جمهورية العراق

وزارة التعليم العالي والبحث العلمي

جامعة القادسية

كلية التربية/ قسم الرياضيات

The chaotic Properties of The Modify of Ikeda Map

بحث تخرج مقدم من قبل الطالب

منتظر سليم كريم

الى مجلس كلية التربية/ جامعة القادسية وهو جزء من متطلبات

نيل شهادة البكالوريوس في الرياضيات

بإشراف الاستاذة

م.م.وفاء هادي

١٤٤٠هـ ٢٠١٩م

بسم الله الرحمن الرحيم

﴿وَقُلْ رَبِّ زِدْنِي عِلْمًا﴾

صدق الله العلي العظيم

سورة طه: ١١٤

شكر وتقدير

بدأنا بأكثر من يد وقاسينا أكثر من هم وعانينا الكثير من
الصعوبات وها نحن اليوم والحمد لله نطوي سهر الليالي وتعب الأيام
وخلاصة مشوارنا بين دفتي هذا العمل المتواضع

أتقدم بخالص شكري وامتناني إلى عمادة كلية التربية في
جامعة القادسية لإتاحتهم الفرصة لي لإكمال البحث ، كما أتقدم بخالص
الامتنان إلى أساتذتي الكرام وبالأخص الأستاذة الفاضلة

((م.م. وفاء هادي))

للمساعدة السديدة والملاحظات الدقيقة التي لولاها لما اكتمل البحث . .

كما اشكر زملائي وزميلاتي للأيام الجميلة التي قضيناها معا

الى كل من ساعدني في معلومة أو نصيحة

لكم مني كل الحب والتقدير

الإهداء

إلى الحبيب المصطفى محمد صلى الله عليه واله وسلم

إلى الذين وجوههم لغير الله ما توجهت ... وأقدامهم لغير الله ما سارت ... إلى

كل من في الوجود بعد الله ورسوله والأئمة الميامين .. إلى نبع الحنان ..

والدتي العزيزة

إلى النور الذي ينير لي درب النجاح ..

أبي العزيز

الى سندي وذخري ومصدر طاقاتي وقوتي عائلتي الحبيبة

اخواتي واخواتي

إلى من كان له الفضل في المساعدة على انجاز هذا البحث الأستاذة الفاضلة

((م.م. وفاء هادي))

إلى... أساتذتنا الكرام جميعاً

Contents

Symbols	*Meaning*
F	*A map*
$DF(v_0)$	*The differential of F at v_0*
$J(F)$	*Jacobian of F*
λ	*The eigen value*
$M_{a,b}$	*The Approximation of Ikeda map*
$\|\| \|\|$	*Norm*
$GL\ (3, R)$	*The set of all 3×3 matrices A where a,b $\in z$ such that det(A)=± 1*
$L(x,v)$	*Lyapunov exponent*

Abstract:-

In the research we introduced Basic definitions which we use through this work we study some necessary properties of the modified of Ikeda map .and we prove chaotic properties of map. We use program of matlab to prove sensitive depended on initial condition and the modified of Ikeda map has positive lyapunov exponent (under some conditions).

Introduction:

The word "chaos" is familiar in everyday speech . it normally means a lack of order or predictability. Thus one says that the weather is chaotic. or that rising particles of smoke are chaotic ,or that the stock market is chaotic. It is the lack of predictability that lies behind the mathematical notion of chaos . both sensitive dependence on initial conditions and the Lyapunov exponent qualify as measures of unpredictability . thus we have the following definition of chaos.

The word "chaos" as it applies to dynamical system has been defined in different ways, two important properties of dynamical systems which have appeared in some of these definitions are the properties of transitivity and positive topological entropy.

Chapter one

1 .General Properties of The Modify of Ikeda Map

In section one, We recall some fundamental definitions important theorem . Also, we study the general properties of $I_{a,b}$. we find the fixed points in different area, we show $I_{a,b}$ has the contraction and expantion area

<u>Definition(1-1-1)[2]</u>:-

Any $p=\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$ for which $f_1(P) = P_1, f_2(P) = P_2, f_3(P) = P_3$ is called
a fixed point.

