Topological transformation d – algebra

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Abstract:

In this paper we will define a topological d – algebra and find some properties of this structure and the most important characteristics and we came to define a new type of spaces called D-periodic space.

Keywords:

Td – algebra, syndetic set , topological transformation d – algebra , periodic space.

1. Introduction

Y. Imai and K. Iseki [4] and K. Iseki [5] introduced two classes of abstract algebras: namely, BCK-algebras and BCI-algebras. It is known that the class of BCK algebras is a proper subclass of the class of BCI-algebras. In [2], [3] Q. P. Hu and X. Li introduced a wide class of abstract algebras: BCH-algebras. They have shown that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. J. Neggers and H. S. Kim [6] introduced the notion of d-algebras which is another generalization of BCK-algebras, and investigated relations between d-algebras and BCK-algebras. They studied the various topologies in a manner analogous to the study of lattices. However, no attempts have been made to study the topological structures making the star operation of d – algebra continuous. Theories of topological groups, topological rings and topological modules are well known and still investigated by many mathematicians. Even topological universal algebraic structures have been studied by some authors.

In this paper we initiate the study of topological d – algebras. We need some preliminary materials that are necessary for the development of the paper. Section 2 contains some basic knowledges of the d – algebras which are needed for studying this topic. And we will define a topological d – algebra and study some general facts for topological d – algebras.

In section 3,we studied topological transformation monoid (D – space) and the most important characteristics. In section 4 we given define a new type of spaces called D - periodic space and study some properties of D – periodic space.

2. Topological d-algebra

In this section, we examine the definition of topological d-algebra and some issues and examples related to the subject.

2.1Definition: A non-empty set D together with a binary operation $*$ and a zero element 0 is said to be a d – algebra if the following axioms are satisfied for all x, $y \in D$

1)) $x * x = 0$

2) $0 * x = 0$

3)$x * y = 0$ and $y * x = 0$ imply that $x = y$.

*2.2 Definition: An element e of D is called a left identity if e*a=a, a right identity if a*e = a for all $a \in D$ and $a \neq e$. If e is both left and right identity then we called e is an identity element. Also we say that (D,*) is d – algebra with identity element*

2.3 Example:

i) Let D be any non – empty set and P(D) is power set of D then (P(D),-) is d – algebra and ϕ is right identity in (P(D) ,-).

ii) let D={ 0,a,b,c} and define the binary operation $*$ on D by the following table:

$*$	0	a	b	c
0	0	0	0	0
a	0	0	b	c
b	0	b	0	a
c	0	c	a	0

Table (1)

Then the pair (D,) is d – algebra with identity element a .*

*2.4 Definition: Let (D, *,0) be a d-algebra and $\phi \neq I \subseteq D$. I is called a d-sub algebra of Dif $x * y \in I$ whenever $x \in I$ and $y \in I$. I is called d – ideal of D if :*

1) $0 \in I$

2) $y \in I$ and $x * y \in I$ imply that $x \in I$.

2.5Definition : Let (D,) be a d – algebra and T be a topology on D. The triple (D,*,T) is called a topological d – algebra (denoted by Td – algebra) if the binary operation * is continuous.*

2.6 Example:

i) Let D={ 0,a,b,c}and $*$be define by the following table:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	b	0	b
c	c	c	c	0

Table (2)

It is clear that (D,) is d – algebra and T={ ϕ,{b},{c},{0,a},{b,c},{0,a,b},{0,a,c},D} is topology on D such that the triple (D,*,T) is a topological d – algebra.*

*ii) Let R be a set of real number and * is a binary operation which define by $a*b = a.(a-b)^2$ then (R,*) is d – algebra and (R,*,T) is Td – algebra where T is usual topology on R.*

2.7 Definition: Let D be a Td – algebra ,U be a non – empty subset of D and a any element in D we define the sets $U_a= \{x\in D/ xa\in U\}$ and $_aU=\{x\in D/ax\in U\}$. Also if $K\subseteq D$ we put $_KU=\cup_{a\in K}\, {}_aU$ and $U_K = \cup_{a\in K}\, U_a$.

2.8 Example: Let D={0,1,2,3,…} and T be the discrete topology on D and $a*b = a.(a-b)^2$. It is clear the triple (D,,T) is a Td – algebra and if $U = \{x\in D/ x<9\}$, K ={0,1,2} and a=2 then $U_2=\{0,1,2,3\}$, $_2U = \{0,1,2,3,4\}$ and $_KU = \{ 0,1,2,3,4\}$.*

2.9 proposition: Let D be a Td – algebra and A,B, W,K are subsets of D then :

1)If $A\subseteq B$ then $_AW\subseteq {}_BW$.

