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**College of Computer Science and Mathematics**

**Department of Mathematics**

**A Study of Some Results on Differential Subordination in Univalent and Multivalent Function Theory**

**A Thesis**

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**By**

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بسم الله الرحمن الرحيم

**( قَالَ رَبِّ اشْرَحْ لِي صَدْرِي \* وَيَسِّرْ لِي أَمْرِي \* وَاحْلُلْ عُقْدَةً مِنْ لِسَانِي \* يَفْقَهُوا قَوْلِي )**

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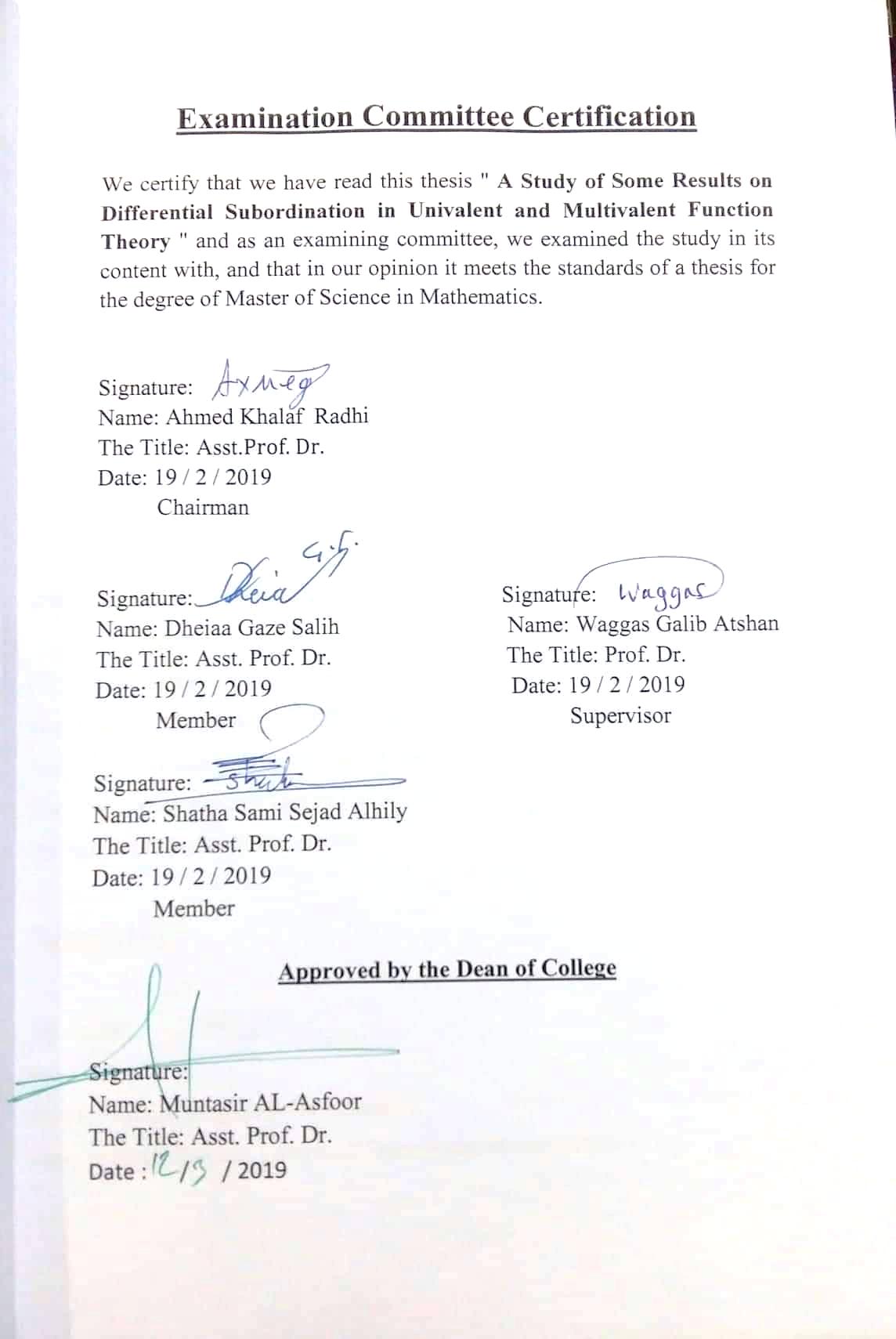
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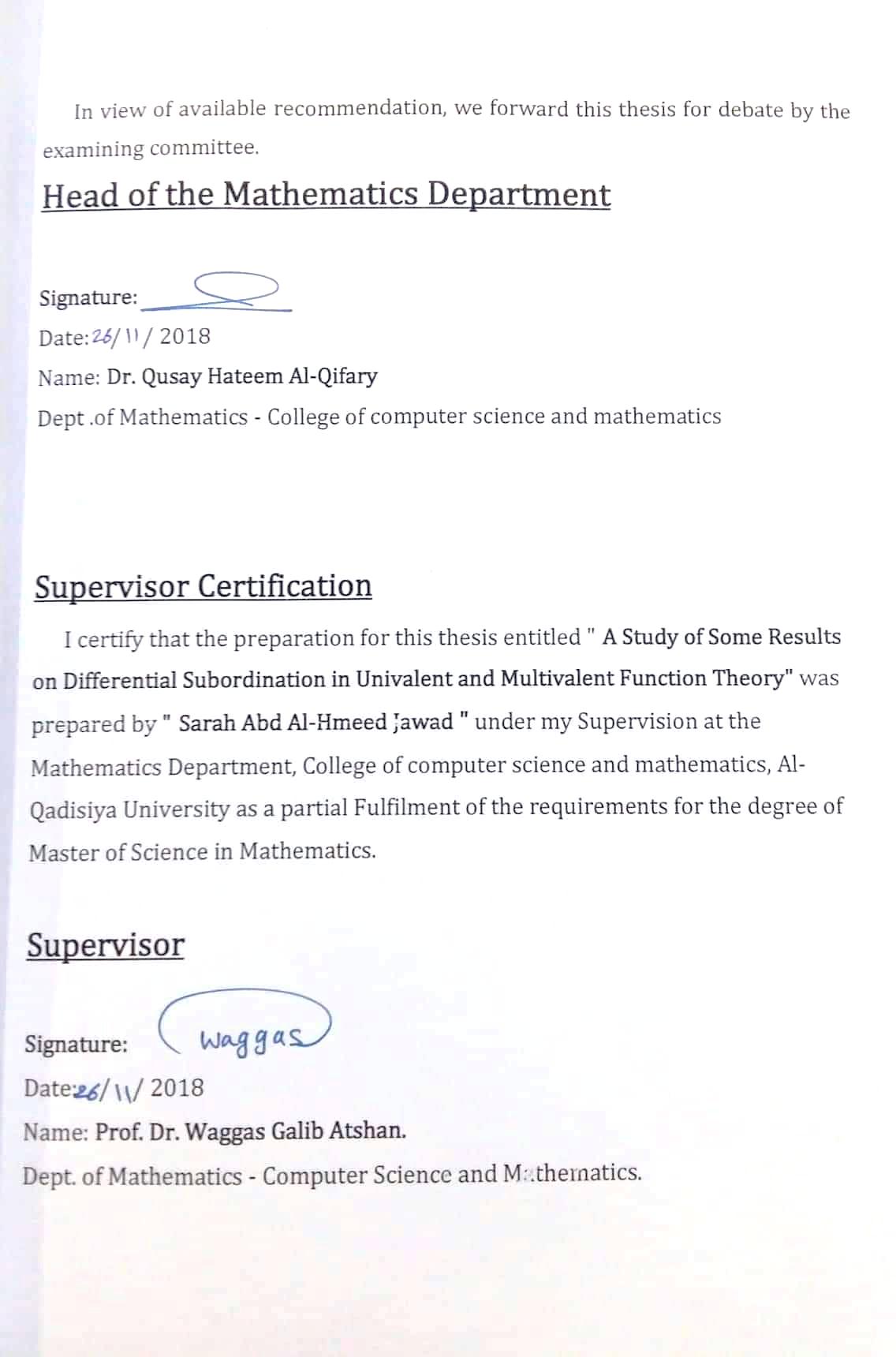
First and foremost, thanks to Allah for everything