<u>Example(1-1-2):-</u>

If $I_{a,b}\begin{bmatrix} x \\ y \\ t \end{bmatrix} = \begin{bmatrix} 1+ax\cos t - ay\sin t \\ ax\sin t + ay\cos t \\ bt \end{bmatrix}$ is the Ikeda map then v=R^3,

$$f_1\begin{bmatrix} x \\ y \\ t \end{bmatrix} = 1 + a\cos t - a\sin t, \qquad f_2\begin{bmatrix} x \\ y \\ t \end{bmatrix} = ax\sin t + ay\cos t,$$

<u>Example(1-1-3):-</u>

Let F: $R^n \to R^n$ be given by $F\begin{bmatrix} x \\ y \\ t \end{bmatrix} = \begin{bmatrix} x^2 \\ 3x+2y-3t \\ -1 \end{bmatrix}$, then $t = -1$

so $x = x^2$ then $x^2 - x = 0$ therefore $x(x-1) = 0$ if $x = 0$ or $x = 1$, if $x = 0$ then $2y + 3 = y$ hence $2y - y = -3$ therefore $y = -3$ If $x = 1$ then $3 + 2y + 3 = y$ hence $2y + 6 = 0$ therefore $y = -3$

so F has fixed points $\begin{bmatrix} 0 \\ -3 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ -1 \end{bmatrix}$

Definition(1-1-4)[3]:-

Let V be a subset of R^3 and $v_0 = \begin{bmatrix} x \\ y \\ t \end{bmatrix}$ be any element in V consider $F:V\rightarrow R^3$ a map. Furthermore assume that the first partials of the coordinate maps f_1 , f_2 and f_3 of F exist at v_0, **the differential of F at v_0** is the linear map $DF(v_0)$ defined on R^3 by :

$$DF(v_0) = \begin{bmatrix} \frac{\partial f_1(v_0)}{\partial x} & \frac{\partial f_1(v_0)}{\partial y} & \frac{\partial f_1(v_0)}{\partial t} \\ \frac{\partial f_2(v_0)}{\partial x} & \frac{\partial f_2(v_0)}{\partial y} & \frac{\partial f_2(v_0)}{\partial t} \\ \frac{\partial f_3(v_0)}{\partial x} & \frac{\partial f_3(v_0)}{\partial y} & \frac{\partial f_3(v_0)}{\partial t} \end{bmatrix}$$

For all v_0 in V. The determinant of DF (v_0) is called the **Jacobian** of F at v_0 and is denoted by $J = det\ \ DF(v_0)$.

Example(1-1-5):-

If $F\binom{x}{y} = \binom{x^2 + y^3 - 1}{x}$ is the map then $V\in R^n$, $f\binom{x}{y} = x^2 + y^3 - 1$

And $g\binom{x}{y} = \ x\ then$ $\frac{\partial f(v0)}{\partial x} = 2x$, $\frac{\partial f(v0)}{\partial y} = 3y^2$, $\frac{\partial g(v0)}{\partial x} = 1$, $\frac{\partial f(v0)}{\partial y} = 0$

$$DF(v_0)=\begin{bmatrix} \frac{\partial f(v0)}{\partial x} & \frac{\partial f(v0)}{\partial y} \\ \frac{\partial g(v0)}{\partial x} & \frac{\partial f(v0)}{\partial y} \end{bmatrix}=\begin{bmatrix} 2x & 3y^2 \\ 1 & 0 \end{bmatrix}$$

Definition(1-1-6) [3]:- :-

Let F: $R^3\rightarrow R^3$ be a map and $v_0 \in R^3$. If $|J\ F(v_0)| < 1$ then F is called **area contracting** at v_0,$|J\ F(v_0)| > 1$ then F is called **area expanding** at v_0.

