_0 in magnitude as well as in sense. Geometrically, images of any two oriented curves taken with their corresponding orientations make the same angle of intersection as the curves at z_0 both in magnitude and direction. A function $w = f(z)$ is said to be conformal in the domain D, if it is conformal at each point of the domain.

Definition (1.8)[5]: A function $f \in \mathcal{A}$ is said to be starlike function of order α if and only if

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, (\, 0 \leq \alpha < 1;\ z \in U). \tag{1.2}$$

Denotes the class of all starlike functions of order α in U by $S^*(\alpha)$ and S^* the class of all starlike functions of order 0, $S^*(0) = S^*$. Geometrically, we can say that a starlike function is conformal mapping of the unit disk onto a domain starlike with respect to the origin. For example, the function

$$f(z) = \frac{z}{(1-z)^{2(1-\alpha)}},$$

is starlike function of order α.

Definition (1.9)[5]: A function $f \in \mathcal{A}$ is said to be convex function of order α if and only if

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha, (\, 0 \leq \alpha < 1;\; z \in U). \tag{1.3}$$

Denotes the class of all convex functions of orderα in U by $C(\alpha)$ and C for the convex function $C(0) = C$.

Definition (1.10)[5]: A function f analytic in the unit disk $U$is said to be close – to – convex of order α $(0 \leq \alpha < 1)$ if there is a convex function g such that

$$Re\left\{\frac{f'(z)}{g'(z)}\right\} > \alpha, \qquad \forall\, z \in U. \tag{1.4}$$

We denote by $K(\alpha)$, the class of close – to – convex functions of order α, $f$is normalized by the usual conditions $f(0) = f'(0) - 1 = 0$. By using argument, we can write the condition (1.4) as

$$\left|arg\frac{f'(z)}{g'(z)}\right| < \frac{\alpha\pi}{2}, \alpha > 0, \forall\, z \in U. \tag{1.5}$$

We note that $C(\alpha) \subset S^*(\alpha) \subset K(\alpha)$.

Definition (1.11)[5]: Let $\mathcal{A}(p)$ denote the class of analytic p–valently functions in U of the form:

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n, (z \in U, p \in \mathbb{N} = \{1, 2, \dots\}). \quad (1.6)$$

We say that f is p–valently starlike of order α, p–valently convex of order α, and p–valently close - to - convex of order $\alpha(0 \le \alpha < p)$, respectively if and only if :

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, \qquad Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha, Re\left\{\frac{f'(z)}{z^{p-1}}\right\} > \alpha.$$

Definition (1.12)[5]: Let us denote by $\mathcal{A}_p^*$ the class of meromorphic function f of the form:

$$f(z) = z^{-p} + \sum_{n=p}^{\infty} a_n z^n, \qquad p \in \mathbb{N} \quad (1.7)$$

which are meromorphic and p–valent in the punctured unit disk $U^* = \{z \in \mathbb{C}: 0 < |z| < 1\} = U - \{0\}$. We say that f is p–valently meromorphic starlike of order $\alpha(0 \le \alpha < p)$ if and only if

$$Re\left\{-\frac{zf'(z)}{f(z)}\right\} > \alpha \text{ for } \ z \in U^*. \quad (1.8)$$

Also, f is p–valently meromorphic convex of order $\alpha(0 \le \alpha < p)$ if and only if

$$Re\left\{-\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} > \alpha, \qquad z \in U^*. \quad (1.9)$$

Note that if $p = 1$, we have defined meromorphic starlike of order $\alpha(0 \le \alpha < 1)$, meromorphic convex of order α $(0 \le \alpha < 1)$ respectively.

Definition (1.13)[5]: Radius of starlikeness of a function f is the largest $R_1, 0 < R_1 < 1$ for which it is starlike in $|z| < R_1$.

Definition (1.14)[5]: Radius of convexity of a function f is the largest $R_2, 0 < R_2 < 1$ for which it is convex in $|z| < R_2$.

