جمهوريه العراق
وزاره التعليم العالي والبحث العلمي
جامعه القادسية/كليه التربية
قسم الرياضيات

some properties of modified of Kaplan map

بحث مقدم إلى

قسم الرياضيات –كلية التربية – جامعة القادسية , وكجزء من متطلبات نيل درجة البكالوريوس في قسم الرياضيات

من قبل الطالب

بشار كاظم حسن

باشراف:

م. م وفاء هادي عبد الصاحب

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بِسْمِ اللَّهِ الرَّحْمَٰنِ الرَّحِيم

(وَقُلِ اعْمَلُوا فَسَيَرَى اللَّهُ عَمَلَكُمْ وَرَسُولُهُ وَالْمُؤْمِنُونَ)

صدق الله العلي العظيم

سورة التوبة الآية (105)

الاهداء

- وطني الحبيب وشهداء العراق والواجب .

- وإلى كل من ضحى من أجل أمن وأمان هذا الوطن الحبيب .

- وإلى كل من ساعدنا في العلم والمعرفة .

- وإلى والدي الحبيب ووالدتي الحبيبة ...

- وإلى اخوتي واخواتي ...

- وإلى أساتذة كلية التربية كافة وبالخصوص الأستاذة المشرفة لما بذلته من جهد في توجيهي ولها الشكر والتقدير .

- أوجه لهم تحياتي وجهدي المتواضع ... لكم منا التحية ...

الباحث

شكر وتقدير

أقدم شكري وتقدير إلى كل من ساهم في انتاج هذا الجهد المتواضع وأخص بالذكر الأستاذة الدكتورة (وفاء هادي) وإلى كافة الأساتذة في كلية التربية – قسم الرياضيات وأخواني الطلبة وإلى شعب العراق الحبيب .

Abstract

The dynamics of modified Kaplan York map studied , it is two dimensional non – linear map

The form of this map is $K_1\binom{x}{y} = \binom{2 \propto x \ mod1}{\beta y + x^2}$

$where\ \alpha\ is\ real\ number$

Some general properties of this map proved as :-

the contracting and expanding area etc .

Also , the chaotic properties of modified Kaplan map studied, we find

1- has sensitive dependence to initial condition in each

2- has positive Lyapunov exponent

Introduction

the chaos is theory an important subject in mathematics, it's study the changes on the behavior of the system with the time, the kaplan york map was introduced in1983[7].

the mofified kaplan -york map has chaotic behavior. it is one at the famous map on discrete dynamical system which has many natural application from physics to chemistry, from medicines to meteorology, from biology to economics

the word "chaos" is familiar in every day speech it normally means a lock of order or predictability. Thus one says that the weather is chaotic. or that rising particles of smoke are chaotic or that the stock market is chaotic, it is lack of predictability that lies behind the mathematical notional chaos both side. lyapunove. exponent qualify as measures of unpredictability thus we have the following definition of chaos

this work consists of chapters

in chapter one:

the concepts and definitions recalled the fixed points, the Jacobian , the eigenvalues found and we proved some necessary properties of the modified Kaplan York map , Also , we determined the contracting and expanding area

in chapter two :

we study the sensitive dependence an initial condition of the modified Kaplan York map , we proved it has positive sensitive dependence to initial condition for all $\propto \in \mathbf{R}$.and you study the Lyapunov exponents, we conclude that the modified Kaplan York map has positive Lyapunov exponents ;for all $\propto \in \mathbf{R}$,so this map is chaotic in sense of Gulick .

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Chapter one

Basic definition

Definition (1.1)[1]

Let V be a subset of R^2,the map M can always be represented in the form M(v) = $\begin{bmatrix} h_1(v) \\ h_2(v) \end{bmatrix}$,for all v in V, when m_1,m_2 are real valued coordinate map of M.

Example (1.2)

If K $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \ mod\ 1 \\ y + cos4\pi x \end{bmatrix}$, then It is Kaplan York map define on R^2

$\mathrm{K1} \begin{bmatrix} x \\ y \end{bmatrix} = 2x$ (mod 1)

$k_2 \begin{bmatrix} x \\ y \end{bmatrix} = y + \ cos4\pi x$.