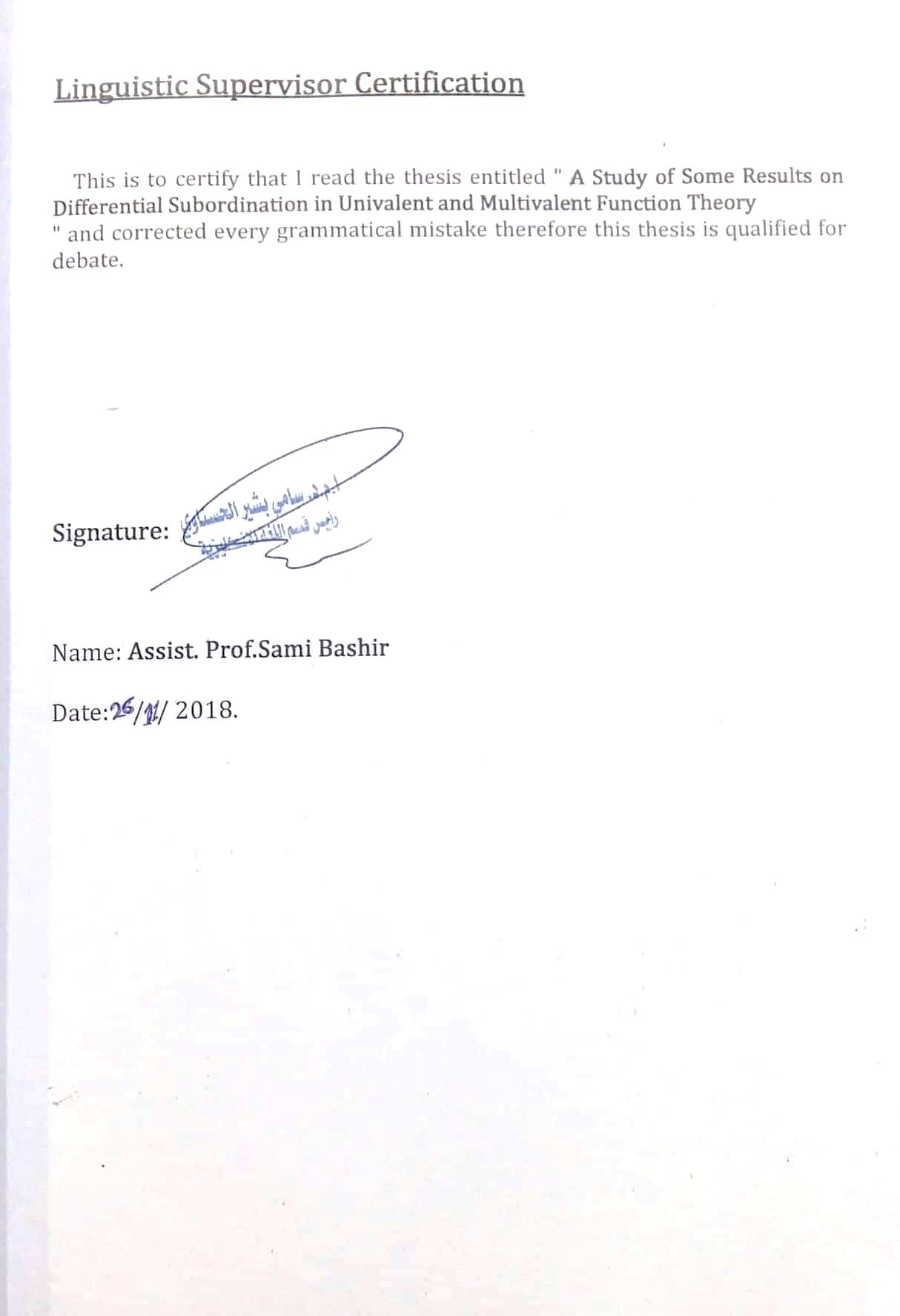
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**List of publications**