2) If $W\subseteq K$ then $_AW\subseteq {}_AK$.

Proof:

1) Let $x\in {}_AW$, then there exist $a\in A$ such that $x\in {}_aW$. Since $A\subseteq B$, so $_AW\subseteq {}_BW$.

2) Let $x\in {}_AW$, then there exist $a\in A$ such that $x\in {}_aW$. Thus $ax\in W$, since $W\subseteq K$, then $x\in {}_aK$, so $_AW\subseteq {}_AK$

2.10 Proposition : Let D be Td – algebra, U and F be two non – empty subset of D, then:

i)If U is open set , then $_aU$ and U_a are open sets for all $a\in D$.

ii) If F is closed set , then $_aF$ and F_a are closed sets for all $a\in D$.

Proof:

i) Let U be an open set, $a\in D$ and let $x\in {}_aU$. Then $ax \in U$, since $*$ is continuous, then there exist two open sets A and B of D Such that $(a,x) \in A\times B$, $ax\in AB= *(A,B) \subseteq U$, thus $aB\subseteq U$. Then $x\in B\subseteq {}_aU$, so $_aU$ is open set of D. By same way we can prove that U_a is an open set.

ii) Let F be an closed set and $a\in D$. Now we prove that $_aF$ is closed set. Let $x \in \overline{{}_aF}$,then there exist a net $\{x_\alpha\}_{\alpha\in\Omega}$ in $_aF$ such that $x_\alpha\to x$. Since D is Td – algebra, then $ax_\alpha\to ax$. Thus $ax\in F$, so $x\in {}_aF$. Hence $_aF$ is closed set and by same way we prove that F_a is closed set.

2.11 Corollary :Let D be Td – algebra, U and A be two non – empty subset of D, then :

i)The sets $_AU$ and U_A are open sets if U is open set.

ii)The sets $_AU$ and U_A are closed sets if U is closed set and A is finite.

2.12Proposition: Let D be Td – algebra and D be a T_2 – compact space . If U is compact set of D then $_aU$ and U_a are compact sets for all $a \in D$.

Proof:

Let U be a compact subset of D. Since D is T_2 then U is closed set in D. Thus by proposition (2.10) then $_aU$ and U_a are closed sets in D for all $a \in D$. Then $_aU$ and U_a are compact sets in D for all $a \in D$.

2.13 Proposition: If H is sub algebra of a Td – algebra D, then $\overline{H}$ is sub algebra.

Proof:

*Let $x,y \in \overline{H}$, then there exist two nets $\{x_\alpha\}_{\alpha\in\Omega}$,$\{y_\alpha\}_{\alpha\in\Omega}$in H such that $x_\alpha \to x$ and $y_\alpha \to y$, since * is continuous then $x_\alpha\, y_\alpha \to xy$. Since $\overline{H}$is closed set, then $xy \in \overline{H}$. Thus $\overline{H}$ is sub d – algebra.*

2.14 Proposition : If {0} is open set of a Td – algebra D , then D is discrete.

Proof:

Let $x \in D$. Since $x * x = 0$ (by definition 2.1) and {0} is open ,then by continuity of binary operation of d – algebra, there exist two open sets V and U of x such that $U * V = \{0\}$. Put $W = U \cap V$. Then $W * W = \{0\}$. This implies that $W = \{x\}$, so D is discrete space.

2.15 Proposition:{0} is closed in a Td – algebra D if and only if D is Hausdorff.

Proof:

$\Rightarrow$) Assume that x and y are different elements in D. Then $x*y \neq 0$ or $y* x \neq 0$ (by definition2.1,3). We can assume $x*y \neq 0$.Since D is a Td – algebra. Then there exist two open sets U and V of x and y respectively, such that

$UV \subseteq X / \{0\}$

Thus $U \cap V = \phi$,so D is Hausdorff.

$\Leftarrow$) clear.

2.16Proposition:If I is an open ideal of a Td – algebra D. Then I is also closed.

Proof: Let $x \notin I$. Then by the continuity of d – algebra there exists an open neighborhood V of x such that $V* V\subseteq I$ (since $x*x = 0$). If for some y is contained in $V \cap I$, then $V \subseteq I$ by definition of d – ideals. This is contradiction. Thus $V \subseteq I^c$. So I is d – ideal.

