Hammadi Alaa Hussein[1], Alia Shani Hassan Kurdi[2], Alaa Hussain Kalil[3]

alaa.hammadi@qu.edu.iq, alia.shani@qu.edu.iq, : alaa.kahlil@qu.edu.iq,

# On Estimates Of Statistically Characteristics Of The Almost Periodic Function Of Control System

## Abstract

We study some properties of the statistical characteristics of the function, almost periodic in the sense of Bohr. Obtained statistical evaluation for different control systems. Also we study the external and internal parallel sets of convex $M$ in the Euclidean space.

**Keywords:** control systems, dynamical systems, differential inclusions, the attainable set , almost periodic functions.

## Introduction

this article is being studied such characteristics of the attainable set of the controlled system as relative frequencies $freq_{[\tau,\tau+\vartheta]}(\varphi, M)$,

$freq_{\vartheta}(\varphi, M)$ absorption of attainability set $D\ (t, X)$ of the control system (1.1) by the given set.

In the works [1,2] the statistical characteristics of the controlled systems are investigated.

$$\dot{x} = f(t, x, u), \qquad (t, x, u) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \qquad (0.1)$$

In this paper, we study the statistical characteristics for functions that are almost periodic in the sense of Bohr and the characteristic $freq_{\vartheta}(\varphi^*, (-\infty, c])$ which is the relative frequency of hit path of the upper solution $\varphi^*(t)$of the Cauchy problem

$$\dot{x} = w(t, x), \qquad x(0) = x_0.$$

in the set $(-\infty, 0]$. There also obtain estimates of statistical charact-

eristics $freq_{[\tau,\tau+\vartheta]}(\varphi, M)$, $freq_{\vartheta}(\varphi, M)$ for different control systems.

## §1.Basic definitions

We obtained characteristic estimates, which reflect the property of uniformity of stay reachable sets of the controlled system

$$\dot{x} = f(t, x, u), \qquad (t, x, u) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \tag{1.1}$$

in the set $M$ on a segment of a given length.

Consider the differential inclusion corresponding to system (1.1)

$$\dot{x} \in F(t, x), \qquad F(t, x) = \overline{co}\, H(t, x), \tag{1.2}$$

where for each fixed point $(t, x) \in \mathbb{R} \times \mathbb{R}^n$ set $H(t, x)$ consists of all limiting values of the function $f\big(t_i, x_i, U(t_i, x_i)\big)$ at $(t_i, x_i) \to (t, x)$,

$\overline{co}\, H(t, x) -$ closure of the convex hull of the set $H(t, x)$. We suppose that that the function $f(t, x, u)$ continuous on set of variables and the function $U(t, x)$ upper semi-continuous by $(t, x)$. Then the function $F(t, x)$ also upper semi-continuous, and the set $F(t, x)$ is non-empty, bounded, closed and convex, so for each initial point $x_0 \in \mathbb{R}^n$ a local solution of the inclusion (1.2) exists (see [3, p. 60]).

We denote by $D(t, X)$ attainability set of system (1.1) at the moment of time $t$ from the initial set X. We assume that for each $X$ the attainable set $D(t, X)$ exists for all $t \geq 0$. This means that for each point $x \in X$ there exists a solution $\varphi(t, x)$ of the inclusion (1.2), satisfying the initial condition $\varphi(0, x) = x$ and continuing for the half-axis $\mathbb{R}_+ = [0, \infty)$.

In works [1, 2, 4] we consider introduced and studied such Characteristics, as relative frequency $freq(D(t, X), M)$, upper and lower relative frequencies $freq^*(D(t, X), M)$, $freq_*(D(t, X), M)$ absorption attainable set $D(t, X)$ of the controlled system (1.1) by a given subset

$$M = \{(t, x) \in \mathbb{R} \times \mathbb{R}^n \ : x \in M(t)\}$$

of space $\mathbb{R}^{n+1}$. To determine these characteristics, we introduce into consideration a set of

$$\alpha(\tau, \vartheta, X) \doteq \{t \in [\tau, \tau + \vartheta]: \ D(t, X) \subseteq M(t) \}.$$