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References

**List of Symbols**

|  |  |
| --- | --- |
| **Symbol** | **Description** |
| **U** | Unit disk. |
|  | Punctured unit disk. |
|  | Closed unit disk. |
|  | Complex plane. |
|  | N |
| **N** | The set of natural numbers. |
|  | The class of all functions analytic in U of the form |
|  | The class of analytic functions by |
|  | The class of normalized mreomorphic functions of the form : |
|  | The class of all p-valent mreomorphic functions of the form: |
|  | The subclass of consisting of functions which are also univalent in |
|  | Class of all starlike functions of order in |
|  | Class of all convex functions of order in . |
|  | Class of all starlike function of order 0. |
|  | Class of all convex functions of order 0. |
|  | Class of close-toconvex functions. |
|  | Hadamard product in and |
|  | subordinate to . |
|  | The class of analytic functions in |
|  | The boundary to . |
|  | The class of the functions of the form: |
|  | Integral operator of meromorphic univalent in |
|  | The class of analytic functions of the form: |
|  | Class of admissible functions of the form: |
|  | Class of admissible functions of the form: |
|  | The class of meromorphic p-valent functions the inequality satisfies: |
|  | Integral operator for **.** |

**Abstract**

The purpose of this thesis is studying of some results on differential subordination in univalent and multivalent function theory. It is the study of differential sandwich results for analytic functions. We obtain some results of differential subordination and superordination of a class of univalent functions.We have also studied and introduced some differential sandwich results of p-valent functions. We obtain results on differential subordination and superordination of a class of p-valent functions in the open unit disk. We have dealt with the third-order differential subordination results of meromorphic univalent functions involving integral operator. Also, we give some applications of differential subordination of a class of meromorphic p-valent functions. We obtain some results, such as; coefficient bounds, growth and distortion bounds, closure theorem, radius of convexity, extreme points, convex linear combination and partial sums.