3. D – space

In this section we will examine the D - space and some simple illustrative examples and their causes and consequences.

3.1Definition:A topological transformation d - algebra is a triple (D,X,φ) where D is a topological d – algebra, X is a topological space and $\varphi : D\times X \to X$ is a continuous function such that $\varphi(d_1,\varphi(d_2, x)) = \varphi(d_1*d_2, x)$ for all $d_1,d_2 \in D$, $x \in X$, and if $(D,*)$ is a topological d – algebra with identity e, we say that the triple (D,X,φ) is a topological transformation d – nalgebra with identity such that $\varphi(e, x) = x$ for all $x \in X$.

3.2 Example: Let $(R,*,U)$ be Td – algebra where $a*b = a(a – b)^2$ for all $a,b \in R$ and (R,U) be usual space then (R,R,φ) is a topological transformation d – algebra where $\varphi(a,b) = b$ for all $a,b \in R$.

3.3 Remark:

(i) The function φ is called an action of D on X and the space X together with φ is called a D – space (or more precisely left D – space) and if $(D,*)$ is a topological d – algebra with identity, then the space X together with φ is called a D – space with identity.

(ii) Since φ is understood from the context we shall often use the notation d.x or x.d for $\varphi(d,x)$ and $d_1.(d_2.x) = (d_1d_2).x$ for $\varphi(d_1,\varphi(d_2,x)) = \varphi(d_1d_2,x)$.

(iii) Similarly, for $H \subseteq D$ and $A \subseteq X$ we put $HA = \{d.a/ d\in H, a\in A\}$ for $\varphi(H, A)$.

(iv) For $d\in D$, let $\varphi_d : X\to X$ be the continuous function defined by $\varphi_d(x) = \varphi(d, x) = d.x$ .Thus $\varphi_{d_1}\varphi_{d_2} = \varphi_{d_1d_2}$ and if X is D – space with identity then $\varphi_e = I_x$, the identity function of X.

3.4 Proposition: *Let X be D – space If $A \subseteq X$, $B \subseteq D$ and $d\in D$ then :*

i) $d\bar{A}\subseteq\overline{dA}$.

ii)$\overline{\bar{B}\bar{A}} = \overline{\bar{B}A} = \overline{AB} = \bar{B}\bar{A}$.

iii) If A,B are compact subset of X and D respectively then BA is compact subset of X.

iv) If A,B are a compact subset of X and D respectively then BA is compact subset of X and if W is a neighborhood of BA then there exist two neighborhoods U and V for A and B respectively such that $VU \subseteq W$.

Proof:

i) Since φ_d is continuous function and $d\bar{A}= \varphi(d, \bar{A})=\varphi_d(\bar{A})\subseteq\overline{\varphi_d(A)}=\overline{dA}$.

ii)Since φ is continuous function then $\bar{B}\bar{A} = \varphi(\bar{B}\times\bar{A}) = \varphi(\overline{B\times A}) \subseteq \overline{\varphi(B\times A)} = \overline{BA} \Rightarrow BA \subseteq \bar{B}A\subseteq\bar{B}\bar{A} \subseteq\overline{BA}$ then $\overline{\bar{B}A}\subseteq\overline{BA}$ and $BA \subseteq B\bar{A}\subseteq\bar{B}\bar{A} \subseteq\overline{BA}$ then $\overline{B\bar{A}}\subseteq\overline{BA}$.

iii) Since φ is continuous function and $\varphi(B\times A)= BA$. Thus BA is compact set.

iv) Clear

3.5 Definition: *Let (D,X,φ) be a topological transformation d – algebra , and $x\in X$. The set $D_x(\varphi)=\{d\in D/ \varphi(d,x)=x\}$ is called the stabilizer of φ at x , and we define the set $D(\varphi) =\cap_{x\in X} D_x(\varphi)$ as the stabilizer of φ.*

3.6 Example: *Let $(Z,*,T)$ be topological d – algebra with discrete topology and a binary operation $*$ where $a*b = a(a - b)^2$ for all $a,b\in Z$. Then (Z,R,φ) is a topological transformation d – algebra where (R,U) is usual topology on real number and $\varphi: Z\times R\to R$ such that $\varphi(z,r)=r$ for all $z\in Z$ and $r\in R$,then the stabilizer of φ at x is $Z_x(\varphi)=\{z\in Z/\varphi(z,x)=x\} =Z$ thus $Z(\varphi) = \cap_{r\in R} Z_r(\varphi)=Z$*

3.7Definition: Let X be a D-space. We called that φ is minimal function if $\varphi(D,x)$ is dense in X for all $x\in X$.