Definition(1-3)[4]

Let F: $R^2 \to R^2$ be a map and S a fixed point if and only if

H $\begin{pmatrix} s1 \\ s2 \end{pmatrix}$=$\begin{pmatrix} s1 \\ s2 \end{pmatrix}$ That is S=$\begin{bmatrix} s1 \\ s2 \end{bmatrix}$ for which $h_1(S)=s_1$, $h_2(S)=s_2$ is called fixed point

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Definition(1-4)[1]

Let V be a subset of $R^2$and $v_0 = \begin{bmatrix} x \\ y \end{bmatrix}$ be any element in V consider F:V→R^2a map. Furthermore assume that the first partials of the coordinate maps f_1 and f_2 of F exist at v_0, the

differential of F at v_0 is the linear map DF(v_0) defined on R^2 by : DG (v_0)=$\begin{bmatrix} \frac{\partial f(v_0)}{\partial x} & \frac{\partial f(v_0)}{\partial y} \\ \frac{\partial g(v_0)}{\partial x} & \frac{\partial g(v_0)}{\partial y} \end{bmatrix}$

For all v_0 in V. The determinant of DF (v_0) is called the Jacobian of F at v_0 and is denoted by $J = det\ \ DF(v_0)$.

Definition(1-5)[5]

Let H: R^2 →R^2 be a map and $v_0 \in R^2$, if $|JH(v_0)| < 1$,then H is called area contracting at v_0 , if $|JH(v_0)| > 1$ then H is called area expanding at v_0

Definition(1-6)[1]:

Suppose that A is a 2*2 matrix , the real number λ is eigenvalue of A Provided that there is anon zero V in R^2 such that AV=λV In this case V is an eigenvalue to V of A (relative to λ)

Definition(1-7)[1]:

Let $\begin{bmatrix} x \\ y \end{bmatrix}$ be a fixed point of F, then $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ is attracting fixed point. If and only if then is a disk centered of $\begin{bmatrix} x \\ y \end{bmatrix}$ such that

$F^n \begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ as $n \rightarrow \infty$ for every $\begin{bmatrix} x \\ y \end{bmatrix}$ in the disk centered of $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$.by contrast $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ is repelling fixed point if and only if there is a disk centered at $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ such that $\left\| F\binom{x}{y} - F\binom{x_0}{y_0} \right\| > \left\| \binom{u}{v} - \binom{x_0}{y_0} \right\|$

, for every $\binom{u}{v}$ in the disk For $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \neq \begin{bmatrix} u \\ v \end{bmatrix}$

Chapter two

Properties chaotic of modified

Kaplan map

Proposition(2-1)

K_1 has a fixed point if $\propto \neq \frac{1}{2}$ *and* $\beta \neq 1$ ..

Proof:-

Since $K_1 \binom{x}{y} = \binom{2 \propto x \; mod 1}{\beta y + x^2}$ so

$x = 2 \propto x \; mod 1$ and $\beta y + x^2 = y$

this imply $x - 2 \propto x = 0$

$(1 - 2 \propto)x = 0$

By hypothesis $\propto \neq \frac{1}{2}$ then

$X = 0$ (mod1), that is

$X = k \; ; \forall k \in Z$

$By + x^2 = y$

$\beta y + k^2 = y \; ; \forall k \in Z$ so

$\beta y = y + k^2 \; ; \forall k \in Z$ then

$\beta y - y = k^2 \; ; \forall k \in Z$ therfore

$(\beta - 1)y = k^2$

Since $\beta \neq 1$ then $y = \frac{k^2}{\beta - 1}$.