**Introduction**

Geometric Function Theory is a fairly old branch of mathematics, specifically complex analysis catch the attention of several mathematicians and researchers for its geometrical aspects and abundant avenues for research work. The study of univalent and multivalent functions is one of the major branches of Geometric Function Theory and aids in developing the complex analysis. One of the essential dilemmas in the study of univalent functions is whether there exists an univalent mapping from a simply connected domain onto a given simply connected domain. However, in view of Riemann mapping Theorem [16] above problem reduces to a problem of mapping an unit disk to onto a given simply connected domain such as starlike, convex, and close-to-convex etc.

Let A refer to the class of all analytic functions on the open unit disk normalized by the conditions and , also let S stand for the subclass of A consisting of univalent functions.

In 1916, Bieberbach [16] studied the second coefficient a2 of a function . He demonstrated that with equality if and only if f is a rotation of the Koebe function and he also stated is generally valid. This statement is known as the Bieberbach conjecture. This conjecture was challenge for several mathematicians for many years , it motivated them to develop a variety of new methods in complex analysis. In 1923, Lwner [35] proved the Bieberbach conjecture for n =3, many investigation have been made regarding the Bieberbach conjecture for specific values of n. Eventually, the conjecture was established by Branges [12] in the summer of 1985. Subordination between analytic functions initiated by Littlewood [32,33] and Lindelf [31], where Rogosinski [52,53] introduced the term and established the basic results involving subordination. Lately, investigations to a variety of interesting properties of the generalized hypergeometric function were made by Srivastava and Owa [62] via applying the concept of subordination.

Research scholars and mathematicians, internationally recognized, such as,Ruscheweyh, Srivastava, Miller, Mocano, Duren, Silveman, Owa, Jahangiri et al., have opened new avenues in the field of complex analysis, specifically in Geometric Function Theory.

The current work unveils differential sandwich, differential subordination and superordination, third- order differential subordination, class of meromorphic p-valent functions and geometric properties. We shall now give the chapters wise breakup.

The first chapter introduces a list of all relevant definitions of analytic, univalent, multivalent (p-valent)functions and some basic results which are needed during chapters for research.

Chapter two is devoted for to studying some results on differential sandwich of univalent and multivalent functions. This chapter is divided into two sections. The first section deals with the study of differential sandwich results for analytic functions. We obtain some results of differential subordination and superordination for univalent functions. The second section is concerned with some differential sandwich results of p-valent functions. We obtain results on differential subordination and superordination of class of p-valent functions on the open unit disk.

Chapter three deals with differential subordination results of meromorphic univalent and multivalent functions with its applications. In section one, we have discussed third- order differential subordination results of meromorphic univalent functions involving integral operator.

The second section deals with the applications of differential subordination of a class of meromorphic p-valent functions. We obtain some geometric properties, such as, coefficient inequality, growth and distortion bounds, closure theorem, radius of convexity, extreme points , convex linear combination and partial sums.

**Chapter One**

**Definition and Fundamental Results**

**Introduction**

In this chapter, we have mentioned all the required definitions, some examples, and basic results of analytic functions, univalent and multivalent(p-valent) functions and also subordination and which are needed in subsequent chapters for research. The detailed proofs and further discussions may be found in standard texts such as Duren [14], Goodman [18] and other references.

**1.1 Univalent and Multivalent Functions [12]**

Let be the complex plane and be the open unit disk in . A function is analytic at a point if it is differentiable in some neighborhood of and it is analytic in a domain if it is analytic at all points in Domain . An analytic function is said to be univalent in a domain if it provides a one-to-one mapping onto its image: . Geometrically, if some horizontal line intersects the graph of function more than once, then the function is not one-to-one. But if no horizontal line intersects the graph of the function more than once, then the function is one-to-one. As examples, the function is univalent in U while is not univalent in U. Also is univalent in U for each positive integer n.

An analytic function is locally univalent at a point if it is univalent in some neighborhood of .

**Remark(1.1.1)[14]:** For an analytic function , the condition is equivalent to the local univalence at .(from Rouch’s theorem)

**Example(1.1.1)[18]:** The function is a locally univalent at each point, since for all . But it is not univalent function since for all .

**Definition (1.1.1)[14]:** A function is said to be conformal at a point if it preserves the angle between oriented curves passing through in magnitude as well as in sense. Geometrically, images of any two oriented curves taken with their corresponding orientations make the same angle of intersection as the curves at both in magnitude and direction.

