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Application of the ZZ Transform in the Differential Equations

A research submitted to the Council of the Department of Mathematics College of Education / University of Qadisiya, as part of a bachelor's degree in Mathematics requirement

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Dedication

To the fountain of patience and optimism and hope To each of the following in the presence of God and His Messenger

..... my mother dear

To those who have demonstrated to me what is the most beautiful

...... of my brother's life

To the big heart

......my dear father

To the people who paved our way of science and knowledge ***....... All our teachers Distinguished***

To those who defended us and without them we couldn't do this work

......All soldiers in Iraqi army and all

Abstract

In this article, a new integral transform similar to Laplace and Sumudu Transforms is initiated. It congregates to Sumudu transform just by changing variables. The advantage of this transform is that it solves the differential equation with variable coefficient, which is an edge on Sumudu transform. A table is offered for existing Sumudu transform. Different examples are given and results are like-minded.

1. Introduction

There are several integral transforms like, Laplace Transform, Fourier Transform, Sumudu Transform, Elzaki Transform, Tariq Transform, Natural Transform and Aboodh Transform, to crack the DEs and IEs. Of these the most widely used transform is Laplace Transform. In view of various appealing properties which craft visualization earlier, we pioneered a new integral transform, named as ZZ Transformation and applied it to the different DEs. Subsequently we developed the ZZ Transform of various functions and derivatives employed in engineering problems. In this paper we thrashed out the basic theory of ZZ transform with supporting examples and presented a table. The specialty of this new transform is the convergence to the Sumudu transform. It works as a check on the mentioned transform.

A differential equation is an equation involving derivatives i.e. is an equation for a function that relates the values of the function to the values of its derivatives. An ordinary differential equation (ode) is a differential equation for a function of a single variable, e.g., x(t), while a partial differential equation (pde) is a differential equation for a function of several variables, e.g., v(x, y, z, t). An ode contains ordinary derivatives and a pde contains partial derivatives. Typically, pde's are much harder to solve than ode's. In applications, the functions usually represent physical quantities, the derivatives represent their rates of change, and the differential equation defines a relationship between the two. Because such relations are extremely common, differential equations play a prominent role in many disciplines including engineering, phsics, economics, and biology. In pure mathematics, differential equations are studied from several different perspectives, mostly concerned with their solutions. A solution to a differential equation on an interval α<t<β is any function y(t) which satisfies the differential equation in question on the interval α<t<β. It is important to note that solutions are often accompanied by intervals and these intervals can impart some important information about the solution. Only the simplest differential equations are solvable by explicit formulas; however, some properties of solutions of a given differential equation may be determined without finding their exact form. If a closed-form expression for the solution is not available, the solution may be numerically approximated using computers. The theory of dynamical systems puts emphasis on qualitative analysis of systems described by differential equations, while many numerical methods have been developed to determine solutions with a given degree of accuracy.

2. The New Transform "ZZ Transform":

Let $f(t)$ be a function defined for all $t \geq 0$ The ZZ transform of $f(t)$ is the function Z(u,s) defined by

$$Z(u,s) = H\{f(t)\} = s\int_0^\infty f(ut)e^{-st}\, dt \qquad (1)$$

provided the integral on the right side exists.

Equation (1) can be written as

$$H\{f(t)\} = \frac{s}{u}\int_0^\infty f(t)e^{-\frac{s}{u}t}\, dt \qquad (2)$$

3. Existence and Linearity of ZZ Transform:

<u>Difintion 3.1</u> Function Of Exponential Order *b*

A function $f(t)$ is of exponential order $b>0$ if there are numbers M and T >0 such that:

$$|f(t)| < Me^{bt} \quad , \quad t > T \qquad (3)$$

Theorem 3.1: (Existence) If $f(t)$ is piecewise continuous in every finite interval $0 \leq t \leq K$ and of exponential order b for $t > K$ then its *ZZ* transform $Z(u, s)$ exists for all $s > b$ and $u > b$.