Therefore $\begin{pmatrix} k \\ \frac{+k^2}{\beta - 1} \end{pmatrix}$ is a fixed point of $k_1 \; ; \forall k \in Z$. ■

Remark (2.2)

If $\propto = \frac{1}{2}$ thus $y + k^2 = y$, so $k^2 = 0$, then k_1 has unique fixed point is $\binom{0}{0}$

Remark (2.3)

If $\propto = \frac{1}{2}$ and $\beta = 1$, then k_1 has infinite points

$$\begin{pmatrix} x \; mod1 \\ \beta y + x^2 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

If $|\propto| = \frac{1}{2}$ and $|\beta| = 1$

$$\begin{pmatrix} 2 \propto \; x \; mod1 \\ y + x^2 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

then k_1 has $\begin{pmatrix} 0 & mod1 \\ 0 & \end{pmatrix}$ which is fixed points

If $|\beta| = 1$ and $|\propto| \neq \frac{1}{2}$

$$\begin{pmatrix} 2 \propto x & mod1 \\ y + x^2 & \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix},$$

then k_1 has $\begin{pmatrix} 0 & mod1 \\ 0 & \end{pmatrix}$ which is fixed points

Proposition(2.4):

The Jacobain of the modified Kaplan York map is $(2 \propto \beta)$, $\forall \propto, \beta \in R$

Proof

The differential matrix of $K_1$is $Dk_1 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{\partial f1}{\partial x} & \frac{\partial f1}{\partial y} \\ \frac{\partial f2}{\partial x} & \frac{\partial f2}{\partial y} \end{pmatrix} = \begin{pmatrix} 2 \propto & 0 \\ 2x & \beta \end{pmatrix}$

J= det $Dk_1 \begin{pmatrix} x \\ y \end{pmatrix} = 2 \propto \beta$; $\forall \propto, \beta \in R$. ■

Proposition(2.5):

1* If either$|\propto| > \frac{1}{2}$ or $|\beta| > \frac{1}{2}$,then k_1 is area expanding map.

2*$|\propto| < \frac{1}{2\,|\beta|}$ or $|\beta| < \frac{1}{2|\propto|}$,$\propto \neq 0$, k_1 is area contracting map.

Proof

If $|\,J\,| = \left|\ \det Dk_1\begin{pmatrix} x \\ y \end{pmatrix}\right|$

= $|2 \propto \beta| > 1$, thus $|\propto \beta| > \frac{1}{2}$.

Thus either $|\propto| > \frac{1}{2}$ or $|\beta| > 1$.

If $|\propto| > \frac{1}{2}$ then $|\propto \beta| > \left|\frac{1}{2}\beta\right| > \frac{1}{2}\,|\beta|$ then $|\beta| > 1$

If $|\beta| > 1$ then $|\propto \beta| > |\propto| > \frac{1}{2}$

Proposition(2.6):

The modified Kaplan York map is C^∞

Proof

Note that

$\frac{\partial k_1}{\partial x} = 2 \propto$, $\frac{\partial k_1}{\partial y} = 0$, $\frac{\partial k_2}{\partial x} = 2x$ and $\frac{\partial k_2}{\partial y} = \beta$

$\frac{\partial^n k_1}{\partial x^n} = 0\,; \forall\, n \in N$ and $\frac{\partial^n k_2}{\partial y^n} = 0\,; \forall\, n \in N$

these partial derivatives are exist and continuous then k_1 is C^∞

Remark (2.7):-

k_1 is not one - to – one, so k_1 is not differentiable map.

Remark (2.8):-

K_1 is not onto if

1* $\propto = 0, \beta = 0$
2*$\propto = 0, \beta \neq 0$ or
3* $\propto \neq 0, \beta = 0$

Remark (2.9):-

The eigen values of Dk_1 at the fixed point are $\lambda_1 = 2 \propto$ and $\lambda_2 = 2k; \forall k \in Z$

Proof

Since $Dk_1\binom{x}{y} = \begin{pmatrix} 2\propto & 0 \\ \beta & 2x \end{pmatrix}$

So that $Dk_1(p) = \begin{pmatrix} 2\propto & 0 \\ \beta & 2k \end{pmatrix}; \forall k \in Z$

The eigen values of $Dk_1$is

Det $\begin{pmatrix} 2\propto -\lambda & 0 \\ \beta & 2k - \lambda \end{pmatrix} = 0$, this imply

$(2\propto -\lambda)(2k - \lambda)=0$,so $\lambda_1 = 2\propto$ and $\lambda_2 = 2k; \forall k \in Z$

To find the type of the fixed points we should prove this proposition .