Definition 1. (see [1,4]). The relative frequency of absorption of the attainability set $D(t, X)$ of system (1.1) by the set $M$ called is the following limit

$$freq(D(t, X), M) \doteq \lim_{\vartheta \to \infty} \frac{mes\ \alpha(0, \vartheta, X)}{\vartheta}$$
$$= \lim_{\vartheta \to \infty} \frac{mes\{t \in [0, \vartheta]: D(t, X) \subseteq M(t)}{\vartheta}, \qquad (1.3)$$

where $mes -$ is the Lebesgue measure on the real line. If the limit (1.3) does not exist, then the characteristics

$$freq_*(D(t, X), M) \doteq \underline{\lim}_{\vartheta \to \infty} \frac{mes\ \alpha(0, \vartheta, X)}{\vartheta},$$
$$freq^*(D(t, X), M) \doteq \overline{\lim}_{\vartheta \to \infty} \frac{mes\ \alpha(0, \vartheta, X)}{\vartheta} \qquad (1.4)$$

are called, respectively, the lower and upper relative frequency of absorption of the attainability set D (t, X) of system (1.1) by the set $M$.

Definition 2 (see [2, 4]). The relative frequency of absorption of the attainability set of system (1.1) given by the set M on the interval $[\tau, \tau + \vartheta]$ is the characteristic

$$freq_{[\tau, \tau+\vartheta]}(D(t, X), M) \doteq \frac{mes\ \alpha(\tau, \vartheta, X)}{\vartheta}$$
$$= \frac{mes\{t \in [\tau, \tau + \vartheta]: D(t, X) \subseteq M(t)}{\vartheta}.$$

It is important to consider the relative frequency $freq_{[\tau,\tau+\vartheta]}(D(t,X),M)$

for any time point $\tau > 0$, therefore, it is natural for a given $\vartheta > 0$ to determine the characteristic

$$freq_{\vartheta}(D(t,X),M) \doteq \inf_{\tau\geq 0} freq_{[\tau,\tau+\vartheta]}(D(t,X),M) =$$
$$= \inf_{\tau\geq 0}\frac{mes\{t\in[\tau,\tau+\vartheta]: D(t,X)\subseteq M(t)}{\vartheta}.$$

This characteristic differs from (1.3), (1.4 ) that it reflects the uniformity property of staying of the attainability set D (t, X) in the set M on interval of predetermined length.

Definition 3. The relative frequency of finding the graph of the function $\varphi\ (t)$ in the set $M\ (t)$ on the interval $[\tau,\tau\ +\ \vartheta]$ is the characteristic

$$freq_{[\tau,\tau+\vartheta]}(\varphi,M) \doteq \frac{mes\{t\in[\tau,\tau+\vartheta]: \varphi(t)\in M(t)\ \}}{\vartheta},$$

such for any given $\vartheta >\ 0$, we define the characteristic

$$freq_{\vartheta}(\varphi,M) \doteq \inf_{\tau\geq 0} freq_{[\tau,\tau+\vartheta]}(\varphi,M)$$
$$= \inf_{\tau\geq 0}\frac{mes\{t\in[\tau,\tau+\vartheta]: \varphi(t)\in M(t)\ \}}{\vartheta}.$$

In this notation

$$freq_{[\tau,\tau+\vartheta]}(\varphi,(-\infty,c]) \doteq \frac{mes\{t\in[\tau,\tau+\vartheta]: \varphi(t)\leq c\}}{\vartheta},$$

$$freq_{\vartheta}(\varphi,(-\infty,c]) \doteq \inf_{\tau\geq 0} freq_{[\tau,\tau+\vartheta]}(\varphi,(-\infty,c])$$
$$= \inf_{\tau\geq 0}\frac{mes\{t\in[\tau,\tau+\vartheta]: \varphi(t)\leq c\ \}}{\vartheta}.$$

## § 2. Estimates of relative frequencies for almost periodic functions.

Recall that the function $\varphi\ (t)$ is called almost periodic in the sense of Bohr, if it is continuous and for every $\varepsilon > 0$ the set of $\varepsilon$-almost periods

$$\theta(\varepsilon) = \{\vartheta \in \mathbb{R} : \sup_{t\in\mathbb{R}} |\varphi(t+\vartheta) - \varphi(t)| \leq \varepsilon\}$$

relatively tightly on the real axis $\mathbb{R}$, that is, there is a number $L > 0$ such that, each interval $[a, a + L]$ length $L$ contains at least one element of the set (see, for example, [5, p. 367-368], [6, p. 7]).