Example(1-1-7):-

If

$F\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x+y\\x\end{pmatrix}$ *then F is area constracting map at* $\begin{pmatrix}0\\0\end{pmatrix}$

Solution:-

DF$\begin{pmatrix}0\\0\end{pmatrix} = \begin{pmatrix}1 & 1\\1 & 0\end{pmatrix}$, *then* $\det DF\begin{pmatrix}x\\y\end{pmatrix} = -1 < 1$ therefore F is area contracting map

Definition(1-1-8) [4]:- :-

A map $F:R^n \to R^n$ is called a **diffeomorpism** provided it is:

1. One-to-one.

2. Onto.

3. C^∞

4. its inverse $F^{-1}: R^n \to R^n$ is C^∞.

Definition(1-1-9) [2]:-:

Suppose that A is a 2×2 matrix. The real number λis an **eigenvalue** of A

provided that there is a nonzero v in R^2such that Av=λv. In this case v is an eigenvector of A (relative to λ) .

Definition(1-1-10) [3]:-:-

Let $\begin{bmatrix}x\\y\\t\end{bmatrix}$ be a fixed point of F, then $\begin{bmatrix}x_0\\y_0\\t_0\end{bmatrix}$ is attracting fixed

point. If and only if then is a disk centered of $\begin{bmatrix}x\\y\\t\end{bmatrix}$ such that

$F^n\begin{bmatrix}x\\y\\t\end{bmatrix}\to\begin{bmatrix}x_0\\y_0\\t_0\end{bmatrix}$ as n→∞ for every $\begin{bmatrix}x\\y\\t\end{bmatrix}$ in the disk centered of $\begin{bmatrix}x_0\\y_0\\t_0\end{bmatrix}$.by contrast $\begin{bmatrix}x_0\\y_0\\t_0\end{bmatrix}$is repelling

fixed point if and only if there is a disk centered at $\begin{bmatrix}x_0\\y_0\\t_0\end{bmatrix}$such that

$$\left\|F\begin{pmatrix}u\\v\\w\end{pmatrix}-F\begin{pmatrix}x_0\\y_0\\t_0\end{pmatrix}\right\| > \left\|\begin{pmatrix}u\\v\\w\end{pmatrix}-\begin{pmatrix}x_0\\y_0\\t_0\end{pmatrix}\right\|$$, for every $\begin{pmatrix}u\\v\\w\end{pmatrix}$ in the disk

for which $\begin{bmatrix}u\\v\\w\end{bmatrix} \neq \begin{bmatrix}x_0\\y_0\\t_0\end{bmatrix}$.

Example(1-1-11):

Let F: $R^n \to R^n$ be given by $F\begin{bmatrix}x\\y\\t\end{bmatrix}=\begin{bmatrix}2x+3t\\x^2+2y+5t\\2t\end{bmatrix}$

$so\ t = 2t\ then\ t = 0\ and\quad x = 2x + 3t\quad then\ \ t = 0\ ,\ x = 2x\quad hence\quad 2x - x\ = 0\quad therefore\quad x = 0\quad so\quad x^2 + 2y + 5t = y\quad if\ x = 0,\ t = 0\quad then\ x^2 + y + 5t = 0\ therefore\ y = 0\ .\ then\ F\begin{bmatrix}x\\y\\t\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}\ is\ the\ fixed\ point$

***Example* (1-1-12):-**

let $F: R^2 \to R^2$ be a map $F\binom{x}{y} = \begin{pmatrix} 2x \\ a\ sinx - y \end{pmatrix}$, then $\binom{0}{0}$ is the saddle fixed point.