3.8 Example: *Let $D = \{0,a,b,c\}$ and $*$ is define by the table:*

$*$	0	a	b	c
0	0	0	0	0
a	b	0	b	c
b	c	a	0	c
c	a	a	b	0

Table (3)

Then $(D,*)$ is d – algebra and $(D,*,\tau)$ is Td – algebra where τ is discrete topology on D and let (D, τ') be indiscrete topological space. The action φ of $(D,*,\tau)$ on (D, τ') such that $\varphi(a,b) = b$ then for every $d \in D$, $\overline{\varphi(D,d)} = \overline{\{d\}} = D$. Thus φ is minimal function.

3.9 Definition: Let X be a D-space. We called that φ is faithful if for any distinct $d_1, d_2 \in D$ there exist $x \in X$ such that $\varphi(d_1,x) \neq \varphi(d_2,x)$.

3.10 Example: Let D = {0,1,2,3} and $*$ be define by the following table:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	1	1
2	2	2	0	2
3	3	3	3	0

Table (4)

Then $(D,*)$ is d – algebra and $(D,*,\tau)$ is Td – algebra where τ is discrete topology on D. $(N^{\#}, \tau')$ is a topological space where $\tau' = \{U_n \ / \ U_n = \{0,1,2,3,n, n+1, \dots\} \cup \{\phi\}$. If $\varphi: D \times N^{\#} \to D$ defined by $\varphi(a,b) = a$, then φ action of $(D,*,\tau)$ on (D, τ'). Then for any $a,b \in D$ then there $n \in N^{\#}$ such that $a = \varphi(a,n) \neq \varphi(b,n) = b$, then φ is faithful.

3.11 Definition: Let D be a Td - algebra , a subset T of D is called right syndetic in D if there exists a compact subset H of D such that $_HT=HT=D$ and T is called left syndetic if there exists a compact subset H of D such that $T_H=TH=D$.

3.12 Example: Let X = {1,2,3} and(P(X), –) be Td – algebra with discrete topology. If $T=\{\phi, X, \{1\}, \{2\}, \{3\}\}$ and $H=\{X, \{1,2\},\{1,3\},\{2,3\}\}$ then $_HT=HT=P(X)$ and T is right syndetic in P(X). If $T = \{\phi, X, \{1\}, \{1,2\}, \{1,3\}\}$, $K = \{\phi, \{1\}, \{2\}, \{3\}, \{2,3\}\}$ then $T_K=TK=P(X)$ and T is left syndetic in P(X).

Notation :- we note that if a Td – algebra D is finite then every subset of D is right (left) syndetic in D.

3.13 Proposition: Let D be a Td – algebra and T be a subset of D then T is right syndetic in D if and only if there are compact subsets H_1, H_2, of D such that $_{H_1}T = D$ and $H_2T=D$.

Proof:

$\Rightarrow$) let T be right syndetic in D then there exists a compact subset H in D such that $_HT = HT = D$ by this completes the proof .

$\Leftarrow$) let H_1,H_2 be compact subsets of D such that $_{H_1}T=D$ and $H_2T= D$.

Put $H= H_1 \cup H_2$ then :

$_HT = {}_{H_1\cup H_2}T = \bigcup_{h\in H_1\cup H_2} {}_hT = (\bigcup_{h\in H_1} {}_hT) \cup (\bigcup_{h\in H_2} {}_hT) = {}_{H_1}T \cup {}_{H_2}T$

$= D \cup {}_{H_2}T = D$ and $HT = (H_1 \cup H_2)T = H_1 T \cup H_2 T = H_1 T \cup D = D$

Thus T is right syndetic in D

3.14 Proposition: Let D be a Td – algebra and A be a right (left) syndetic subset in D then $\bar{A}$ is right (left) syndetic subset in D

Proof:-

Since A is a right syndetic subset in D, then there exists a compact subset H of D such that $_HA$ =HA=D by proposition (2.7)and $A\subseteq \bar{A}$ then $_H\bar{A}$ = $H\bar{A}$ =D thus $\bar{A}$ is right (left) syndetic.

3.15 Definition: Let (D,X,φ)be a topological transformation d- algebra. The point $x\in X$ is called periodic relative toφ if D_x (φ) is right syndetic in D and φ is called periodic if $D(\varphi)$ is right syndetic.

3,16 Proposition: Let (D,X,φ) be topological transformation d- algebra if φ is periodic then any element of X is periodic relative to the function φ.