**Lemma 1.** Let the function $\varphi\ (t)$ be almost periodic in the sense of Bohr and $L$ is a number such that each interval $[a, a + L]$ contains at least one ε-almost period and $\vartheta \in \theta\ (\varepsilon) \geq L$. Then for any $\tau > 0$, the inequality holds

$$freq_{[0,\vartheta]}(\varphi, (-\infty, -2\varepsilon]) \leq freq_{[\tau,\tau+\vartheta]}(\varphi, (-\infty, 0]) \leq$$

$$\leq freq_{[0,\vartheta]}(\varphi, (-\infty, 2\varepsilon]) \qquad (2.1)$$

Therefore $freq_{[0,\vartheta]}(\varphi, (-\infty, -2\varepsilon]) \leq freq_{\vartheta}(\varphi, (-\infty, 0]) \leq$

$$\leq freq_{[0,\vartheta]}(\varphi, (-\infty, 2\varepsilon])\ .$$

Proof: Since $\vartheta > L$, then the interval $[\tau, \tau + \vartheta]$ contains at least one $\varepsilon$ —almost period of the function $\vartheta\ (t)$. We fix one of them and denote it by $\vartheta$. Consider the case when $\theta \in (\tau, \tau + \vartheta)$ and represent the interval $[\tau, \tau + \vartheta]$ as a union

$$[\tau, \tau + \vartheta] = [\tau, \theta) \cup [\theta, \tau + \vartheta].$$

Let $t \in [\tau, \theta)$. Since $\theta - \vartheta$ is $2\varepsilon$-almost the period of the function $\varphi\ (t)$, that performed inequality $|\varphi(t) - \varphi(t - (\theta - \vartheta))| \leq 2\varepsilon$, i.e.

$$\varphi(t - \theta + \vartheta) - 2\varepsilon \leq \varphi(t) \leq \varphi(t - \theta + \vartheta) + 2\varepsilon.$$

Therefore,

$$mes\{t \in [\tau, \theta) : \varphi(t - \theta + \vartheta) \leq -2\varepsilon\} \leq mes\{t \in [\tau, \theta) : \varphi(t) \leq 0\} \leq$$
$$\leq mes\{t \in [\tau, \theta) : \varphi(t - \theta + \vartheta) \leq 2\varepsilon\},$$

from which we obtain the inequality

$$mes\{t \in [\tau - \theta + \vartheta, \vartheta) : \varphi(t) \leq -2\varepsilon\} \leq mes\{t \in [\tau, \theta) : \varphi(t) \leq 0\} \leq$$
$$\leq mes\{t \in [\tau - \theta + \vartheta, \vartheta) : \varphi(t) \leq 2\varepsilon\}. \qquad (2.2)$$

Let $t \in [\theta, \tau + \vartheta]$, then $\varphi(t - \theta) - \varepsilon \leq \varphi(t) \leq \varphi(t - \theta) + \varepsilon$.

Hence,

$$mes\{t \in [0, \tau - \theta + \vartheta) : \varphi(t) \leq -\varepsilon\} = mes\{t \in [\theta, \tau + \vartheta) : \varphi(t - \theta) \leq -\varepsilon\} \leq mes\{t \in [\theta, \tau + \vartheta) : \varphi(t) \leq 0\} \leq mes\{t \in [\theta, \tau + \vartheta) : \varphi(t - \theta) \leq \varepsilon\} = mes\{t \in [0, \tau - \theta + \vartheta) : \varphi(t) \leq \varepsilon\} \qquad (2.3)$$

adding term by term inequalities (2.2) and (2.3) we get

$$\frac{mes\{t \in [0, \vartheta] : \varphi(t) \leq -2\varepsilon\}}{\vartheta} \leq \frac{mes\{t \in [\tau, \tau + \vartheta] : \varphi(t) \leq 0\}}{\vartheta} \leq$$
$$\leq \frac{mes\{t \in [0, \vartheta] : \varphi(t) \leq 2\varepsilon\}}{\vartheta} \qquad (2.4)$$

that is, inequality (2.1) is proved.