Definition (1.15)[5]: The convolution (or Hadamard product) of functions f and g denoted by $f * g$ is defined as following for the functions in $\mathcal{A}(p)$ and $\mathcal{A}^*(p)$ respectively:

(i) If

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n, \qquad g(z) = z^p + \sum_{n=p+1}^{\infty} b_n z^n,$$

then

$$(f * g)(z) = z^p + \sum_{n=p+1}^{\infty} a_n b_n z^n.$$

(ii) If

$$f(z) = z^{-p} + \sum_{k=1}^{\infty} a_{k-p} z^{k-p}, \qquad g(z) = z^{-p} + \sum_{k=1}^{\infty} b_{k-p} z^{k-p},$$

then

$$(f * g)(z) = z^{-p} + \sum_{k=1}^{\infty} a_{k-p} b_{k-p} z^{k-p}.$$

and if $p = 1$ in (i), then the convolution (or Hadamard product) for the functions in $\mathcal{A}$.

Definition (1.16)[5]: An analytic function f is said to be bi-univalent in a domain U, if f and its inverse f^{-1} are both univalent in U, the class of all bi-univalent analytic functions f in U is denoted by Σ.

Chapter Two

Coefficients Estimate for a Certain Subclass of m-Fold Symmetric Bi-Univalent Functions

Chapter Two

2.1. Introduction

Denote by $\mathcal{A}$ the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (2.1.1)$$

which are analytic in the open unit disk $U = \{z: z \in \mathbb{C}\ and\ |z| < 1\}$. Further, denote by $\mathcal{S}$ the class of all functions in $\mathcal{A}$ which are univalent and normalized by $f(0) = 0 = f'(0) - 1$ in U. The well investigated subclasses of the univalent function class $\mathcal{S}$ are the class of starlike functions of order α $(0 \leq \alpha < 1)$, denoted by $\mathcal{S}^*(\alpha)$ and the class of convex functions of order α denoted by $\mathcal{K}(\alpha)$ in U.

The Koebe One-Quarter Theorem [5] states that the image of U under every function f from $\mathcal{S}$ contains a disk of radius 1/4. Thus every such univalent function has an inverse f^{-1} which satisfies

$$f^{-1}(f(z)) = z \qquad (z \in U)$$

and

$$f(\ f^{-1}(w)) = w \quad \left(|w| < r_0\,(f);\ r_0(f) \geq \frac{1}{4}\right),$$

where

$$f^{-1}(w) = g(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots. \qquad (2.1.2)$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in U if both f and f^{-1} are univalent in U. We denote by Σ the class of all bi-univalent functions in U given by the Taylor-Maclaurin series expansion (2.1.1).

For each function $f \in \mathcal{S}$, the function

$$h(z) = \sqrt[m]{f(z^m)}\big(z \in U, m \in \mathbb{N}\big) \qquad (2.1.3)$$

is univalent and maps the unit disk U into a region with m-fold symmetry. A function is said to be m-fold symmetric (see [10, 14]) if it has the following normalized form:

$$f(z) = z + \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1} \left(z \in U, m \in \mathbb{N}\right). \qquad (2.1.4)$$

We denote by $\mathcal{S}_m$ the class of m-fold symmetric univalent functions in U, which are normalized by the series expansion (2.1.4). In fact, the functions in the class $\mathcal{S}$ are one-fold symmetric (that is, m=1).

In [16] Srivastava et al. defined m-fold symmetric bi-univalent function analogues to the concept of m-fold symmetric univalent functions. They gave some important results, such as each function $f \in \Sigma$ generates an m-fold symmetric bi-univalent function for each $m \in \mathbb{N}$, in their study. Furthermore, for the normalized form of f given by (2.1.4), they obtained the series expansion for f^{-1} as follows:

$$f^{-1}(w) = g(w) = w - a_{m+1}w^{m+1} + [(m+1)a_{m+1}^2 - a_{2m+1}]w^{2m+1} -$$

$$\left[\frac{1}{2}(m+1)(3m+2)a_{m+1}^3 - (3m+2)a_{m+1}a_{2m+1} + a_{3m+1}\right]w^{3m+1} + \cdots, \qquad (2.1.5)$$

where $f^{-1} = g$. We denote by Σ_m the class of m-fold symmetric bi-univalent functions in U. For $m = 1$, formula (2.1.5) coincides with formula (2.1.2) of the class Σ.