Proof :

let φ be periodic. Then by Definition (3,13), we get D (φ) is right syndetic . Then there exist a compact subset H of D such that HD (φ)= $_H$D (φ)=D.

But D (φ)= $\cap_{x\in X}D_x(\varphi)$and $\cap_{x\in X}D_x(\varphi) \subseteq D_x(\varphi)$. thus $_H(\cap_{n\in X} D_x(\varphi))$ = $H(\cap_{x\in X} D_X(\varphi))$ =D, then H $D_x(\varphi)$= $_H D_x(\varphi)$=D. hence any element in X is periodic relative to φ.

3.17 Proposition: Let (D, X ,φ) be a topological transformation d- algebra and let $x\in X$ be a periodic relative to φ. Then $\varphi(D,x)$ is compact and $\varphi(D,x) = \varphi(D,y)$ for any $y\in\varphi(D,x)$.

Proof :

let $x \in X$ such that x is a periodic relative to φ. Then there exists a compact subset $H \subseteq D$ such that H $D_x(\varphi)$ = D=$_H D_x(\varphi)$. First we prove that $\varphi(D, x)$= $\varphi(H, x)$. Let $y \in\varphi(H, x) \Rightarrow\exists h \in H$ such that y= φ (h,x) . since $H\subseteq D$, thus y= φ(h,x) $\in\varphi(D,x) \Rightarrow\varphi(H, x) \subseteq\varphi(D,x)$. Let $z\in\varphi(D, x)$. Then there exists $d_1\in D$ such that z = $\varphi(d_1,x)$. Since H D_x (φ) = D , then there exist $h_1\in H$ such that d_1 = h_1d_2 and $d_2\in D_x(\varphi)$, then z= $\varphi(d_1,x)$= $\varphi(h_1d_2,x)$= $\varphi(h_1$, $\varphi(d_2,x)$ =$\varphi(h_1,x)$.$\Rightarrow z\in\varphi(H,x)$, hence φ(D ,x) $\subseteq$ φ (H,x) thus φ (D, x) = φ(H, x) . Since φ(H, x) is compact , then φ(D, x) is compact. Second let $y\in\varphi$(D ,x) . we prove that φ(D ,x) =φ(D ,y) . let $z\in\varphi(D,x)$ then there exist $d_1,d_2\in D$. Such that y= φ(d, x) and z =$\varphi(d_1, x)$. Since D= $_HD_x$ (φ) then there exist $h_3\in H$ such that $hd\in D_x(\varphi)$[by define of $_H D_x(\varphi)$] then φ(hd, x) = x. Then z = $\varphi(d_1,x)$, $\Rightarrow$ z = $\varphi(d_1, \varphi(hd ,x))$ = $\varphi(d_1, (h , \varphi(d ,x))$ = $\varphi(d_1h , \varphi(d ,x))$ = $\varphi(d_2, \varphi(d_2 ,x))$=$\varphi(d_2 ,y)\in\varphi(D ,y)$ then $\varphi(D ,x)\subseteq\varphi(D ,y)$.Let w $\in\varphi(D ,x) \Rightarrow\exists d_3\in D$ such that W= $(d_3 ,y)\Rightarrow$ w= $(d_3 , \varphi(d ,x)$ = $\varphi(d_3d ,x)$ = $\varphi(d_4 ,x) \in\varphi(D ,x) \Rightarrow\varphi(D ,y)\subseteq\varphi(D ,x)$ then $\varphi(D ,x)$= $\varphi(D ,y)\, \forall y \in\varphi(D ,x)$.

3.18 Proposition: Let (D, X, φ) be a topological transformation d- algebra , let X be a T_2 – space and the point $x \in X$ is periodic point relative to φ, then φ is minimal function if and only if X = $\varphi(D,x)$.

Proof:

$\Rightarrow$)Suppose that φ is minimal. Since X is T_2 – space and by proposition (3.15) we get that $\varphi(D, x)$ is closed set in X. Thus $\varphi(D,x)$ =X (since φ is minimal).

$\Rightarrow$) let X = $\varphi(D, x)$, by proposition (3.15) we get that $\varphi(D,y)$= $\varphi(D,x)$ =X for all $y \in X$. So φ is minimal.