From (2.4) and definitions

$freq_{\vartheta}(\varphi, (-\infty, 0]) = \inf_{\tau \geq 0} \frac{mes\{t \in [\tau, \tau + \vartheta] : \varphi(t) \leq 0\}}{\vartheta}$ also follows inequality

$$\frac{mes\{t \in [0, \vartheta] : \varphi(t) \leq -2\varepsilon\}}{\vartheta} \leq freq_{\vartheta}(\varphi, (-\infty, 0])$$
$$\leq \frac{mes\{t \in [0, \vartheta] : \varphi(t) \leq 2\varepsilon\}}{\vartheta}.$$

If $\theta = \tau$, that is, τ is $\varepsilon$ —almost the period of $\varphi(t)$, then

$|\varphi(t) - \varphi(t - \tau)| \leq \varepsilon$. Writing inequalities similar to (2.3), we obtain

$$freq_{[0,\vartheta]}(\varphi, (-\infty, -2\varepsilon]) \leq\leq freq_{\vartheta}(\varphi, (-\infty, 0]) \leq freq_{[0,\vartheta]}(\varphi, (-\infty, 2\varepsilon]).$$

If $\theta = \tau + \vartheta$ is a $2\varepsilon$ —almost period of the function $\varphi(t)$, then for all $t \in (-\infty, \infty)$ performed inequality Then, similarly to (2.2), we obtain

$$|\varphi(t) - \varphi(t - (\theta - \vartheta))| \leq \varepsilon.$$

Then, similarly to (2.2), we obtain

$$\frac{mes\{t \in [0,\vartheta]: \varphi(t) \leq -2\varepsilon\}}{\vartheta} \leq \frac{mes\{t \in [\tau, \tau + \vartheta]: \varphi(t) \leq 0\}}{\vartheta} \leq$$
$$\leq \frac{mes\{t \in [0,\vartheta]: \varphi(t) \leq 2\varepsilon\}}{\vartheta},$$

that is, inequality (2.1) holds.

Consider the Cauchy scalar problem

$$\dot{x} = w(t, x), \qquad x(0) = x_0 \qquad (2.5)$$

under the assumption that the following condition is satisfied.

Condition 1. For each $t \geq 0$, the function $x \rightarrow w(t, x)$ is continuous in the totality(see [6] ) of variables and the inequality holds

$$\overline{\lim_{|x|\to\infty}} \frac{|w(t,x)|}{|x|} < \infty$$

Recall that the upper solution $\varphi^*(t)$ of the Cauchy problem (2.5) It called a solution that for any other solution φ (t) of this problem on a common interval existence inequality holds $\varphi^*(t) \geq \varphi(t)$. In [7, p. 38] it is shown that if the function $w(t, x)$ is continuous, then the upper solution $\varphi^*(t)$ of the Cauchy problem (2.5) exists for all $t \geq 0$. Let $\tilde{\varphi}(t)$ be some almost periodic function. Then there is a limit, (see [8, p. 150]).

$$freq(\tilde{\varphi}, (-\infty, 0]) \doteq \lim_{\vartheta \to \infty} \frac{mes\{t \in [0, \vartheta]: \tilde{\varphi}(t) \leq 0\}}{\vartheta}.$$

**Lemma 2**. Let the following conditions be satisfied:

1) for each x ∈ R, the function t → w (t, x) is almost periodic in the sense of Bohr and is satisfied equality

$$freq(w, \{0\}) \doteq \lim_{\vartheta \to \infty} \frac{mes\{t \in [0, \vartheta]: w(t, x) = 0\}}{\vartheta} = 0;$$