Proposition(2.10):

1* If $|\propto| < \frac{1}{2}$ *and* $k = 0$ then the fixed point of k_1 are attractor

2* either $|\propto| > \frac{1}{2}$ *or* $|k| > 1, \forall k \in Z$ then the fixed point of k_1 are saddle

3* If $|\propto| > \frac{1}{2}$ *and* $|k| > 1, \forall k \in Z$ then the fixed points of k_1 are repelling

Proof

Since the eigen values of Dk_1 are $\lambda_1 = 2\propto$ and $\lambda_2 = 2k; \forall k \in Z$ and by the proposition holds

Chapter two

Properties chaotic of modified Kaplan map:-

1. *Sensitive dependence on initial condition of* modified *Kaplan York map*

Chaos is characterized by a sensitive dependence of a dynamical system variables on the initial conditions . Orbits starting with slightly different initial conditions locally diverge from each other at an exponential rate ,to provide a vigorous characterization as well as a way of measuring sensitive dependence on initial conditions let (x,d) be any metric space .

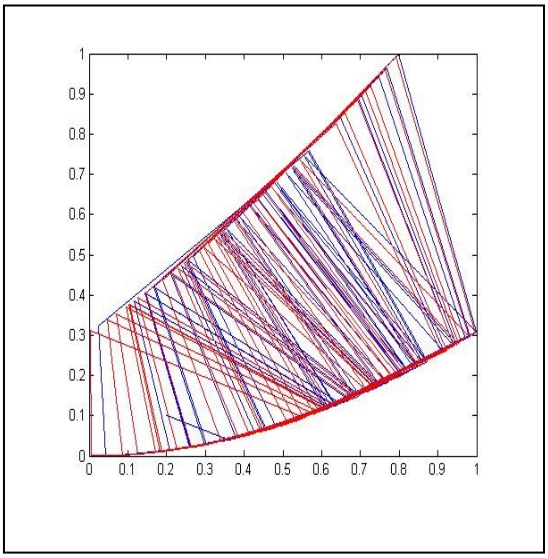
<u>Definition (2.1)[5]</u>

Let (X,d) be a metric space .A map f: (X,d) →(X,d) is said to be sensitive dependence on initial conditions if there exit $\varepsilon > 0$ such that for any $x_0 \in x$ and any open set U⊆ X containing x_0 there exists $y_0 \in U$ and n$\in Z^+$such that $d\ (f^n\ (x_0), f^n(y_0)) > \in$ that

is$\exists \in > 0, \forall\ x, > 0, \exists\ y \in B_0(x), \exists\ n \in N, d(f^n(x_0), f^n(y_0)) > \varepsilon$.

Although, there is no universal agreement on definition of chaos , it is generally agreed that a chaotic dynamical system should exhibit

now , we draw some figures to The modified of Kaplan York map to show or approve the sensitivity dependence to initial condition



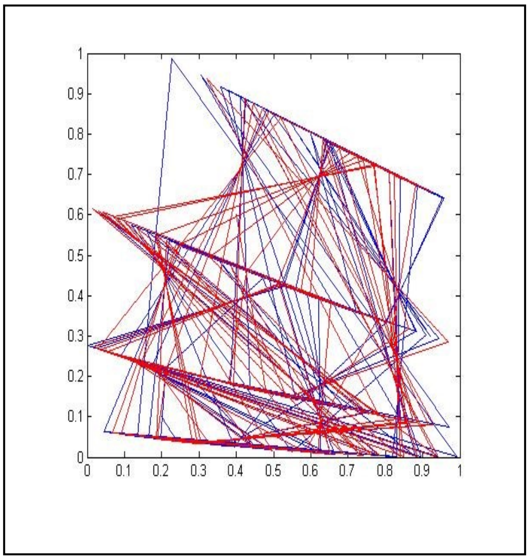
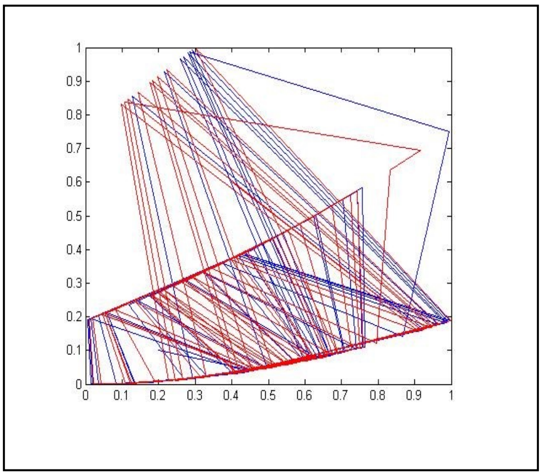