Lewin [11] investigated the bi-univalent function class Σ and showed that $|a_2| < 1.51$. Subsequently, Brannan and Clunie [2] conjectured that $|a_2| < \sqrt{2}$. Netanyahu [13] showed that max $|a_2| = 4/3$ if $f(z) \in \Sigma$. The best known estimate for functions in Σ were obtained in 1984 by Tan [17], that is, $|a_2| \le 1.485$. The coefficient estimate problem involving the bound of $|a_2|$ $(n \in \mathbb{N}\{1,2\})$ for each $f \in \Sigma$ given by (2.1.4) is still an open problem. In fact, the aforecited work of Srivastava et al. [15] essentially revived the investigation of various subclasses of the bi-univalent function class Σ in recent years. Recently, many authors investigated bounds for various subclasses of bi-univalent functions (see [3, 4, 7, 8, 15].

Dziok and Srivastava [6] considered a linear operator :

$H(a_1, \dots, a_q;\ b_1, \dots, b_s; z): A \to A$ defined by the following Hadamard product :

$H(a_1, \dots, a_q;\ b_1, \dots, b_s)f(z) = h(a_1, \dots, a_q;\ b_1, \dots, b_s; z) * f(z)$

For $q \le s+1$, $z \in \mathbb{U}$. $f \in A$ is given by (2.1.1), then we have

$$H_{q,s} = H(a_1, \dots, a_q;\ b_1, \dots, b_s)f(z) = z + \sum_{n=2}^{\infty} \Gamma_n\,[a_1; b_1]\, a_n z^n$$

Where

$$\Gamma_n = \Gamma_n\,[a_1; b_1] = \frac{(a_1)_{n-1} \cdots (a_q)_{n-1}}{(b_1)_{n-1} \cdots (b_s)_{n-1}}\,\frac{1}{(n-1)!}\ (n \in \mathbb{N})$$

And

$$(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} = \begin{cases} 1\ (n=0) \\ \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1)(n = 1,2,3,\cdots) \end{cases}$$

The linear operator $H_{q,s}[a_1; b_1; z] := H(a_1, \dots, a_q;\ b_1, \dots, b_s)f(z)$ is a generalization of many other linear operators considered earlier .

Recently, Atshan and AL-Ziadi [1] study a new general subclasses $R_{\Sigma_m}(\tau, \lambda, \eta; \alpha)$ *and* $R_{\Sigma_m}(\tau, \lambda, \eta; \beta)$ of Σ_m . In this work , we introduce and investigate a general subclasses $R_{\Sigma_m}^{q,s}(\tau, \lambda, \eta; \alpha)$ and $R_{\Sigma_m}^{q,s}(\tau, \lambda, \eta; \beta)$ of Σ_m. Consisting of analytic and m-fold symmetric bi-univalent functions in the open unit disk U , which are associated with the linear operator. We obtain estimates on the Taylor - Maclaurin coefficient $|a_{m+1}|$ and $|a_{2m+1}|$. Also, we obtain special cases for our results.

In order to derive our main result, we have to recall here the following lemma [5].