A function is said to be conformal in the domain if it is conformal at each point of the domain. Any analytic univalent function is a conformal mapping because of its angle-preserving property.

The well-known Riemann mapping theorem states that every simply connected domain (which is not the whole complex plane ), can be mapped conformally onto the unit disk U.

**Theorem (1.1.1)( Riemann Mapping Theorem)[14,p.11]:** Let be a simply connected domain which is a proper subset of the complex plane. Let be a given point in . Then there is a unique univalent analytic function which maps onto the unit disk U satisfying .

In view of this theorem, the study of analytic univalent functions on a simply connected domain can be restricted to the open unit disk U. Let H=H(U) be the class of analytic functions defined on U Let H[a, n] be the subclass of H(U) consisting of functions of the form:

,

|  |
| --- |
| with H=H[1,1]. Also, let denote the class of all functions analytic in the open unit disk U, and normalized by and.  A function has the Taylor series expansion of the form:    **Definition(1.1.2)[20]:** Let be a function analytic in the unit disk. If the equation have never more than p-solutions in U, then is said to be p-valent in U.  For a fixed let be the class of all analytic function of the form:  that are p-valent (multivalent) in the open unit disk , with =.  The subclass of consisting of univalent functions is denoted by S. The function given by  is called the Koebe function, which maps U onto the complex plane except for a slit along the half-line (- and is univalent . It plays a very important role in the study of the class S. The Koebe function and its rotations ( for and the only extremal functions for various problem in the class S.  **Remark(1.1.2)[16]:** The Koebe function can be written as . In Figure (1.1), the sequence of mapping used in building the koebe function is shown. The function maps U univalently onto the right half-plane . Then the function takes this half-plane onto the entire minus the part of the negative real axis from to infinity so that the Koebe function is establishe.    Figure (1.1):The mapping of Koebe function.  In 1916, Bieberbach[14] conjectured that for He proved only for the case when n=2.  **Theorem (1.1.2) (Bieberbach′s Conjecture)[14]:** If , then with equality if and only if is the rotation of the Koebe function.  For the cases n=3, and n=4 the conjecture was proved by Lwner [32] and Garabedian and Schiffer [15], respectively. Later, Pederson and Schiffer [43] proved the conjecture for n=5, and for n=6, it was proved by Pederson[44]and Ozawa[42 ], independently. In 1985, Louis de Branges [10], proved Bieberbach′s Conjecture for all the coefficients n.  **Theorem (1.1.3)(de Branges Theorem or Bieberbach′s Theorem)[10]:**  If , then  ,  with equality if and only if is the Koebe function or one of its rotations. Bieberbach′s Theorem has many important properties in univalent function.  These include the well known covering theorem :If ,then the image of U under contains a disk of radius 1/4 .  **Theorem (1.1.4) ( Koebe One-Quarter Theorem)[14,p.31]:** The range of every function contains the disk { }.  The Distortion theorem, being another consequence of the Bieberbach theorem gives sharp upper and lower bounds for .  **Theorem (1.1.5)(Distortion Theorem) [14.p.32]:**  For each    The distortion theorem can be used to obtain sharp upper and lower bounds for which is known the Growth theorem.  **Theorem (1.1.6)(Growth Theorem )[14,p.33]:** For each    Another consequence of the Bieberbach theorem is the Rotation theorem.  **Theorem (1.1.7)(Rotation Theorem )[14,p.99]:** For each  where . The bound is sharp.  1.2 **Subclasses of Univalent Function**s  The long gap between the Bieberbach's conjecture in 1916 and its proof by de Branges in 1985 motivated researchers to consider classes defined by geometric conditions. Notable among them are the classes of convex functions, starlike functions and close-to-convex functions.  A set in the complex plane is called convex if for every pair of points and lying in the interior of , the line segment joining and also lies in the interior of [14],i.e.  for  If a function maps U onto a convex domain, then is a convex function. The class of all convex functions in is denoted by [1]. An analytic description of the class is given by  .  **Example(1.2.1)[18]:** The Mbius function is a convex function because it maps onto a half-plane.  **Remark(1.2.1)[16]:** Figure 1.2 describes the image for a convex function.    Figure (1.2) :The image of convexity  Let be an interior point of . A set in the complex plane is called starlike with respect to if the line segment joining to every other point lies in the interior of [14], i.e.  for  If a function maps U onto a starlike domain, then is a starlike function. The class of starlike functions with respect to origin is denoted [16]. Analytically,  **Example(1.2.2)[18]:** The Koebe function is a starlike function and the domain is starlike with respect to each  **Remark(1.2.2)[16]:** The image of a starlike function is show in Figure 1.3    Figure 1.3: The image of starlikeness.  In 1936, Robertson [47] introduced the concepts of convex functions of order and starlike functions of order for . A function is said to be convex of order if  and starlike of order    These classes are respectively denoted by ( and ().  Note that = and (0)=.  **Definition(1.2.1)[14]**: Radius of convexity of a function is the largest for which it is convex in .  **Definition(1.2.2)[14]:** Radius of starlikeness of a function is the largest for which it is starlike in .  An important relationship between convex and starlike functions was first observed by Alexander [8] in 1915 and know later as Alexander’s Theorem.  **Theorem (1.2.1) (Alexander’s Theorem)[14.p.43]:** Let .Then if and only if  From this, it is easily proven that () if and only if Another subclass of that has an important role in the study of univalent functions is the class of close-to-convex functions introduced by Kaplan [24] in 1952. A function is close-to-convex in U if there is a convex function and a real number such that    The class of all such functions is denoted by . The subclass of ,namely convex , starlike and close-to-convex functions are related as follows:    The well known Noshiro-Warschawski theorem states that a function with positive derivative in U is univalent.  **Theorem(1.2.2)[40,64]**: For some real ,if a function is analytic in a convex domain and    then is univalent in .  Kaplan[24] applied Noshiro- Warschawski theorem to prove that every close-to-convex function is univalent.  The class of mreomorphic functions is yet another subclass of univalent functions.Let denote the class of normalized mreomorphic functions of the form:    that are analytic in the punctured unit disk except for a simple pole at .  **Definition(1.2.3)[20]:** Let be a function analytic in the punctured unit disk If the equation has never more than p-solution in , then is said to be p-valent in . The class of all p-valent meromorphic functions is denoted by, and expressed by the form:  **Definition (1.2.4)[49]:** The convolution (or Hadamard product) of functions and denoted byis defined as following for the functions in and respectively:  (i)If then  (ii)If then    **Definition (1.2.5)[34]:** Let be a subset of X .A point is called an extreme point of if it has no representation of the form as a proper convex combination of two distinct points y and z in .  **Theorem (1.2.3)[15]:** Assume that is analytic and not constant in a domain of the complex z-plane. For any point for which ,this mapping is conformal, that is, it preserves the angle between two differentiable arcs.  **Theorem(1.2.4)[14]: (Maximum Modulus Theorem):** Suppose that a function is continuous on a boundary of ( any disk or region). Then, the maximum value of on occurs on and never in the interior (i.e; only on if is not constant) .  **1.3 Differential Subordination:**  A function is said to be a Schwarz function, if for all , then where "capital " is defined as follows:  Let and be any two sequences and ≥ for all 𝑛. If there exists a constant number 𝜂 > such that ≤ 𝜂 (for all 𝑛), then, we write [14]  A function is said to be subordinate to in U, written , if there exists a schwars function , analytic in U with ()= and such that (z)= ((z)). If the function is univalent in U , then if and .[36]  Ma and Minda [33]have given a unified treatment of various subclasses consisting of starlike and convex functions by replasing the superordinate function by a more general analytic function .For this purpose,they considered an analytic function with positive real part on U with ()=1, and maps the unit disk U onto a region starlike with respect to 1, symmetric with respect to the real axis. The class of Ma – Minda starlike functions denoted by consists of functions satisfying  and similarly the class of Ma-Minda convex functions denoted by consists of functions satisfying the subordination  respectively.  The basic definitions and theorems in the theory of subordination and certain applications subordination were developed by Miller and Mocanu [36].  **Definition (1.3.1) [36]:** Let and be univalent in Ιf is analytic in and satisfies the second–order differential subordination:  then is called a solution of the differential subordination An univalent function is called a dominant of the solutions of the differential subordination , more over simply dominant, if for all satisfying A univalent dominant that satisfies for all dominants of is said to be the best dominant of  **Definition (1.3.2) [37]:** Let and the function be analytic in. If the functions and are univalent in and if satisfies the second–order differential superordination:  then is called a solution of the differential superordination An analytic function is called a subordinant of the solutions of the differential superordination or more simply a subordinant, if for all satisfying An univalent subordinant that satisfies for all subordinants of is said to be the best subordinant.  **Definition (1.3.3**) **[36]:** Let the set of all functions that are analytic and injective on ,where  and  ,  and are such that for . Futher, let the subclass of for which be denoted by , and  **Lemma (1.3.1) [36] :** Let q be univalent in the unit disk U and let θ and be analytic in a domain containing q(U) with when Set  Suppose that  (i) is starlike univalent in ,   1. (ii)Re for .   If is analytic in withand  (1.3)  then and is the best dominant of (1.3).  **Lemma (1.3.2) [37]:** Let q be convex univalent in function in U and let with    If is analytic in , and  (1.4)  then and is the best dominant of (1.4).  **Lemma (1.3.3) [37]:** Let q be convex univalent in U and let , further assume that Re . If Q and is univalent in U, then  (1.5)  which implies that and q is the best subordinant of (1.5).  **Lemma (1.3.4) [12]:** Let q be convex univalent in the unit disk U and let be analytic in domain containing q . Suppose that   1. (i)Re 2. (ii)Q.   If  is univalent in U and  , (1.6)  then and is the best subordination of (1.6).  **Lemma (1.3.5)[50]:** The function is univalent in unit disc U if and only if or  **Theorem (1.3.6)[6]:** Let and let and satisfy the following conditions  ,  where . if Ω a set in  ( ,  then  **Some Results on Differential Sandwich of Univalent and Multivalent Functions**  **Chapter Two** |