Fig (1-1) $\propto$=0.9 ,β=0 with initial points (0.2,0.1) , (0.4,0.6)

Fig (1-2) $\propto$=-1.9 ,β=0 with initial points (0.2,0.1) , (0.4,0.6)



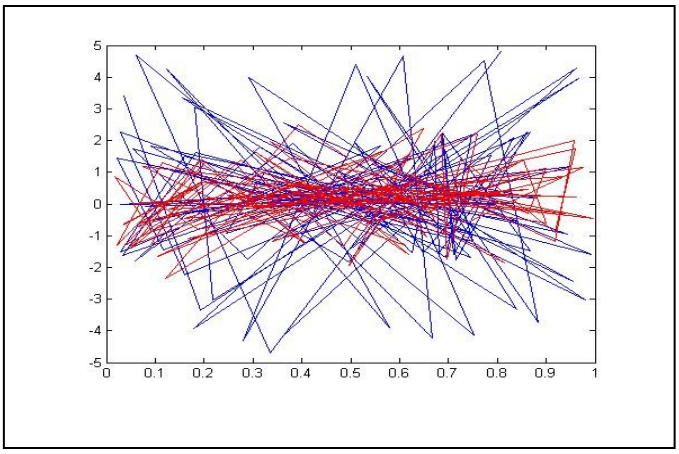
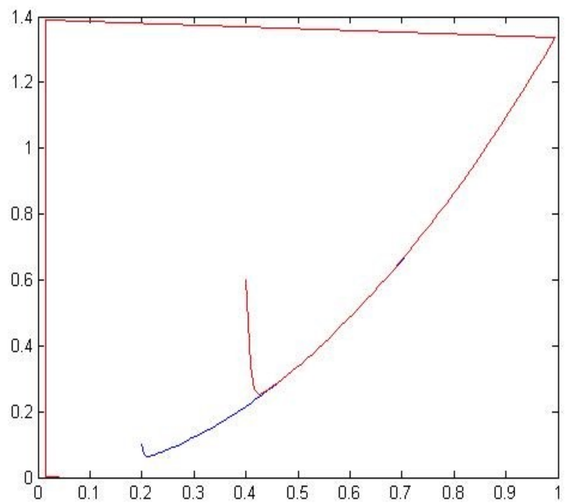


Fig (1-3) $\propto$=1.15 ,β=0 with initial points (0.2,0.1) , (0.4,0.6)

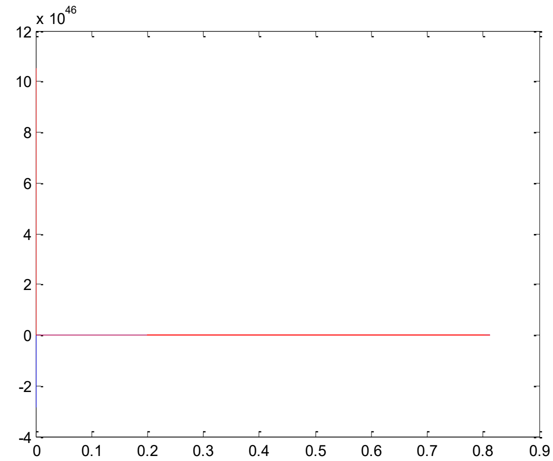
Fig (1-4)$\propto$=2.7 ,β=-1 with initial points (0.2,0.1) , (0.4,0.6)