2) there exists an almost periodic solution $\varphi(t)$ of the equation $\dot{x} = w(t, x)$ such that

$$\lim_{t \to \infty} (\varphi^*(t) - \tilde{\varphi}(t)) = 0.$$

Then there is an inequality

$$freq_{\vartheta}(\tilde{\varphi}, (-\infty, 0]) \leq freq(\varphi, (-\infty, 0])$$

Proof: Since φ (t) is the upper solution of the Cauchy problem (2.5), then inequality holds

$$freq(\varphi, (-\infty, 0]) \doteq \lim_{\vartheta \to \infty} \frac{mes\{t \in [0, \vartheta]: \varphi(t) \leq 0\}}{\vartheta}$$

$$\geq \lim_{\vartheta \to \infty} \frac{mes\{t \in [0, \vartheta]: \varphi^*(t) \leq 0\}}{\vartheta}. \qquad (2.6)$$

In work [2], it was shown that the equalities $freq\ (w, \{0\}) = 0$ and the

$$\lim_{t \to \infty} (\varphi^*(t) - \tilde{\varphi}(t)) = 0$$

must be equality

$$\lim_{\vartheta\to\infty}\frac{mes\{t\in[0,\vartheta]:\varphi^*(t)\le 0\}}{\vartheta}\le\lim_{\vartheta\to\infty}\frac{mes\{t\in[0,\vartheta]:\tilde{\varphi}(t)\le 0\}}{\vartheta}\quad(2.7)$$

Consequently, from (2.6) and (2.7) we get the inequality

$$freq(\varphi,(-\infty,0])\ge\lim_{\vartheta\to\infty}\frac{mes\{t\in[0,\vartheta]:\tilde{\varphi}\le 0\}}{\vartheta}$$
$$=freq(\tilde{\varphi},(-\infty,0])\,.\qquad(2.8)$$

Since (see [4])there exist  is an inequality

$$freq_\vartheta(\tilde{\varphi},(-\infty,0])=\inf_{\tau\ge 0}\frac{mes\{t\in[\tau,\tau+\vartheta]:\tilde{\varphi}(t)\le 0\ \}}{\vartheta}$$
$$\le freq(\tilde{\varphi},(-\infty,0]),\qquad(2.9)$$

that inequality $freq_\vartheta(\tilde{\varphi},(-\infty,0])\le freq(\varphi,(-\infty,0])$ aright.

## § 3. Estimates of the characteristics of the attainability set of the controlled system

**Definition 4 (see [9]).** External parallel set of a convex set $M$ in the Euclidean space is called the following set

$$M^c\doteq M+O_c(0)=\bigcup_{x\in M}O_c(x)\,,\quad where\quad c>0.$$

$O_c(x)-$ close the ball of radius $c$ with centre at $x$.

Internal parallel set of a convex set M in a Euclidean space called the set

$$M^c\doteq M-O_c(0)=M\setminus\bigcup_{x\in\partial M}O_c(x)\,,\quad where\quad c>0.$$

(here $\partial M$ is the boundary of the set $M$). Will consider

$$M=\{(t,x)\in\mathbb{R}\times\mathbb{R}^n\ :x\in M(t)\}$$

in suppose that for each $t \in R$ the set $M\ (t)$ is convex and has a non-empty interior.

**Theorem 1. 1)** Suppose that there exists is a number $c$ such that $|\varphi(t) - \psi(t)| \leq c$ for all $t\ \in\ [\tau, \tau\ +\ \vartheta]$. Then the inequality holds

$$freq_{[\tau,\tau+\vartheta]}(\psi, M^{-c}) \leq freq_{[\tau,\tau+\vartheta]}(\varphi, M) \leq freq_{[\tau,\tau+\vartheta]}(\psi, M^{c}). \quad (3.1)$$

**2)** If there exists a $c_1$ such that $|\varphi(t) - \psi(t)| \leq c_1$ for all $t\ \in\ [0, +\infty)$, then

$$freq_{\vartheta}(\psi, M^{-c_1}) \leq freq_{[\tau,\tau+\vartheta]}(\varphi, M) \leq freq_{\vartheta}(\psi, M^{c_1}). \quad (3.2)$$