**Introduction:**

In [35] Miller and Mocanu extended the study of differential inequalities of real-valued functions to complex –valued functions defined in the unit disk.Following Miller and Mocanu [36,37], Bulboac [11]and others [7,9,38,39,54] studied different classes of analytic functions, be means of differential subordination and superordination.

In this chapter, we concentrate in particular on the study of applications of differential sandwich of univalent and multivalent functions.

This chapter consists of two sections. Section one deals with the study of differential sandwich results for analytic functions. Here, we obtain some results,such as,let q be convex univalent function in U with q()=1, ,z and suppose that q satisfies:

Re . If satisfies the subordination

then

and q is the best dominant.

Section two, is devoted to a study of some differential sandwich results of p-valent functions. We obtain results on differential subordination and superordination of a class of p-valent functions in the open disk, such as, let q be univalent function in U, with q()=1, and suppose that

Re (1+ ) max {0; – Re (} z U, where 0 , 1 ,

(z) q(z) + ( ) z q(z),

(z) = (1-) + ,

then

q(z) ,

and q is the best dominant.

**2.1 On Differential Sandwich Results For Analytic Functions**

Let H=H(U) be the class of analytic functions in the open unit disk For n a positive integer and Let H[a, n] be the subclass of H consisting of functions of the form:

|  |
| --- |
| (a ). (2.1) |

Also, let A be the subclass of H consisting of functions of the form:

|  |
| --- |
| (2.2) |

Let : If p and are univalent functions in U and if p satisfies the second-order differential superordination.

|  |
| --- |
| (2.3) |

then p is called a solution of the differential superordination of(2.3).( If is subordinate to , then is superordinate to ). An analytic function q is called a subordinant of (2.3), if for all the functions p satisfying (2.3).

An univalent subordinant that satisfies for all the subordinants q of (2.3) is called the best subordinant. Miller and Mocanu [36] have obtained conditions on the functions and for which the following implication holds :

|  |
| --- |
| (2.4) |

For  ,Al-shaqsi [5] defined the following integral operator:

(2.5)

We also note that the operator   defined by ([2.5](https://www.hindawi.com/journals/ijmms/2014/260198/#EEq1.1)) can be expressed by the series expansion as follows :

|  |
| --- |
| . (2.6) |

Moreover, from ([2.6](https://www.hindawi.com/journals/ijmms/2014/260198/#EEq1.2)), it follows that

(2.7)

Ali et al.[2] obtained sufficient conditions for certain normalized analytic functions to satisfy

where and are given univalent functions in U with . Also, Tuneski [62] obtained sufficient conditions for starlikeness of in terms of the quantity . Recently, Shanmugam et al.[54,55], Goyal et al.[19] also obtained sandwich results for certain classes of analytic functions.

The main object here to find sufficient conditions for certain normalized analytic functions to satisfy:

,

and

,

where q1 and q2 are given univalent functions in U with q1()= q2( 1.

**Theorem (2.1.1):** Let q be convex univalent function in U with

Re . (2.8)

If the subordination

(2..9)

then

(2.10)

and is the best dominant of (2.9).

**Proof** : Define the function p by

(2.11)

Differentiating (2.11) with respect to z logarithmically, we get

(2.12)

Now, in view of (2.7), we obtain the following subordination

therefore ,

The subordination (2.9) from the hypothesis becomes

An application of Lemma(1.3.2) with and

Putting in Theorem( 2.1.1) ,we obtain the following

**Corollary (2.1.1)**: Let

Re .

If satisfies the subordination

then

and is the best dominant.

**Theorem (2.1.2)**: Let q be convex univalent in U with and assume that q satisfies

Re , (2.13)

where and .