3.19 Definition:Let X be a D – space . we say that isφ called topological transitive if for any two non – empty open subsets U, $V \subseteq X$, there exist $d\in D$ such that $\varphi(d,U) \cap V \neq\varphi$

3.20 Example: Let D = {0, a, b, c} such that $*$ is define by the following table :

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	b	0	b
c	c	c	c	0

Table (5)

then $(D,*,\tau)$ is Td – algebra where $\tau = \{\phi, D, \{b\}, \{c\}, \{b,c\}, \{a,0\}, \{0,a,b\}, \{0,a,c\}\}$ and (D,τ) is topological space where $\tau = \{\phi, D, \{a\}, \{a,b\}\}$ then D is D –space where $\varphi(a,b)=b$ $\forall a,b \in D$ .Thus φ is topologically transitive..

3.21 Definition :Let X be D – space we say that X is a topological point transitive if there exist a point x such that $\overline{Dx}=X$ and x is called point transitive

3.22 Definition: Let X be a D- space we say that X is densely point transitive if there exist dense set $Y \subseteq X$ of point transitive.

3.23 Proposition :Every densely point transitive D- space is a topologically transitive.

Proof:

Let U and V be two open non-empty subset of X such that and Y be set of point transitive such that $\overline{Y} = X$ then $Y \cap V \neq \phi$, so there exist $y \in Y \cap V$. By transitively of y , then there exists $d \in D$ such that $U \cap dV \neq \phi$,hence (D,X,φ) is topological transitive d – algebra.

4. the orbits and minimal sets in d – algebra

In this section we study the invariant sets, minimal sets, the orbits of element and the relationship between these concepts, with some specific issues and illustrative examples.

Notation: Let (D,X,φ) be a topological transformation D – algebra, $C \subseteq D$ and $Y \subseteq X$ then $C(Y) = \varphi(C,Y) = \{\varphi(c, y), c \in C, y \in Y\}$.

4.1 Definition: Let (D,X, φ) be a topological transformation of X .we say that A is invariant under D if and only if D (A)=A

4.2 Example: Let (R,R, φ) be a topological transformation d – algebra (where $(R,*, U)$ is a topological D- algebra such that $*$ is define by $a*b= a(a-b)^2$ for all $a,b \in R$ and U is usual topology on R and φ is define by $\varphi(r_1,r_2)= r_2$ for all $r_1, r_2 \in R$. Then any subset of R is invariant set.

4.3 Proposition: Let (D,X,φ) be a topological transformation then :

1) If Ω is a family of all invariant subsets under D then $\bigcup_{A\in\Omega} A$ is invariant set under D and if φ is one to one the $\bigcap_{A\in\Omega} A$ is an invariant set under D.

2) If A and B are an invariant subsets of X under D then A^C and A – B are invariant subsets of X under D.

3) If φ is closed function and A is invariant subset of X under D then $\bar{A}$ is invariant subset under D .

4) If φ is open function , D is d- algebra with identity e and A is invariant of X under D - algebra then A° is invariant subset under D.

Proof:

1) Let $A \in \Omega$ Then A is an invariant of X under D , then $\varphi(D\times A) = A$ for all $A \in \Omega$. $D(\bigcup_{A\in\Omega} A)= \varphi (D \times\bigcup_{A\in\Omega} A)= \varphi(\bigcup_{A\in\Omega} D \times A) = \bigcup_{A\in\Omega} A$ Then $\bigcup_{A\in\Omega} A$ is invariant set.

And :

$D(\bigcap_{A\in\Omega} A) = \varphi (D\times\bigcap_{A\in\Omega} A) = \varphi (\bigcap_{A \in\Omega} (D\times A))$ since φ is one to one then $\varphi (\bigcap_{A \in\Omega} (D\times A))= \bigcap_{A\in\Omega} \varphi (D\times A) = \bigcap_{A \in\Omega} A$ then $\bigcap_{A\in\Omega} A)$ is invariant set.

2) i)Let A be invariant the $\varphi(D\times A) =A$ then($\varphi(D\times A))^c =A^c$. Since φ is one to one then $(\varphi(D\times A))^c = \varphi((D\times A)^c) = \varphi(D\times A^c)$. Hence $\varphi(D\times A^c) =A^c$ thus A^c is invariant.

ii) let A and B be two invariant subsets of X under D then $\varphi(D\times(A \cap B^c)) =A \cap B^c$, since $A - B = A\cap B^c$ then from (1) and (2,i) we get that A-B is invariant.

3)Since φ is continuous and closed function. Then $\bar{A} =\overline{\varphi(D\times A)} = \varphi(\overline{D\times A}) = \varphi(D\times\bar{A})$ thus $\bar{A}$ is invariant.