Proof: Define the sets $M^c = \{(t, x) \in \mathbb{R} \times \mathbb{R}^n\ : x \in M^c(t)\}$ and $M^{-c} = \{(t, x) \in \mathbb{R} \times \mathbb{R}^n\ : x \in M^{-c}(t)\}$. from the inequality

$|\varphi(t) - \psi(t)| \leq c$, which is true for all $t \in [\tau, \tau\ +\ \vartheta]$, followed by the inclusion

$$\{t \in [\tau, \tau + \vartheta) : \psi \in M^{-c}(t)\} \subseteq \{t \in [\tau, \tau + \vartheta) : \varphi \in M(t)\} \subseteq$$
$$\subseteq \{t \in [\tau, \tau + \vartheta) : \psi \in M(t)\},$$

from which we get inequalities

$$mes\{t \in [\tau, \tau + \vartheta) : \psi \in M^{-c}(t)\} \leq mes\{t \in [\tau, \tau + \vartheta) : \varphi \in M(t)\} \leq$$
$$\leq mes\{t \in [\tau, \tau + \vartheta) : \psi \in M^{c}(t)\}. \quad (3.3)$$

Therefore, for all $t \in [\tau, \tau\ +\ \vartheta]$ we have:

$$freq_{[\tau,\tau+\vartheta]}(\psi, M^{-c}) \doteq \frac{mes\{t \in [\tau, \tau + \vartheta] : \psi(t) \in M^{-c}\ \}}{\vartheta}$$
$$\leq \frac{mes\{t \in [\tau, \tau + \vartheta] : \varphi(t) \in M\ \}}{\vartheta} = freq_{[\tau,\tau+\vartheta]}(\varphi, M)(3.4)$$

$$freq_{[\tau,\tau+\vartheta]}(\varphi, M) = \frac{mes\{t \in [\tau, \tau+\vartheta]: \varphi(t) \in M\}}{\vartheta}$$
$$\leq \frac{mes\{t \in [\tau, \tau+\vartheta]: \psi(t) \in M^{c}\}}{\vartheta} \doteq freq_{[\tau,\tau+\vartheta]}(\psi, M^{c}), (3.5)$$

from (3.4) and (3.5) we obtain inequality (3.1).

Let us prove the second statement of the theorem. Similarly, as in inequality (3.3) we get

$$mes\{t \in [\tau, \tau+\vartheta): \psi \in M^{-c_1}(t)\} \leq mes\{t \in [\tau, \tau+\vartheta): \varphi \in M(t)\} \leq$$
$$\leq mes\{t \in [\tau, \tau+\vartheta): \psi \in M^{c_1}(t)\}. \qquad (3.6)$$

From inequality (3.6) for all $t \in [0, +\infty)$ we get

$$\inf_{\tau \geq 0} \frac{mes\{t \in [\tau, \tau+\vartheta]: \psi(t) \in M^{-c}\}}{\vartheta}$$
$$\leq \inf_{\tau \geq 0} \frac{mes\{t \in [\tau, \tau+\vartheta]: \varphi(t) \in M\}}{\vartheta}$$
$$\leq \inf_{\tau \geq 0} \frac{mes\{t \in [\tau, \tau+\vartheta]: \psi(t) \in M^{c_1}\}}{\vartheta}, \qquad (3.7)$$

from the fact that

$$\inf_{\tau \geq 0} freq_{[\tau,\tau+\vartheta]}(\psi, M^{-c_1}) \leq \inf_{\tau \geq 0} freq_{[\tau,\tau+\vartheta]}(\varphi, M)$$
$$\leq \inf_{\tau \geq 0} freq_{[\tau,\tau+\vartheta]}(\psi, M^{c_1}).$$

Since

$$freq_{\vartheta}(\varphi, M) \doteq \inf_{\tau \geq 0} freq_{[\tau,\tau+\vartheta]}(\varphi, M)$$
$$= \inf_{\tau \geq 0} \frac{mes\{t \in [\tau, \tau+\vartheta]: \varphi(t) \in M\}}{\vartheta},$$

then inequality (3.7) is equivalent to (3.2).