<u>Proof:</u>

We have for every positive number k

$$\frac{s}{u}\int_{0}^{\infty} f(t)\, e^{-\frac{s}{u}t}\ dt = \frac{s}{u}\int_{0}^{k} f(t)\, e^{-\frac{s}{u}t}\ dt + \frac{s}{u}\int_{k}^{\infty} f(t)\, e^{-\frac{s}{u}t}\ dt$$

Since $f(t)$ is piece wise continuous in every finite interval $0 \leq t \leq k$, the first integral on the right side exists. Also the second integral on the right side exists. So $f(t)$ is of exponential order b for $t > k$. To see this we have only to observe that in such case:

$$\left|\frac{s}{u}\int_{0}^{\infty} f(t)e^{-\frac{s}{u}t}\ dt\right| \leq \frac{s}{u}\int_{0}^{\infty}\left|f(t)e^{-\frac{s}{u}t}\right|\ dt = \frac{s}{u}\int_{0}^{\infty}|f(t)|e^{-\frac{s}{u}t}\ dt$$

$$\leq \frac{s}{u}\int_{0}^{\infty} M\, e^{bt}\, e^{-\frac{s}{u}t}\ dt = \frac{s}{u}\int_{0}^{\infty} M\, e^{-\left(\frac{s}{u}-b\right)t}\ dt$$

$$= \frac{s}{u}\left.\frac{-Me^{-\left(\frac{s}{u}-b\right)t}}{\frac{s}{u}-b}\right|_{0}^{\infty} = \frac{sM}{(s-bu)}$$

$$\therefore \left|\frac{s}{u}\int_{0}^{\infty} f(t)e^{-\frac{s}{u}t}\ dt\right| \leq \frac{sM}{(s-bu)}$$

Theorem 3.2: (Linearity) If a and b are any constants and $f(t)$ and $g(t)$ are functions, then

$$H\{a\, f(t) + b\, g(t)\} = aH\{f(t)\} + bH\{g(t)\}$$

proof

$$H\{a\, f(t) + b\, g(t)\} = \frac{s}{u}\int_0^\infty \left(a\, f(t) + b\, g(t)\right) e^{-\frac{s}{u}t}\ dt$$

$$= a\ \frac{s}{u}\int_0^\infty f(t)\, e^{-\frac{s}{u}t}\ dt + b\ \frac{s}{u}\int_0^\infty g(t)\, e^{-\frac{s}{u}t}\ dt$$

$$= aH\{f(t)\} + bH\{g(t)\}$$

4. ZZ Transform for Basic Function:

For any function $f(t)$ we assume that the integral (2) exists. The sufficient conditions for the existence of ZZ transform are that $f(t)$ for $t \geq 0$ be piecewise continuous and of exponential order, other wise ZZ transform may or may no.

Theorem 4.1 : Let $f(t)$=1 then $H\{f\} = 1$

proof

$$H\{1\} = \frac{s}{u}\ \int_0^\infty e^{-\frac{s}{u}t}\ dt = \frac{s}{u}\int_0^\infty\ e^{-\frac{s}{u}t}\ dt = \frac{s}{u}\left[-\frac{u}{s}\ e^{-\frac{s}{u}t}\right]_0^\infty = 1$$

$$\therefore H\{1\} = \ 1$$

Theorem 4.2 : Let $f(t)$=t then $H\{f\} = \frac{u}{s}$

Proof

$$H\{t\} = \frac{s}{u}\ \int_0^\infty t\, e^{-\frac{s}{u}t}\ dt$$

$$suppose\ that\ U = t \to dU = dt \quad \&\ dV = e^{-\frac{s}{u}t}\, dt \to V = -\frac{u}{s}e^{-\frac{s}{u}t}$$

$$\therefore \int_0^\infty t\, e^{-\frac{s}{u}t}\, dt = -\frac{u}{s}\ te^{-\frac{s}{u}t}\,]_0^\infty + \frac{u}{s}\int_0^\infty e^{-\frac{s}{u}t}\, dt$$

$$= 0 + \frac{u}{s}[-\frac{u}{s}e^{-\frac{s}{u}t}]_0^\infty$$

$$= 0 + \frac{u}{s}\left[0 + \frac{u}{s}\right] = \frac{u^2}{s^2}$$

$$\therefore\ H(t) = \frac{s}{u}\ \int_0^\infty t\ e^{-\frac{s}{u}t}\ dt = \frac{s}{u}\ \frac{u^2}{s^2} = \frac{u}{s}$$