is not sensitive depended on initial condition:-



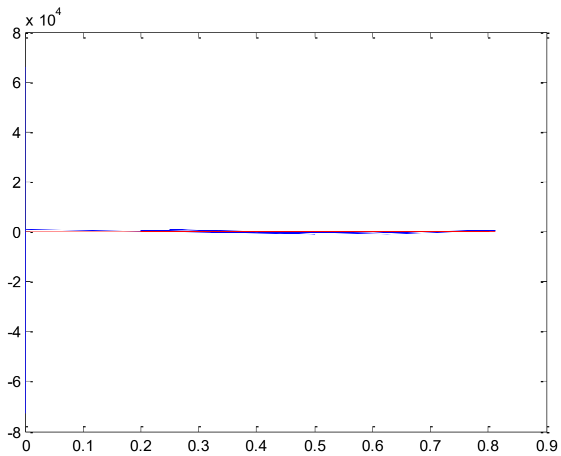
Fig(1-5)

A1=0.003, a2=0.001with initial points (0.2,0.1) , (0.4,0.6)



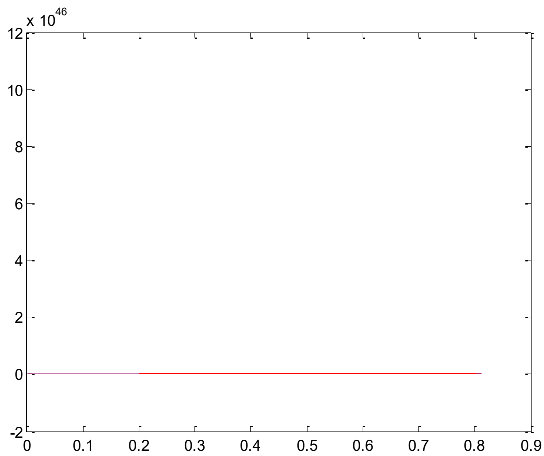
Fig(1-6)

A1=3.0003 , a2=3.00001 with initial points (0.2,0.1) , (0.4,0.6)



Fig(1-7)

A 1=2.4, a2=0.2with initial points (0.2,0.1) , (0.4,0.6)



Fig(1-8)

A1=-1.1 , a2=1 with initial points (0.2,0.1) , (0.4,0.6)

2. The Lyapunov Exponents of modified Kaplan York Map

The Lyapunov exponents give the average exponential rate of divergence or convergence of nearly orbital in the phase – space .

Definition (2-2-1)[3]

Let F: X→X be continuous differential map, where X is any metric space. Then all x in X in direction V the Lyapunov exponent was defined of a map F at X by L(x,v)= $\lim_{n\to\infty} \frac{1}{n}\, ln\|\, DF_x^n v\|$ whenever the limit exists in higher dimensions for example in R^n the map F will have n Lyapunov exponents, say

$L_1^{\pm}(x,v_1), L_2^{\pm}(x,v_2), \dots, L_n^{\pm}(x,v_n)$, for a maximum Lyapunov exponent that is

$L_{\pm}(x,v) = Max\left\{L_1^{\pm}(x,v_1), L_2^{\pm}(x,v_2), L_3^{\pm}(x,v_3), \dots, L_n^{\pm}(x,v_n)\right\}$,

where v=(v_1 ,v_2,…,v_n)

Proposition(2-2-2):

$k_1$has a positive Lypunov exponent

Proof

Since $\lambda_1 = 2 \propto$ and $\lambda_2 = 2\,k$ and L (k_1) = log $|2 \propto|$ or L (k_1) = log $|k|$

; $k \in Z$

If $|\propto| > \frac{1}{2}$ then L (n)> 0 and if $k \neq 0$ then L (n_1) >0

Definition (2-2-3) [1]

A map H is chaotic in sense of Gulick if it satisfies at least one of the following conditions:-

1* H has a positive Lyapunov exponent at each point in its domain

2* H has sensitive dependence on initial conditions on its domain.

By Proposition (2-2-2) and by draw of the sensitive dependence we get the Kaplan York map is chaotic in sense of Gulick

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