**Example 1:** Consider the set $M = \{(t, x) \in [0, +\infty) \times [0,1]\}$ and take as $\varphi$ and $\psi$ are continuous functions and defined by them at $t > 0$

$$\varphi(t) = \sin t + 1 + \frac{1}{8(t+1)}, \qquad \psi(t) = \sin t + 1.$$

Obviously, for all $t > 0$, the inequality holds $|\varphi(t) - \psi(t)| \leq \frac{1}{8}$.

We find estimates of the characteristics $freq_{\vartheta}(\psi, M^{c_1})$ and $freq_{\vartheta}(\psi, M^{c_1})$, where $c_1 = \frac{1}{8}$ and $\vartheta = 2\pi$. Since

$$M^{\frac{1}{8}} = \{(t, x) \in [0, +\infty) \times \left[0,1\tfrac{1}{8}\right]\},$$

$$M^{-\frac{1}{8}} = \left\{(t, x) \in [0, +\infty) \times \left[0, \frac{7}{8}\right]\right\},$$

it is easy to calculate that $freq_{2\pi}\left(\psi, M^{\frac{1}{8}}\right) = \frac{1}{2} + \frac{1}{\pi} \cdot sin^{-1}\frac{1}{8}$ and

$freq_{2\pi}\left(\psi, M^{-\frac{1}{8}}\right) = \frac{1}{2} - \frac{1}{\pi} \cdot sin^{-1}\frac{1}{8}$, therefore, by using Theorem 1, we obtain

$$\frac{1}{2} - \frac{1}{\pi} \cdot sin^{-1}\frac{1}{8} \leq freq_{2\pi}(\psi, M) \leq \frac{1}{2} + \frac{1}{\pi} \cdot sin^{-1}\frac{1}{8}.$$

## REFERENCES


1. Rodina L.I., Tonkov E.L. The statistically weak invariant sets of control systems, Vestn. Udmurt. Univ.Mat. Mekh. Komp'yut. Nauki, 2011, no. 1, pp. 67–86 .

2. Rodina L.I. Estimation of statistical characteristics of attainability sets of controllable systems, Russian Mathematics, 2013, vol. 57, issue 11, pp. 17–27.

3. Filippov A.F. Differentsial'nye uravneniya s razryvnoi pravoi chast'yu (Differential equations with discontinuous right-hand side), Moscow: Nauka, 1985, 223 p.

4.Rodina L.I., Hammadi A. H. The characteristics of attainability set connected with invariancy of control systems on the finite time interval, Vestn. Udmurt. Univ. Mat. Mekh. Komp'yut. Nauki, 2013, no. 1,

pp. 35–48 .

5. Demidovich B.P. Lektsii po matematicheskoi teorii ustoichivosti (Lectures on the mathematical theory of stability), Moscow: Nauka, 1967, 472 p.

6. Levitan B. M. and Zhikov V. V. , Almost periodic functions and differential equations,Moscow University Publishing House, Moscow, 1978.

7.Hartman Ph. Ordinary differential equations, New York–London–Sydney: John Wiley and Sons, 1964.

8.Levitan B. M. , "Some questions of the theory of almost periodic functions I," Uspekhi Matematicheskikh Nauk, 1947 vol. 2, no. 5, pp. 21, (Russian).

9.Leichtweiss K.Vypuklye mnozhestva (Convex sets), Moscow: Nauka, 1985, 335 p.

1. Hammadi Alaa Hussein, , Lecturer in College of Computer Science and Information Technology, Department of Mathematics, university of Al-Qadisiyah.

Email: alaa.hammadi@qu.edu.iq, aliiraqmath@gmail.com.

2. Alia Shani Hassan Kurdi, Lecturer in College of Science, Department of Mathematics, university of Al-Qadisiyah.

Email: alia.shani@qu.edu.iq, aliashany@gmail.com.

3.Alaa Hussain Kalil, Lecturer in College of Science, Department of Mathematics, university of Al-Qadisiyah.

Email: alaa.kahlil@qu.edu.iq, alaahessaininkk@gmail.com