Solution :-

$F\binom{x}{y} = \begin{pmatrix} 2x \\ a\ sinx - y \end{pmatrix}$, so $DF\binom{0}{0} = \begin{pmatrix} 2 & 0 \\ a\ cosx & -1 \end{pmatrix}$ then to find eigenvalues

$det(F - \lambda I) = det\begin{pmatrix} 2-\lambda & 0 \\ a\ cosx & -1-\lambda \end{pmatrix} = (2-\lambda)(-1-\lambda) = 0\ therefore$

$\lambda 1 = 2 > 1,\ \lambda 2 = -1 < 1$ then

$\binom{0}{0}$ $is\ the\ saddle\ fixed\ point$.

<u>Example(1-1-13) :-</u>

let $F: R^2 \to R^2$ be a map $F\binom{x}{y} = \begin{pmatrix} 2x \\ a\ sinx + 3y \end{pmatrix}$, then $\binom{0}{0}$is the repelling fixed point.

Solution:-

$F\binom{x}{y} = \begin{pmatrix} 2x \\ a\ sinx + 3y \end{pmatrix}, so\ DF\binom{0}{0} = \begin{pmatrix} 2 & 0 \\ a\ cosx & 3 \end{pmatrix}$ then to find eigenvalues

$$det(F - \lambda I) = det\begin{pmatrix} 2-\lambda & 0 \\ a\ cosx & 3-\lambda \end{pmatrix} = (2-\lambda)(3-\lambda) = 0\ therefore$$

$\lambda 1 = 2 > 1, \lambda 2 = 3\ > 1$ then $\binom{0}{0}$ is the repelling fixed point

<u>Example (1.1.14):-</u>

let F: $R^2 \rightarrow R^2$ be a map $F\binom{x}{y} = \binom{-2x}{a\, sinx - y}$, then $\binom{0}{0}$is the attracting fixed point.

Solution:-

$F\binom{x}{y} = \binom{-2x}{a\, sinx - y}$, so $DF\binom{0}{0} = \begin{pmatrix} -2 & 0 \\ a\, cosx & -1 \end{pmatrix}$ then to find eigenvalues

$$det\ (F - \lambda I) = det \begin{pmatrix} -2-\lambda & 0 \\ a\, cosx & -1-\lambda \end{pmatrix}$$
$$= (-2-\lambda)(-1-\lambda) = 0\ therefore$$

$\lambda 1 = -2 < 1, \lambda 2 = -1 < 1$ then $\binom{0}{0}$ is the attracting fixed point.

proposition(1-1-15):-

If $b > 0$ then $M_{a,b}$ has two fixed point.

Proof: -

by the definition of fixed points, we get:-

$\begin{bmatrix} x \\ y \\ t \end{bmatrix} = \begin{bmatrix} 1 - axt^2 - yt^3 \\ axt^3 - ayt^2 \\ bt^2 \end{bmatrix}$ $bt^2 = t$ then $t(bt-1) = 0$ either $t = 0\ or\ (bt-1) = 0$ then $t = \frac{1}{b}$. If t = $\frac{1}{b}$

$\frac{ax}{b^3} - \frac{a\,y}{b^2} - y = 0$ so $\frac{a\,x - a\,b\,y - b^3\,y}{b^3} = 0$, therefore $a\,x - a\,b\,y - b^3\,y = 0$

$ax = y(b^3 + ab)$, $x = \frac{(b^3+ab)y}{a}$ then $1 - \frac{ay(b^3+ab)}{ab^2} - \frac{ay}{b^3} - \frac{(b^3+ab)y}{a} = 0$

, $1-\frac{y(b^3+a)}{b}-\frac{ay}{b^3}+\frac{yb^3+yab}{a}=0$ since $\frac{ab^3-ab^2(b^2+a)y-a^2y-b^6y-ab^4y}{ab^3}=0$