Lemma 2.1. If $p \in \mathcal{P}$, then $|b_k| \leq 2$ for each $k \in \mathbb{N}$, where $\mathcal{P}$ is the family of all functions $\mathcal{P}$, analytic in U, for which

$$R\big(p(z)\big) > 0 \qquad (z \in U),$$

where

$$h(z) = 1 + c_1 z + c_2 z^2 + \cdots \qquad (z \in U).$$

2.2. Coefficient bounds for the function class $\mathcal{R}_{\Sigma_m}^{a,b,c}(\tau, \lambda, \eta; \alpha)$

Definition 2.1. A function $f \in \Sigma_m$ given by (1.1.4) is said to be in the class $\mathcal{R}_{\Sigma_m}^{q,s}(\tau, \lambda, \eta; \alpha)$ if the following conditions are satisfied:

$$\left|\arg\left(1 + \frac{1}{\tau}\left[(1-\lambda)\frac{H_{q,s}f(z)}{z} + \lambda H_{q,s}f'(z) + \eta z H_{q,s}f''(z) - 1\right]\right)\right| < \frac{\alpha\pi}{2} (z \in U) \qquad (2.2.1)$$

$$\left|\arg\left(1 + \frac{1}{\tau}\left[(1-\lambda)\frac{H_{q,s}g(w)}{w} + \lambda H_{q,s}g'(w) + \eta w H_{q,s}g''(w) - 1\right]\right)\right| < \frac{\alpha\pi}{2} (w \in U), \qquad (2.2.2)$$

where($\tau \in \mathbb{C}\backslash\{0\};\ \lambda \geq 1;\ 0 \leq \eta \leq 1;\ 0 < \alpha \leq 1$)and the function g is given by (2.1.5).

Theorem 2.1. Let the function $f(z)$, given by (2.1.4), be in the class $\mathcal{R}_{\Sigma_m}^{q,s}(\tau, \lambda, \eta; \alpha)$. Then

$$|a_{m+1}| \leq \frac{2\alpha|\tau|}{\sqrt{\left|\alpha\tau(m+1)\big(1+2\lambda m+2\eta m(2m+1)\big)\Gamma_{2m+1} + (1-\alpha)\big(1+\lambda m+\eta m(m+1)\big)^2\Gamma_{m+1}^2\right|}} \quad (2.2.3)$$

and

$$|a_{2m+1}| \leq \frac{2\alpha^2|\tau|^2(m+1)}{\big(1+\lambda m+\eta m(m+1)\big)^2\Gamma_{m+1}^2} + \frac{2\alpha|\tau|}{\big(1+2\lambda m+2\eta m(2m+1)\big)\Gamma_{2m+1}}. \qquad (2.2.4)$$

Proof. Let $f \in \mathcal{R}_{\Sigma_m}^{q,s}(\tau,\lambda,\eta;\alpha)$. Then

$$1+\frac{1}{\tau}\left[(1-\lambda)\frac{H_{q,s}f(z)}{z}+\lambda H_{q,s}f'(z)+\eta z H_{q,s}f''(z)-1\right]=[p(z)]^{\alpha} \qquad (2.2.5)$$

and

$$1+\frac{1}{\tau}\left[(1-\lambda)\frac{H_{q,s}g(w)}{w}+\lambda H_{q,s}g'(w)+\eta w H_{q,s}g''(w)-1\right]=[q(w)]^{\alpha}, \qquad (2.2.6)$$

where $g=f^{-1}, p(z),\ q(z)$ in $\mathcal{P}$ and have the forms

$$p(z)=1+p_m z^m+p_{2m}z^{2m}+p_{3m}z^{3m}+\cdots \qquad (2.2.7)$$

and

$$g(w)=1+q_m z^m+q_{2m}z^{2m}+q_{3m}z^{3m}+\cdots. \qquad (2.2.8)$$

Now, equating the coefficients in (2.2.5) and (2.2.6), we get

$$\left(\frac{1+\lambda m+\eta m(m+1)}{\tau}\right)\Gamma_{m+1}a_{m+1}=\alpha p_m, \qquad (2.2.9)$$

$$\left(\frac{1+2\lambda m+2\eta m(2m+1)}{\tau}\right)\Gamma_{2m+1}a_{2m+1}=\alpha p_{2m}+\frac{1}{2}\alpha(\alpha-1)p_m^2, \qquad (2.2.10)$$

$$-\left(\frac{1+\lambda m+\eta m(m+1)}{\tau}\right)\Gamma_{m+1}a_{m+1}=\alpha q_m \qquad (2.2.11)$$

and

$$\left(\frac{1+2\lambda m+2\eta m(2m+1)}{\tau}\right)\Gamma_{2m+1}[(m+1)a_{m+1}^2-a_{2m+1}]=\alpha q_{2m}+\frac{1}{2}\alpha(\alpha-1)q_m^2. \qquad (2.2.12)$$