Theorem 4.3: *Let* $f(t) = t^n$ then $H(f) = n!\ \frac{u^n}{s^n}$

Proof:

$$H(t^n) = \frac{s}{u}\int_0^\infty t^n e^{-\frac{s}{u}t}\, dt$$

$$suppose\ that\ \ U = t^n \rightarrow dU = nt^{n-1}dt \quad \&\ dV = e^{-\frac{s}{u}t}\, dt$$

$$\rightarrow V\ = -\frac{u}{s}e^{-\frac{s}{u}t}$$

$$\therefore \int_0^\infty t^n e^{-\frac{s}{u}t}\, dt = -\ \frac{u}{s}t^n e^{-\frac{s}{u}t}\,]_0^\infty + n\ \frac{u}{s}\int_0^\infty t^{n-1}\ e^{-\frac{s}{u}t}\, dt = n\ \frac{u}{s}\int_0^\infty t^{n-1}\ e^{-\frac{s}{u}t}\, dt$$

$$\therefore\ A(t^n) = n\ \frac{u^2}{s^2}\int_0^\infty t^{n-1}\ e^{-\frac{s}{u}t}\, dt$$

$$now\ suppose\ that\ \ U = t^{n-1} \rightarrow dU = (n-1)t^{n-2}dt \quad \&\ dV = e^{-\frac{s}{u}t}\, dt$$

$$\rightarrow V\ = -\frac{u}{s}e^{-\frac{s}{u}t}$$

$$\therefore \int_0^\infty t^{n-1}e^{-\frac{s}{u}t}\, dt = -\frac{u}{s}t^{n-1}e^{-\frac{s}{u}t}\,]_0^\infty + (n-1)\frac{u}{s}\int_0^\infty t^{n-2}\ e^{-\frac{s}{u}t}\, dt$$

$$= (n-1)\frac{u}{s}\int_0^\infty t^{n-2}\ e^{-\frac{s}{u}t}\, dt$$

$$\therefore\ H(t^n) = n(n-1)\frac{u^3}{s^3}\int_0^\infty t^{n-2}\ e^{-\frac{s}{u}t}\, dt$$

If we continue to calculate integration in the same way then we have:

$$H(f) = n!\ \frac{u^n}{s^n}$$

Theorem 4.4: Let $f(t) = e^{at}\ then\ H(f) = \frac{s}{s-ua}$

Proof

$$H(e^{at}) = \frac{s}{u}\int_0^\infty e^{at}\, e^{-\frac{s}{u}t}\, dt\ = \frac{s}{u}\int_0^\infty e^{-(\frac{s}{u}-a)t} dt = \frac{s}{u}.\frac{-u}{(s-u\,a)}.e^{-(\frac{s}{u}-a)t}]_0^\infty$$

$$= \frac{s}{u}.\frac{u}{(s-u\,a)} = \frac{s}{s-ua}$$

Theorem 4.5: $H(\sin at) = \frac{aus}{s^2+a^2u^2}$ & $H(\cos at) = \frac{s^2}{s^2+a^2u^2}$

Proof:

By Euler's formula for the complex numbers

$$e^{iat} = \cos at + i \sin at$$

By theorem 4.4

$$H\{e^{iat}\} = \frac{s}{s-i\,ua} = \frac{s}{s-i\,ua} * \frac{s+i\,ua}{s+i\,ua} = \frac{s^2+i\,sua}{s^2+\ u^2a^2} = \frac{s^2}{s^2+\ u^2a^2} + i\ \frac{sua}{s^2+\ u^2a^2}$$

But by theorem 3.2: (Linearity)

$$H\{e^{iat}\} = H\{\cos at\} + i\ H\{\sin at\} = \frac{s^2}{s^2+\ u^2a^2} + i\ \frac{sua}{s^2+\ u^2a^2}$$

And this give as:

$$H(\sin at) = \frac{aus}{s^2+a^2u^2} \quad \& \quad H(\cos at) = \frac{s^2}{s^2+a^2u^2}$$

In the Appendex A we illustarate in a table the ZZ transform for some function.

5. Some Properties of ZZ Transform [10]:

If the ZZ transform of the function $f(t)$ is $H(f)$ then:

1. $H\{f^{(n)}(t)\} = \frac{s^n}{u^n} H(f) - \sum_{k=0}^{n-1} \frac{s^{n-k}}{u^{n-k}} f^{(k)}(0)$
2. $H\left\{\int_0^t f(p)dp\right\} = \frac{u}{s} H\{f\}$
3. $H\{t\, f(t)\} = \frac{u^2}{s}\frac{d}{du} H(f) + \frac{u}{s} H(f)$
4. $H\{t\, f'(t)\} = \frac{u^2}{s}\frac{d}{du}\left[\frac{s}{u} H(f)\right] + H(f)$
5. $H\{t\, f''(t)\} = s\frac{d}{du} H(f) - \frac{s}{u} H(f) + \frac{s}{u} f(0)$

6. Inverse of ZZ Transform

If the $H\big(f(t)\big) = Z(u,s)$ then $f(t)$ is called the inverse ZZ transform of $Z(u,s)$ and mathematically it is defined as:

$$f(t) = H^{-1}(Z(u,s))$$

where H^{-1}denotes the inverse ZZ transform operator.