$then\ \ ab^2-ab^4-a^2b^2y-a^2y-b^6y-ab^4y=0$
$therefore$

$y=\frac{ab^3}{ab^4-a^2b^2-a^2-b^6-ab^4}$

, $x=\frac{(b^3+ab)ab^3}{a(ab^4-a^2b^2-a^2-b^2-ab)}\ and\ \ t=\frac{1}{b}$

Hence p1 $\begin{pmatrix}\frac{(b^3+ab)ab^3}{a(ab^4-a^2b^2-a^2-b^2-ab)}\\ \frac{ab^3}{ab^4-a^2b^2-a^2-b^6-ab^4}\\ \frac{1}{b}\end{pmatrix}$ is the fixed point of approximation of Ikeda map . If t=0 then $\begin{pmatrix}1\\0\\0\end{pmatrix}$ is fixed point.■

Proposition(1-1-16):-

The Jacobain of Modify of Ikeda map is $\frac{2a^2(b^2+1)}{b^6}\ at\ p_1$.

proof:-

The differential matrix of $M_{a,b}$ is

$$DM_{a,b}=\begin{bmatrix}\frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial t}\\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial t}\\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial t}\end{bmatrix}$$

so $DM_{a,b}=\begin{bmatrix}-at^2 & -at^3 & 2axt-3ayt^2\\ at^3 & -at^2 & 3axt^2-2ayt\\ 0 & 0 & 2bt\end{bmatrix}=0$

then

$$J = detDM_{a,b}(v_0) = 2det\begin{bmatrix} -\frac{a}{b^2} & -\frac{a}{b^3} \\ \frac{a}{b^3} & -\frac{a}{b^2} \end{bmatrix} = \frac{2a^2(b^2+1)}{b^6}$$

Proposition (1-1-17):-

(1) If $|a| < \mp\frac{b^2\sqrt{1+b^2}}{\sqrt{2}}$ then $M_{a,b}$ is area contracting map at p_1 .

(2) If $|a| > \mp\frac{b^2\sqrt{1+b^2}}{\sqrt{2}}$ then $M_{a,b}$ is area expanding map at p_1

Proof:-

(1) If $|a| < \mp\frac{b^2\sqrt{1+b^2}}{\sqrt{2}}$ so the absolute value of Jacobain of $M_{a,b}$ is least then 1 so from definition then $M_{a,b}$ is area contracting map.

(2) If b≠0 since $|J| = \left|det\left(DI_{a,b}\begin{bmatrix} x \\ y \\ t \end{bmatrix}\right|\right| = \left|\frac{2a^2(b^2+1)}{b^6}\right|$ by hypothesis $|a| > \mp\frac{b^2\sqrt{1+b^2}}{\sqrt{2}}$ then $M_{a,b}$ is area expanding $M_{a,b}$ ■

Proposition (1-1-18):-

The Modify of Ikeda map is diffeomorphism

1. $M_{a,b}$ is onto

Proof:-

Let $\begin{bmatrix} v \\ w \\ s \end{bmatrix}$ any element in R^3 such that $\begin{bmatrix} v \\ w \\ s \end{bmatrix} = \begin{bmatrix} 1 - axt^2 - yt^3 \\ axt^3 - ayt^2 \\ bt^2 \end{bmatrix}$

then

$v = 1 - axt^2 - ayt^3$ then $axt^2 = 1 - v - ayt^3$

therefore

$x = \frac{1-v-ayt^3}{at^2}$ so $w = axt^3 - ayt^2$ then $axt^3 - w = ayt^2$ therefore

$y = \frac{\frac{at^3-vat^3-a^2t^6y}{at^2} - w}{at^2} = \frac{at^3-vat^3-a^2t^6y-wat^2}{a^2t^4}$ then $ya^2t^4 = at^3 - vat^3 - a^2t^6y - wat^2$ therefore $at^3 - vat^3 - a^2t^6y - wat^2 - ya^2t^4 = 0$ then