4) Since D is d- algebra with identify e, then $A^{\circ}\subseteq\varphi (D \times A^{\circ})$. Now we prove that $\varphi(D \times A^{\circ})\subseteq A^{\circ}$. Since φ is open function and $\varphi(D \times A) =A$ then $\varphi(D \times A^{\circ})\subseteq (\varphi(D\times A))^{\circ}=A^{\circ}$,so $\varphi(D\times A^{\circ})=A^{\circ}$ then A° is invariant.

<u>4.4 Definition:</u> Let (D ,X, φ) be a topological transformation d- algebra and $x \in X$. The orbit of x by D is the set $\{d x / d\in D\}$ and we denoted by Dx or D (x). The orbit closure of x by D is $\overline{Dx}$.

<u>4.5 Proposition</u>: Let $(D\ \ X, \varphi)$ be a topological transformation d- algebra such that D is d- algebra with identity e then :

1) If $x \in X$ then Dx is minimal invariant subset of X contain x.

2) If $x \in X$ and φ is closed then the orbit closure of x by D is minimal invariant closed subset of X by D and contain x.

<u>Proof:</u>

1) Let $x\in X$, then Dx is orbit of x by D. Since (D, X, φ) is topological transformation d- algebra and $(D,*)$ is a d-algebra with identity then $D(Dx) = \varphi (D\times Dx) =\varphi (D\times\varphi(D,x)) =\varphi(*(D\times D),x)= \varphi(D,x) = Dx$.

Then Dx is an invariant under D and contains x. Let A be a subset of X such that A is invariant under D, $x\in A$ and $A \subseteq Dx$. Since D (A) = A then $D(A)\subseteq D(x)$. But $D (x) \subseteq D(A)$ (since $x \in A$) then A=D x . Thus Dx is minimal.

2) Let $x \in X$, then $\overline{Dx}$ is the orbit closure of x by D , from (1) we get that Dx is an invariant under D . then by proposition (4.3). $\overline{Dx}$ is an invariant under D . Now we prove that $\overline{Dx}$ is minimal sebst of X.

Let A be closed subset of X and it is invariant under D such that $x \in A$. Then $Dx \subseteq D (A)$, then $\overline{Dx} \subseteq \overline{D(A)} = \bar{A}$ then $\overline{Dx} = \bar{A}$ (since A is closed) then $\overline{Dx}$ is minimal set.

<u>4.6 Definition</u>: Let (D, X, φ) be topological transformation d- algebra , A be a subset of X and S be a subset of D. Then the set A is called minimal set by the set S if A is orbit closure by the set S and if $B \subseteq A$ such that B is any orbit closure by S then B=A. The set A is called closure minimal orbit if S = D

<u>4.7 Remark</u> :Let (D, X, φ) be a topological transformation. Then X is a closure minimal orbit if and only if $\overline{Dx} =X$ for all $x\in X$

Proof

$\Rightarrow$)Let X be closure orbit minimal then $\overline{Dx} = X$ for some $x \in X$. let $y \in X$ since $\overline{Dy} \subseteq X$. Then by Definition (4.6) we have $\overline{Dy} = X$.

4.8 proposition: Let (D, X, φ) be a topological transformation d- algebra with identity e such that φ is closed function .Then the following are equivalent :-

i) A is closure minimal orbit by D

ii) A is a closure non – empty and invariant under D and it is smaller set satisfy this property.

iii) A is close non – empty set and A = DU for all closed non empty set U of A.

Proof (i) $\rightarrow$ (ii) let A be closure minimal orbit by D . Then $A = \overline{Dx}$ for some $x \in X$, thus A is non-empty and closed . Then A is invariant under D (by proposition (4.5)). Now , let $B \neq \phi$ and it is invariant under D and closed such that $B \subseteq A$. Since $A = \overline{Dx}$ and $\overline{Dx}$ is smaller invariant under D. Hence A =B .

ii)$\rightarrow$(iii) let $U \neq \phi$ and U is closed set of A then $D\,U \subseteq DA = A \Rightarrow DU \subseteq A$ and since $\overline{DU} = \overline{\varphi(D \times U)} = \varphi(\overline{D} \times \overline{U}) = \varphi\ (D \times U) = Du \Rightarrow Du$ is closed and since $D(DU) = \varphi(D \times \varphi(D \times U)) = \varphi(*(D,D)\ , U) = \varphi(D, U) = DU$. Then Du is closed and invariant by D. Since A is smaller and satisfy this property, the set $A \subseteq DU$ the A = DU.