7. Application Of ZZ Transform

Example 7.1: Consider the initial value problem

$Y''(t) + 2y'(t) + 5y(t) = e^{-t} \sin t$

With the initial conditions

$Y(0) = 0$ and $y'(0) = 1$

Solution:

$Y''(t) + 2y'(t) + 5y(t) = e^{-t} \sin t \qquad (1)$

Applying the ZZ transform to both sides of (1) we have

$H\{Y''(t)\} + 2H\{y'(t)\} + 5H\{y(t)\} = H\{e^{-t} \sin t\}$

$$\frac{s^2}{u^2} Z(u,s) - \frac{s^2}{u^2} y(0) - \frac{s}{u} y'(0) + 2\left(Z(u,s) - \frac{s}{u} y(0)\right) + 5Z(u,s) = \frac{\frac{s}{u}}{(\frac{s}{u}+1)^2+1}$$

$$\left(\frac{s^2}{u^2} + 2\frac{s}{u}\, 5\right) Z(u,s) = \frac{s}{u} + \frac{\frac{s}{u}}{(\frac{s}{u}+1)^2+1}$$

$$Z(u,s) = \frac{\frac{s}{u}}{((\frac{s}{u}+1)^2+1)((\frac{s}{u}+1)^2+4)} + \frac{\frac{s}{u}}{((\frac{s}{u}+1)^2+4)}$$

$$Z(u,s) = \frac{1}{3}\left(\frac{\frac{s}{u}}{\left(\left(\frac{s}{u}+1\right)^2+1\right)} - \frac{\frac{s}{u}}{\left(\left(\frac{s}{u}+1\right)^2+4\right)}\right) + \frac{\frac{s}{u}}{((\frac{s}{u}+1)^2+4)}$$

Now applying the inverse ZZ transform, we get

$Y(t) = \frac{1}{3} e^{-t} \sin t + \frac{1}{3} e^{-t} \sin 2t$

$Y(t) = \frac{1}{3} e^{-t} (\sin t + \sin 2t)$

Example 7.2: Consider the initial value problem

$$y'' - 2y' + y = 6\, t\, e^t \qquad , \qquad y'(0) = 0,\ \ y(0) = 1$$

$$H\{y''\} - 2H\{y'\} + H\{y\} = 6\, H\{t\, e^t\}$$

$$\frac{s^2}{u^2} H\{y\} - \frac{s^2}{u^2} y(0) - \frac{s}{u}\ y'(0) - 2\ \frac{s}{u} H\{y\} + 2\frac{s}{u} y(0) + H\{y\} = \frac{6\, s/u}{(\frac{s}{u}-1)^2}$$

$$\left(\frac{s^2}{u^2} - 2\frac{s}{u} + 1\right) H\{y\} - \frac{s^2}{u^2} + 2\frac{s}{u} = \frac{6\, s/u}{(\frac{s}{u}-1)^2}$$

$$\left(\frac{s}{u} - 1\right)^2 H\{y\} = \frac{6\, s/u}{(\frac{s}{u}-1)^2} + \frac{s^2}{u^2} - 2\frac{s}{u}$$

$$\therefore H\{y\} = \frac{6\, s/u}{(\frac{s}{u}-1)^4} + \frac{\frac{s^2}{u^2} - 2\frac{s}{u}}{(\frac{s}{u}-1)^2}$$

$$H^{-1}\{H\{y\}\} = H^{-1}\left\{\frac{6\, s/u}{(\frac{s}{u}-1)^4}\right\} + H^{-1}\left\{\frac{\frac{s^2}{u^2} - 2\frac{s}{u}}{(\frac{s}{u}-1)^2}\right\}$$