$y = \frac{at^3-vat^3-wat^2}{t^4a^2\,(1+t^2)}$ since

$s = bt^2$ then $t^2 = \frac{s}{b}$ then $t = \sqrt{\frac{s}{b}}$ hence

$$\begin{bmatrix} \dfrac{1-v-\frac{as}{b}\sqrt{\frac{s}{b}}\left(\frac{at^3-vat^3-wat^2}{2a^2t^6}\right)}{a\frac{s}{b}} \\ \dfrac{\frac{as}{b}\sqrt{\frac{s}{b}}-\frac{vas}{b}\sqrt{\frac{s}{b}}-wa\sqrt{\frac{s}{b}}}{a^2\frac{s^2}{b^2}(1+\frac{s}{b})} \\ \sqrt{\frac{s}{b}} \end{bmatrix} \text{is onto}$$

(2) The $M_{a,b}$ is not one to one

Proof :-

$1 - axt^2 - ayt^3 = 0$, so $axt^3 - axt^2 = 0$ then $bt^2 = 0$ therefore $t = 0\,, y = 0$ hence $x = 0$ then the $M_{a,b}$ is not one to one

(3) The $M_{a,b}$ is C^∞ then

$\frac{\partial f_1}{\partial x} = -at^2$, $\frac{\partial^2 f_1}{\partial x^2} = 0$, ………. $\frac{\partial^n f_n}{\partial x^n} = 0$ for all n $\in N$ and $n \geq 2$

$\frac{\partial f_2}{\partial y} = -at^3$, $\frac{\partial^2 f_2}{\partial y^2} = 0$, ……… $\frac{\partial^n f_n}{\partial y^n} = 0$ for all n $\in N$ and $n \geq 2$

$\frac{\partial f_3}{\partial t} = 2at$, $\frac{\partial^2 f_3}{\partial t^2}$ =2b, ……… $\frac{\partial^n f}{\partial y^n} = 0$ for all n$\in N$ and n ≥3

Then $M_{a,b}$is not diffeomorphism map.■

Proposition(1-1-19):-

The eigenvalues of D$M_{a,b}$ at the point p_1 is $\lambda_1 = 2$,

$\lambda_2 = -\ at^2 \pm at^3 i$.

Proof:-

So by proposition (fixed point) $M_{a,b}$ has fixed point $v_0 = \begin{pmatrix} \frac{(b^3+ab)ab^3}{a(ab^4-a^2b^2-a^2-b^2-ab)} \\ \frac{ab^3}{ab^4-a^2b^2-a^2-b^6-ab^4} \\ \frac{1}{b} \end{pmatrix}$

$det(A - \lambda I)v_0 =$

$det \begin{bmatrix} -at^2 - \lambda & -at^3 & 2axt - 3ayt^2 \\ at^3 & -at^2 - \lambda & 3axt^2 - ayt \\ 0 & 0 & 2bt - \lambda \end{bmatrix}$=0

$then\ (2 - \lambda)((at^2 + \lambda)^2 + a^2t^6) = 0$ therefore $(\lambda - 2)(a^2t^4 + \lambda 2at^2 + \lambda^2 + a^2t^6) = 0\ \ then\ \lambda_1 = 2\ and\ \ \lambda^2 + 2at^2\lambda + a^2 + a^2t^6 = 0\ since$

$$\lambda_{2,3} = \frac{-(2at^2) \mp \sqrt{4a^2t^4 - 4a^2t^4 - 4a^2t^6}}{2}$$

then $\lambda_{2,3} = -at^2 \pm at^3 i$ therefore $\lambda_1 = 2$ and $\lambda_{2,3} = -at^2 \pm at^3 i$ is the eigenvalues of approximation of Ikeda map .■

Proposition(1-1-20):-

(1)If $|-at^2 \mp at^3 i| > 1$ and $|2| > 1$ then $M_{a,b}$has repelling fixed point at p_1.