From (2.2.9) and (2.2.11) we find

$$p_m=-q_m \qquad (2.2.13)$$

and

$$2\left(\frac{1+\lambda m+\eta m(m+1)}{\tau}\right)^2 \Gamma_{m+1}^2 a_{m+1}^2=\alpha^2(p_m^2+q_m^2). \qquad (2.2.14)$$

From (2.2.10), (2.2.12) and (2.2.14), we get

$$\left(\frac{1+2\lambda m+2\eta m(2m+1)}{\tau}\right)\Gamma_{2m+1}(m+1)a_{m+1}^2=\alpha(p_{2m}+q_{2m})+\frac{\alpha(\alpha-1)}{2}(p_m^2+q_m^2)$$

$$=\alpha(p_{2m}+q_{2m})+\frac{(\alpha-1)}{\alpha}\left(\frac{1+\lambda m+\eta m(m+1)}{\tau}\right)^2 \Gamma_{m+1}^2 a_{m+1}^2. \qquad (2.2.15)$$

Therefore, we have

$$a_{m+1}^2=\frac{\alpha^2\tau^2(p_{2m}+q_{2m})}{\alpha\tau(m+1)\big(1+2\lambda m+2\eta m(2m+1)\big)\Gamma_{2m+1}+(1-\alpha)((1+\lambda m+\eta m(m+1))^2\Gamma_{m+1}^2}. \qquad (2.2.16)$$

Applying Lemma (2.1) for the coefficients p_{2m} andq_{2m}, we immediately have

$$|a_{m+1}|\le\frac{2\alpha|\tau|}{\sqrt{\left|\alpha\tau(m+1)\big(1+2\lambda m+2\eta m(2m+1)\big)\Gamma_{2m+1}+(1-\alpha)\big(1+\lambda m+\eta m(m+1)\big)^2\Gamma_{m+1}^2\right|}} \qquad (2.2.17)$$

The last inequality gives the desired estimate on $|a_{m+1}|$ given in (2.2.3).

Next, in order to find the bound on $|a_{2m+1}|$, by subtracting (2.2.12) from (2.2.10), we obtain

$$2\left(\frac{1+2\lambda m+2\eta m(2m+1)}{\tau}\right)\Gamma_{2m+1}a_{2m+1}-\left(\frac{1+2\lambda m+2\eta m(2m+1)}{\tau}\right)\Gamma_{2m+1}(m+1)a_{m+1}^2$$

$$=\alpha(p_{2m}-q_{2m})+\frac{\alpha(\alpha-1)}{2}(p_m^2-q_m^2). \qquad (2.2.18)$$

It follows from (2.2.13), (2.2.14) and (2.2.18) that

$$a_{2m+1}=\frac{\alpha^2\tau^2(m+1)(p_m^2+q_m^2)}{4\big(1+\lambda m+\eta m(m+1)\big)^2\Gamma_{m+1}^2}+\frac{\alpha\tau(p_{2m}-q_{2m})}{2\big(1+2\lambda m+2\eta m(2m+1)\big)\Gamma_{2m+1}}. \qquad (2.19)$$

Applying Lemma (2.1) once again for the coefficients p_m, p_{2m}, q_m and q_{2m}, we readily get

$$|a_{2m+1}| \leq \frac{2\alpha^2|\tau|^2(m+1)}{\big(1+\lambda m+\eta m(m+1)\big)^2 \Gamma_{m+1}^2} + \frac{2\alpha|\tau|}{\big(1+2\lambda m+2\eta m(2m+1)\big)\Gamma_{2m+1}}.$$

This completes the proof of Theorem (2.2.1).