$$y = t^3 e^t + (1-t)e^t = (t^3 - t + 1)e^t$$

Example 7.3: y'' + 2y' +5y = 0 , y(0) = 1 , y'(0) = 5

$$\frac{s^2}{u^2} \text{ H } \{y\} - \frac{s^2}{u^2} \text{ y(0)} - \frac{s}{u} \text{ y}' (0) + 2\frac{s}{u} \text{ H(y)} - 2\frac{s}{u} \text{ y(0)} + 5 \text{ H(y)} = 0$$

$$\left(\frac{s^2}{u^2} + 2\frac{s}{u} + 5\right) \text{H(y)} - \frac{s^2}{u^2} - 5\frac{s}{u} - 2\frac{s}{u} = 0$$

$$\left(\frac{s^2}{u^2} + 2\frac{s}{u} + 5\right) \text{H(y)} = \frac{s^2}{u^2} + 7\frac{s}{u}$$

$$\text{H(y)} = \frac{\frac{s^2}{u^2} + 7\frac{s}{u}}{(\frac{s^2}{u^2} + 2\frac{s}{u} + 1 + 4)} = \frac{\frac{s^2}{u^2} + \frac{s}{u} + 6\frac{s}{u}}{\left(\frac{s}{u} + 1\right)^2 + 4}$$

$$= \frac{\frac{s^2}{u^2} + \frac{s}{u}}{\left(\frac{s}{u} + 1\right)^2 + 4} + 3\,\frac{2\,\frac{s}{u}}{\left(\frac{s}{u} + 1\right)^2 + 4}$$

$\therefore$ y = e^{-t} cos2t + 3 e^{-t} sin2t

Example 7.4 $\mathbf{y'' - 2y' + 2y = 2\,e^{-t}\sin t}$, y(0) = y' (0) = 0

$$\frac{s^2}{u^2} H\{y\} - \frac{s^2}{u^2} y(0) - \frac{s}{u} y'(0) - 2\frac{s}{u} H(y) + 2\frac{s}{u} y(0) + 2H(y) = 2\frac{\frac{s}{u}}{\left(\frac{s}{u}+1\right)^2+1}$$

$$\left(\frac{s^2}{u^2} - 2\frac{s}{u} + 2\right) H(y) = 2\frac{\frac{s}{u}}{\left(\frac{s}{u}+1\right)^2+1}$$

$$H(y) = \frac{2\frac{s}{u}}{\left(\left(\frac{s}{u}+1\right)^2+1\right)\left(\left(\frac{s}{u}-1\right)^2+1\right)}$$

$$\frac{2\frac{s}{u}}{\left(\left(\frac{s}{u}+1\right)^2+1\right)\left(\left(\frac{s}{u}-1\right)^2+1\right)} = \frac{A\frac{s^2}{u^2}+B\frac{s}{u}}{\left(\frac{s}{u}+1\right)^2+1} + \frac{C\frac{s^2}{u^2}+D\frac{s}{u}}{\left(\frac{s}{u}-1\right)^2+1}$$

And this give as:

$A = \frac{1}{4}$, $B = \frac{1}{2}$, $C \frac{-1}{4}$, $D = \frac{1}{2}$

$$\therefore H(y) = \frac{\frac{1}{4}\frac{s^2}{u^2}+\frac{1}{2}\frac{s}{u}}{\left(\frac{s}{u}+1\right)^2+1} + \frac{\frac{-1}{4}\frac{s^2}{u^2}+\frac{1}{2}\frac{s}{u}}{\left(\frac{s}{u}-1\right)^2+1}$$

$$= \frac{1}{4}\frac{\frac{s^2}{u^2}+2\frac{s}{u}}{\left(\frac{s}{u}+1\right)^2+1} - \frac{1}{4}\frac{\frac{s^2}{u^2}-2\frac{s}{u}}{\left(\frac{s}{u}-1\right)^2+1}$$

$$= \frac{1}{4}\left[\frac{\frac{s^2}{u^2}+2\frac{s}{u}}{\left(\frac{s}{u}+1\right)^2+1} + \frac{\frac{s}{u}}{\left(\frac{s}{u}+1\right)^2+1}\right] - \frac{1}{4}\left[\frac{\frac{s^2}{u^2}-\frac{s}{u}}{\left(\frac{s}{u}-1\right)^2+1} - \frac{1}{4}\frac{\frac{s}{u}}{\left(\frac{s}{u}-1\right)^2+1}\right]$$

$$\therefore y = \frac{1}{4}\left[e^{-t}\cos t + e^{-t}\sin t\right] - \frac{1}{4}\left[e^{t}\cos t - 3\,e^{t}\sin t\right]$$