(2)If $|-at^2 \mp at^3 i| < 1$ and $|2| > 1$ then $M_{a,b}$ has saddle fixed point at p_1.

proof:-

By proposition (1-1-19) and definition it's satisfying (1-1-2)■

Chapter Tow

1.Sensitive of Modify of Ikeda map:-

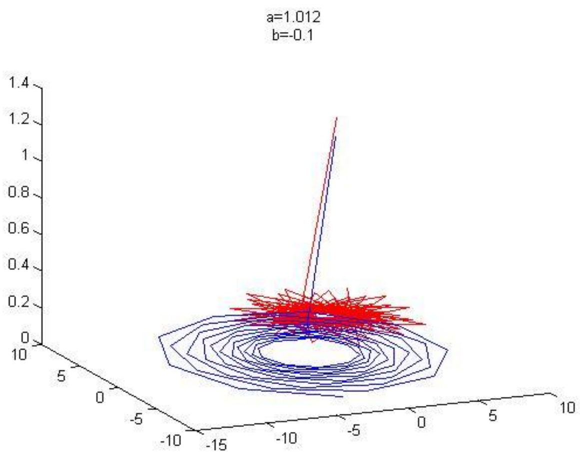
Chaos is character ited by a sensitive dependence of system dynamical variables on the initial conditions . trajectories starting with slightly different initial conditions locally diverge from each other at an exponential rates to provide a rigorous characterization as well as a way of measuring sensitive dependence on initial conditions.

Definition (2-1-1):-

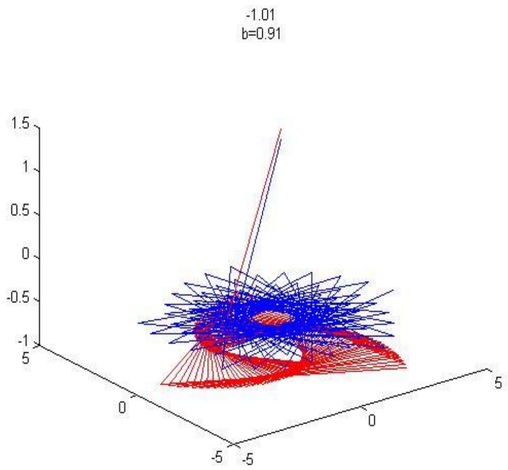
Let (X,d) be a metric space .A map f: (X,d) →(X,d) is said to be sensitive dependence on initial conditions if there exit $\varepsilon > 0$ such that for any $x_0 \in x$ and any open set U⊆ X containing x_0 there exists $y_0 \in U$ and n∈ Z^+such that

$d\ (f^n\ (x_0), f^n(y_0)) >$∈that is $\exists \in > 0, \forall\ x\ , > 0, \exists\ y \in B_0(x), \exists\ n \in N, d\big(f^n(x_0), f^n(y_0)\big) > \varepsilon$.

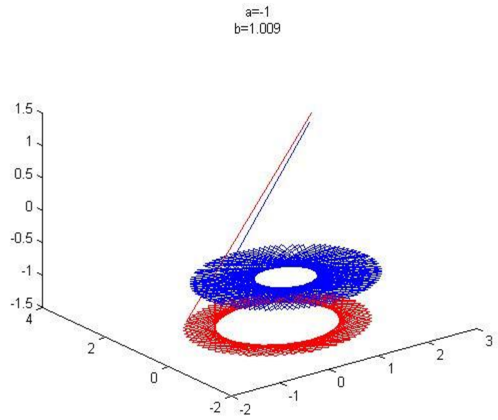
now , we draw some figures to The Ikeda map to show or approve the sensitivity dependence to initial condition



Figure(1.2) a = -1.012 , b = 0.1 with initial points (1.7,1.3,1.2) and (1.8,1.4,1.2)