2.3. Coefficient bounds for the function class $\mathcal{R}_{\Sigma_m}^{q,s}(\tau,\lambda,\eta;\beta)$

Definition 2.3.1. A function $f \in \Sigma_m$ given by (2.1.4) is said to be in the class $\mathcal{R}_{\Sigma_m}^{q,s}(\tau,\lambda,\eta;\beta)$ if the following conditions are satisfied:

$$Re\left(1+\frac{1}{\tau}\left[(1-\lambda)\frac{H_{q,s}f(z)}{z}+\lambda H_{q,s}f'(z)+\eta z H_{q,s}f''(z)-1\right]\right) > \beta \quad (z \in U) \qquad (2.3.1)$$

and

$$Re\left(1+\frac{1}{\tau}\left[(1-\lambda)\frac{H_{q,s}g(w)}{w}+\lambda H_{q,s}g'(w)+\eta w H_{q,s}g''(w)-1\right]\right) > \beta \;\; (w \in U), \qquad (2.3.2)$$

Where $(\tau \in \mathbb{C}\backslash\{0\};\ \lambda \geq 1;\ 0 \leq \eta \leq 1;\ 0 \leq \beta < 1)$and the function g is given by (2.1.5).

Theorem 2.3.1. Let the function $f(z)$, given by (2.1.4), be in the class $\mathcal{R}_{\Sigma_m}^{q,s}(\tau,\lambda,\eta;\beta)$. Then

$$|a_{m+1}| \leq \sqrt{\frac{4|\tau|(1-\beta)}{(m+1)\big(1+2\lambda m+2\eta m(2m+1)\big)\Gamma_{2m+1}}} \qquad (2.3.3)$$

and

$$|a_{2m+1}| \leq \frac{2|\tau|^2(1-\beta)^2(m+1)}{\big(1+\lambda m+\eta m(m+1)\big)^2\Gamma_{m+1}^2} + \frac{2|\tau|(1-\beta)}{\big(1+2\lambda m+2\eta m(2m+1)\big)\Gamma_{2m+1}}. \qquad (2.3.4)$$

Proof. It follows from (2.3.1) and (2.3.2) that there exists $p, q \in \mathcal{P}$ such that

$$1+\frac{1}{\tau}\left[(1-\lambda)\frac{H_{q,s}f(z)}{z}+\lambda H_{q,s}f'(z)+\eta z H_{q,s}f''(z)-1\right]=\beta+(1-B)p(z) \qquad (2.3.5)$$

and

$$1+\frac{1}{\tau}\left[(1-\lambda)\frac{H_{q,s}g(w)}{w}+\lambda H_{q,s}g'(w)+\eta w H_{q,s}g''(w)-1\right]=\beta+\ (1-B)q(w) \qquad (2.3.6)$$

where $p(z)$ and $q(w)$ have the forms (2.2.7) and (2.2.8), respectively. Equating coefficients in (2.3.5) and (2.3.6), we get

$$\left(\frac{1+\lambda m+\eta m(m+1)}{\tau}\right)\Gamma_{m+1}a_{m+1}=(1-\beta)p_m, \qquad (2.3.7)$$

$$\left(\frac{1+2\lambda m+2\eta m(2m+1)}{\tau}\right)\Gamma_{2m+1}a_{2m+1}=(1-\beta)p_{2m}, \qquad (2.3.8)$$

$$-\left(\frac{1+\lambda m+\eta m(m+1)}{\tau}\right)\Gamma_{m+1}a_{m+1}=(1-\beta)q_m \qquad (2.3.9)$$

and

$$\left(\frac{1+2\lambda m+2\eta m(2m+1)}{\tau}\right)\Gamma_{2m+1}[(m+1)a_{m+1}^2-a_{2m+1}]=(1-\beta)q_{2m}. \qquad (2.3.10)$$