E xample 7.5 y' + y = sin t, y (0) = 1

H{y'} + H{y} = H{sin t}

$$\left(1+\frac{s}{u}\right) H\{y\} = \frac{us}{s^2+u^2} + \frac{s}{u}$$

$$\left(\frac{u+s}{u}\right) H\{y\} = \frac{us}{s^2+u^2} + \frac{s}{u}$$

$$H\{y\} = \frac{u^2 s}{(s+u)(s^2+u^2)} + \frac{s}{u+s}$$

$$\frac{u^2 s}{(s+u)(s^2+u^2)} = \frac{As}{u+s} + \frac{Bs^2+csu}{s^2+u^2}$$

$\therefore u^2 s = As^3 + Asu^2 + Bs^3 + cs^2u + Bs^2u + csu$

And this give us :

A = $\frac{1}{2}$, B = $\frac{1}{2}$, C = $\frac{1}{2}$

∴ H{ y} = $\frac{1}{2}\ \frac{s}{s+u}$ - $\frac{1}{2}\ \frac{s^2}{s^2+u^2}$ + $\frac{1}{2}\ \frac{su}{s^2+u^2}$ + $\frac{s}{s+u}$

∴ H^{-1} (H{y}) = $\frac{3}{2}$ H^{-1} ($\frac{s}{s+u}$) - $\frac{1}{2}$ H^{-1} ($\frac{s^2}{s^2+u^2}$) + $\frac{1}{2}$ H^{-1} ($\frac{su}{s^2+u^2}$)

∴ y = $\frac{3}{2}$ e^{-1} - $\frac{1}{2}$ cos t + $\frac{1}{2}$ sin t

Example 7.6

$$y'' + 9\,y = 6\cos 3t \qquad , \qquad y'(0) = 3,\ \ y(0) = 0$$

$H\{y''\} + 9\,H\{y\} = 6\,H\{\cos 3t\}$

$\frac{s^2}{u^2} H\{y\} - \frac{s^2}{u^2}\,y(0) - \frac{s}{u}\ y'(0) + 9\,H\{y\} = \frac{6\,s^2}{s^2+9\,u^2}$

$\frac{s^2}{u^2} H\{y\} - 3\,\frac{s}{u}\ + 9\,H\{y\} = \frac{6\,s^2}{s^2+9\,u^2}$

$\left(\frac{s^2}{u^2} + 9\right) H\{y\} = \frac{6\,s^2}{s^2+9\,u^2} + 3\,\frac{s}{u}$

$\left(\frac{s^2+9\,u^2}{u^2}\right) H\{y\} = \frac{6\,s^2}{s^2+9\,u^2} + 3\,\frac{s}{u}$

$\therefore\ H\{y\} = \frac{6\,s^2 u^2}{(s^2+9\,u^2)^2} + \frac{3\,s\,u}{(s^2+9\,u^2)}$

$H^{-1}\{H\{y\}\} = H^{-1}\left\{\frac{6\,s^2 u^2}{(s^2+9\,u^2)^2}\right\} + H^{-1}\left\{\frac{3\,s\,u}{(s^2+9\,u^2)}\right\}$

$$y = t\sin 3t + \sin 3t = (t+1)\sin 3t$$

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Appendix :

Special zz transforms and the conversion to sumudu transform

S.No.	f(t)	H{f(t)}
1	1	1
2	t	$\frac{u}{s}$
3	e^{at}	$\frac{s}{s - au}$
4	cos at	$\frac{s^2}{s^2 - a^2u^2}$
5	sin at	$\frac{aus}{s^2 - a^2u^2}$
6	t^n	n! $\frac{u^n}{s^n}$
7	e^{at} sin bt	$\frac{b\,s\,u}{(s - a\,u)^2 + b^2\,u^2}$
8	e^{at} cos bt	$\frac{s^2 - \,a\,s\,u}{(s - \,\mathrm{a}\,u)^2 + b^2\,u^2}$
9	t cos at	$\frac{s\,u\,(s^2 - \,\mathrm{a}^2\,u^2\,)}{(s^2 + \mathrm{a}^2\,u^2)^2}$
10	t sin at	$\frac{2\,a\,s^2\,u^2}{(s^2 + \mathrm{a}^2\,u^2)^2}$