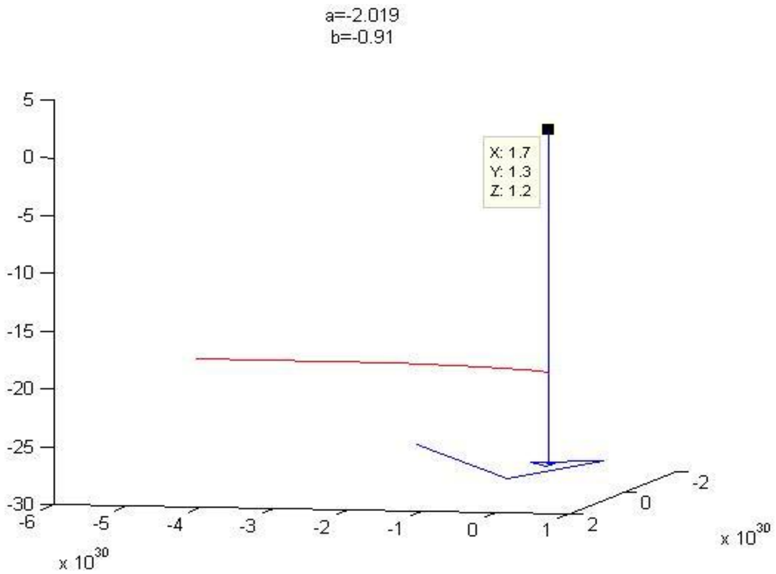


Figure(1.3) a = -1.01 , b = 0.91 with initial point (1.7,1.3,1.2) and (1.8,1.4,1.2)



Figure(1.4) a = -1, b = -1.009 with initial point (1.7,1.3,1.2) and (1.8,1.4,1.2)

No S.D.I



2.The lyapunov Exponents of Ikeda map:

The Lyapunov exponents give the average exponential rate of divergence or convergence of nearly orbital in the phase – space .in system exhibiting exponential orbital divergence, small initial differences which we may not be able to resolve get magnified rapidly leading to less of predictability any system containing at least one positive Lyapunov exponent and it is defined to be chaotic with the magnitude of the exponent reflecting the time scale on which dynamics system become unpredictable.

Definition (2-2-1)[3]

Let F: X→X be continuous differential map, where X is any metric space. Then all x in X in direction V the Lyapunov exponent was defined of a map F at X by L(x,v)= $\lim_{n\to\infty}\frac{1}{n}\, ln\left\|\, DF_x^n v\right\|$ whenever the limit exists in higher dimensions for example in R^n the map F will have n Lyapunov exponents, say

$L_1^{\pm}(x,v_1)$,$L_2^{\pm}(x,v_2),\dots,L_n^{\pm}(x,v_n)$, for a maximum Lyapunov exponent that is

$$L_{\pm}(x,v) = Max\left\{L_1^{\pm}(x,v_1),L_2^{\pm}(x,v_2),L_3^{\pm}(x,v_3),\dots,L_n^{\pm}(x,v_n)\right\},$$

where v=(v_1 ,v_2,…,v_n)

Proposition (2-2-2):-

let $M_{a,b}$:R^3→R^3 , the Lyapunov exponent of $M_{a,b}$ is positive if |a+$\sqrt[2]{t^2+1}| > 1$

Proof :-

By properties (1-1-19) , ∴ |λ1|>1 then $M_{a,b}$ is the positive Lyapunov exponent

Definition (2-2-3) [1]:-:-

A map f is **chaotic** if it satisfies at least one of the following conditions:-

1. F has a positive Lyapunov exponent at each point in its domain that is not eventually periodic.
2. F has sensitive dependence on initial conditions on its domain

Reference:

[1] Al-Dewah,K.A., "Generalogy of period –Doubling Bifurcations of Unimodal and Odd Symmetric Bimodal Maps," M.Sc., University of Baghbab, 2001.

[2] Alligood K.T., Sauer T.D., Yorke J.A., "Chaos : An Introduction

to Dynamical Systems", Springer-Verlag, Inc., New York, 1996 .

[3] Arrowsmith, D.K." Ordinary Differential Equations" , Chapman and Hall USA, 1982.

[4] Beltrami, E. Mathematics for Dynamical Medeling. Boston, Academic press 1989.