From (2.3.7) and (2.3.9), we find

$$p_m=-q_m \qquad (2.3.11)$$

and

$$2\left(\frac{1+\lambda m+\eta m(m+1)}{\tau}\right)^2\Gamma_{m+1}^2a_{m+1}^2=(1-\beta)^2(p_m^2+q_m^2). \qquad (2.3.12)$$

Adding (2.3.8) and (2.3.10), we have

$$\left(\frac{1+2\lambda m+2\eta m(2m+1)}{\tau}\right)\Gamma_{2m+1}(m+1)a_{m+1}^2=(1-\beta)(p_{2m}+q_{2m}). \qquad (2.3.13)$$

Therefore, we obtain

$$a_{m+1}^2 = \frac{\tau(1-\beta)(p_{2m}+q_{2m})}{(m+1)\big(1+2\lambda m+2\eta m(2m+1)\big)\Gamma_{2m+1}}\,. \qquad (2.3.14)$$

Applying Lemma (2.1) for coefficients p_{2m} and q_{2m}, we obtain

$$|a_{m+1}| \le \sqrt{\frac{4|\tau|(1-\beta)}{(m+1)\big(1+2\lambda m+2\eta m(2m+1)\big)\Gamma_{2m+1}}}$$

This gives the bound on $|a_{m+1}|$ as asserted in (2.3.3).

In order to find the bound on $|a_{2m+1}|$, by subtracting (2.3.10) from (2.3.8), we get

$$2\left(\frac{1+2\lambda m+2\eta m(2m+1)}{\tau}\right)\Gamma_{2m+1}a_{2m+1} - \left(\frac{1+2\lambda m+2\eta m(2m+1)}{\tau}\right)\Gamma_{2m+1}(m+1)a_{m+1}^2$$

$$= (1-\beta)(p_{2m}-q_{2m}) \qquad (2.3.15)$$

or equivalently

$$a_{2m+1} = \frac{(m+1)}{2}a_{m+1}^2 + \frac{\tau(1-\beta)(p_{2m}-q_{2m})}{2\big(1+2\lambda m+2\eta m(2m+1)\big)\Gamma_{2m+1}}\,. \qquad (2.3.16)$$

Upon substituting the value of a_{m+1}^2 from (2.3.12), we get

$$a_{2m+1} = \frac{\tau^2(1-\beta)^2(m+1)(p_m^2+q_m^2)}{4\big(1+\lambda m+\eta m(m+1)\big)^2\Gamma_{m+1}^2} + \frac{\tau(1-\beta)(p_{2m}-q_{2m})}{2\big(1+2\lambda m+2\eta m(2m+1)\big)\Gamma_{2m+1}}. \qquad (2.3.17)$$

Applying Lemma (2.1) once again for the coefficients p_m, p_{2m}, q_m *and* q_{2m}, we find

$$|a_{2m+1}| \le \frac{2|\tau|^2(1-\beta)^2(m+1)}{\big(1+\lambda m+\eta m(m+1)\big)^2\Gamma_{m+1}^2} + \frac{2|\tau|(1-\beta)}{\big(1+2\lambda m+2\eta m(2m+1)\big)\Gamma_{2m+1}}\,.$$

This completes the proof of Theorem (2.3.1).

2.4. Application of the main results

For one-fold symmetric bi-univalent functions, the classes $\mathcal{R}_{\Sigma_m}^{q,s}(\tau,\lambda,\eta;\alpha)$ and $\mathcal{R}_{\Sigma_m}^{q,s}(\tau,\lambda,\eta;\beta)$ reduceto the classes $\mathcal{R}_{\Sigma}^{q,s}(\tau,\lambda,\eta;\alpha)$ and $\mathcal{R}_{\Sigma}^{q,s}(\tau,\lambda,\eta;\beta)$ and thus, Theorem (2.2.1) and Theorem (2.3.1) reduce to Corollary (2.4.1) and Corollary (2.4.2), respectively.