(iii) $\rightarrow$(i) let A be closed non –empty set and A = DU for all closed set U of A then there exist $x \in A \Rightarrow \{x\}$ is closed set of A (by X is T_2 – space), then $A = D\,\{x\} = Dx \Rightarrow \overline{Dx} = \bar{A} = A$. Then A is closure orbit. Let $y \in A$ such that $\overline{Dy} \subseteq A$. Since $\{y\}$ is closed set of A then Dy = A $\Rightarrow \overline{Dy} = A$ thusA is closure minimal orbit by D.

4.9 Remark: Let X be a compact D – space than every closure orbit by D is compact.

4.10 Definition: Let (D, X , φ) be a topological transformation d- algebra withleft identity . we say that X is D – periodic space if any point x in X is periodic point .

4.11 Example: Let $(D , *, \tau)$ a topological with left identity 0 where D={0, 1, 2, 3} ,τ is discrete topology on D and $*$ is binary operation which define by the following table:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Table (6)

And (R, U) is usual topological space then (D, R, φ) is a topological transformation d – algebra with left identity o where $\varphi(d,r)=r$ for all $(d,r) \in D \times R$ then $D_x = D$ for all $x \in R$ and $0 \in D$ such that ${}_0D = 0D = D$ then (D, R , φ) is D- periodic space.

4.12 Proposition: Let X be D- periodic space where D is d- algebra with left identity then the collection Ω of all orbits by D is partition for X.

Proof:

Let $\Omega = \{ Dx / Dx$ is orbit of x by D$\}$ now we want to prove that $Dx \cap Dy = \phi$ for all $x,y \in X$ such that $Dx \neq Dy$. Suppose that $Dx \cap Dy \neq \phi \Rightarrow$ there exist $z \in X$ such that $z \in Dx$ and $z \in Dy$ then Dx = Dz and Dy= Dz $\Rightarrow$ Dx= Dy= Dz by proposition(3.17). Since $e \in D$ then $x \in Dx$ for all $x \in X$, $Dx \subseteq X$, thus $\cup\ Dx = X$, then Ω is partial for X.

4.13 Proposition: *Let X be D- periodic space with left identity e then relation* $P= \{(x,y) \in X\times X / y \in D x\}$ *is equivalent relation on X*

Proof:

i) Since $e \in D$ and $\varphi(e,x) = ex = x$ then $(x,x) \in D$.

ii) Let $(x,y) \in P$, then $y \in Dx \Rightarrow Dy = Dx$ and by proposition (2.17) we get $x \in Dy \Rightarrow (y,x) \in P$.

iii) Let (x,y) and $(y,z) \in P$ then $y \in Dx$ and $z \in Dy$. Thus $Dx = Dy = Dz$ (by proposition (2.17)) $\Rightarrow z \in Dx \Rightarrow (x, z) \in P$ then P is equivalent relation on X.

4.14 Proposition: *Let X be D – periodic* T_2 *– space then Dx is closed and minimal set for all* $x \in X$.

Proof:

Let $x \in X$, then x is periodic point (since X is D- periodic space). Then $\varphi(D,x) = Dx$ is compact by proposition (2.17). Since X is T_2 then Dx is closed then $Dx = \overline{Dx}$ for all $x \in X$. Thus Dx is closure orbit , let $y \in Dx$. Since $\overline{Dy} = Dy = Dx \Rightarrow \overline{Dy} \subseteq Dx$. then Dx is minimal set.

4.15 Proposition: *Let X be a D-periodic –* T_2 *– space where D is d- algebra with left identity e and* $A \subseteq X$, *then A is invariant set under D if and only i f* $\overline{Dy} \subseteq A$ *for all* $y \subseteq A$.

Proof:

$\Rightarrow$) Let $y \in A$, then $Dy \subseteq D(A) = A$. Since y is periodic point, thus Dy closed by proposition (4.14) , then $Dy = \overline{Dy} \subseteq A$.

$\Leftarrow$) Since $e \in D$, then $A \subseteq D(A)$ and since $\overline{Dy} \subseteq A$ for all $y \in A$, then $\cup_{y \in A}\overline{Dy} \subseteq A$ thus $D(A) = \cup_{y \in A}\overline{Dy} = A$. Hence $D(A) = A$, thus A is invariant set by D.

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الخلاصة:
في هذا البحث قدمنا تعريف طوبولوجيا د - الجبر و سلطنا الضوء على بعض خصائص هذا الموضوع و ثم أعطينا نوع جديد من الفضاءات يسمى بفضاء – D الدوري.