Corollary 2.4.1. Let $f(z)$ given by (2.1.1) be in the class $\mathcal{R}_{\Sigma}^{q,s}(\tau,\lambda,\eta;\alpha)$.Then

$$|a_{m+1}| \le \frac{2\alpha|\tau|}{\sqrt{|2\alpha\tau(1+2\lambda+6\eta)\Gamma_{2m+1}+(1-\alpha)(1+\lambda+2\eta)^2\Gamma_{m+1}^2|}} \quad (2.4.1)$$

and

$$|a_{2m+1}| \le \frac{4\alpha^2|\tau|^2}{(1+\lambda+2\eta)^2\Gamma_{m+1}^2}+\frac{2\alpha|\tau|}{(1+2\lambda+6\eta)\Gamma_{2m+1}}\,. \quad (2.4.2)$$

Corollary 2.4.2. Let $f(z)$ given by (2.1.1) be in the class $\mathcal{R}_{\Sigma}^{q,s}(\tau,\lambda,\eta;\beta)$. Then

$$|a_{m+1}| \le \sqrt{\frac{2|\tau|(1-\beta)}{(1+2\lambda+6\eta)\Gamma_{2m+1}}} \quad (2.4.3)$$

and

$$|a_{2m+1}| \le \frac{4|\tau|^2(1-\beta)^2}{(1+\lambda+2\eta)^2\Gamma_{m+1}^2}+\frac{2|\tau|(1-\beta)}{(1+2\lambda+6\eta)\Gamma_{2m+1}}\,. \quad (2.4.4)$$

The classes $\mathcal{R}_{\Sigma}^{q,s}(\tau,\lambda,\eta;\alpha)$ and $\mathcal{R}_{\Sigma}^{q,s}(\tau,\lambda,\eta;\beta)$are defined in the following way:

Definition 2.4.1.A function $f(z)\in\Sigma$ given by (2.1.1) is said to be in the class $\mathcal{R}_{\Sigma}^{q,s}(\tau,\lambda,\eta;\alpha)$ if the following conditions are satisfied:

$$\left|arg\left(1+\frac{1}{\tau}\left[(1-\lambda)\frac{H_{q,s}f(z)}{z}+\lambda H_{q,s}f'(z)+\eta z H_{q,s}f''(z)-1\right]\right)\right| < \frac{\alpha\pi}{2}\,(z\in U) \quad (2.4.5)$$

and

$$\left|arg\left(1+\frac{1}{\tau}\left[(1-\lambda)\frac{H_{q,s}g(w)}{w}+\lambda H_{q,s}g'(w)+\eta w H_{q,s}g''(w)-1\right]\right)\right|<\frac{\alpha\pi}{2}(w\in U), \quad (2.4.6)$$

where($\tau\in\mathbb{C}\backslash\{0\};\ \lambda\geq 1;\ 0\leq\eta\leq 1;\ 0<\alpha\leq 1$)and the function g is given by (2.1.2).

Definition 2.4.2. A function $f(z)\in\Sigma$ given by (2.1.1) is said to be in the class $\mathcal{R}_{\Sigma}^{q,s}(\tau,\lambda,\eta;\beta)$ if the following conditions are satisfied:

$$Re\left(1+\frac{1}{\tau}\left[(1-\lambda)\frac{H_{q,s}f(z)}{z}+\lambda H_{q,s}f'(z)+\eta z H_{q,s}f''(z)-1\right]\right)>\beta(z\in U) \quad (2.4.7)$$

and

$$Re\ \left(1+\frac{1}{\tau}\left[(1-\lambda)\frac{H_{q,s}g(w)}{w}+\lambda H_{q,s}g'(w)+\eta w H_{q,s}g''(w)-1\right]\right)>\beta(w\in U), \quad (2.4.8)$$

where($\tau\in\mathbb{C}\backslash\{0\};\ \lambda\geq 1;\ 0\leq\eta\leq 1;\ 0\leq\beta<1$)and the function g is given by (2.1.2).

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