Suppose that -is starlike univalent in U, if satisfies:

, (2.14)

where

, (2.15)

then

, (2.16)

and q(z)is the best dominant of (2.14).

**Proof**: Define the function p by

, (2.17)

by setting :

.

We see thatis analytic in and that . Also, we get

and

It is clear that is starlike univalent in U,

By a straightforword computation,we obtain

, (2.18)

where is given by (2.15).

From (2.14) and (2.18), we have

. (2.19)

Therefore by Lemma (1.3.1), we get . By using (2.17), we obtain the result .

Putting (-1 ) in Theorem (2.1.2), we obtain the following Corollary:

**Corollary (2.1.2):** Let -1 and

where and if satisfies

and is given by (2.15),

and is the best dominant.

**Theorem (2.1.3):** Let q be convex univalent in U with

,

and ,

be univalent in U. If

, (2.20)

then

(2.21)

and q is the best subordinant of (2.20).

**Proof**: Define the function p by

. (2.22)

Differentiating (2.22) with respect to z logarithmically, we get

(2.23)

After some computations and using (2.7), from (2.23), we obtain

=

and now, by using Lemma(1.3.3), we obtain the desired result .

Putting in Theorem (2.1.3), we obtain the following Corollary :

**Corollary (2.1.3):** Let and Re

and

,

be univalent in U. If

,

then

and

is the best subordinant.

**Theorem (2.1.4):** Let q be convex univalent in U with and assume that q satisfies

(2.24)

where z.

Suppose that is starlike univalent in U, let

,

and where is given by (2.15). If

then

and q is the best subordinant of

**Proof**: Define the function p by

, (2.27)

by setting

and

we see that is analytic in . Also we get

.

It is clear that is starlike univalent in U,

By a straightforword computation, we obtain

(2.28)

where is given by (2.15).

From (2.25) and (2.28), we have

.

Therefore, by Lemma (1.3.4), we get . By using (2.27), we obtain the result.

Concluding the results of differential subordination and superordination we arrive at the following ''sandwich result''.

**Theorem (2.1.5)**: Let q1 be convex univalent in U with q1(1, Re{} and let q2 be univalent in U, q2(1,z and , let

,

and

be univalent in U. If

,

and are respectively, the best subordinant and the best dominant.

**Theorem (2.1.6):** Let q1 be convex univalent in U with q1(, and satisfies (2.24), let q2 be univalent in U q2(1,z satisfies (2.13), let

and is univalent in U, where is given by (2.15). If

then

and are respectively. The best subordinant and the best dominant.

**2.2 Some Differential Sandwich Results of p-valent Functions**

Let H(U) denote the class of analytic functions in the open unit disc U = {z : z ∈ : |z| < 1} and let H[a, p] denote the subclass of the functions ∈ H(U) of the form:

(a) .

Also, let A(p) be the subclass of the functions ∈ H(U) of the form:

(p∈N), (2.29)

and set A ≡ A(1). For functions (z) ∈ A(p), given by (2.29), and given by

(p∈N), (2.30)

the Hadamard product (or convolution) of (z) and is defined by

(z∈U;p∈N). (2.31)

Supposing that p and k are two analytic functions in U, let : If h and are univalent functions in U and if p satisfies the second-order superordination

k(z)≺, (2.32)

then h is called to be a solution of the differential superordination (1.4). A function q ∈ H(U) is called a subordinant of (2.32), if q(z) ≺ p(z) for all the functions p satisfying (2.32). A univalent subordinant that satisfies q(z) ≺ (z) for all of the subordinants q of (2.32), is said to be the best subordinant.

Recently, Miller and Mocanu [37] obtained sufficient conditions on the functions k, q and ϕ for which the following implication holds:

k(z) ≺ ⇒ q(z) ≺ p(z).

Using these results, Bulboaca [11] considered certain classes of first-order differential superordinations, as well as superordination-preserving integral operators [25]. Ali et al. [2], using the results from [11], obtained sufficient conditions for certain normalized analytic functions to satisfy

≺ ≺ ,

where and are given univalent normalized functions in U. Very recently, Shanmugam et al. [55-57] obtained the sandwich results for certain classes of analytic functions. Further subordination results can be found in [41-63].

We now define the linear operator is defined by:

(2.33)

For

It is easily verified from (2.33) that

z=.

(2.34)

Differentiating (2.34) j-times with respect to z we get

. (2.35)

Note that linear operator unifies many other operators considered earlier.In particular:

1. (see Cho et al.[13]).
2. (see Srivastava and Aouf [59]).
3. (see Hohlov [21]).
4. (a,c) (see Saitoh [52]).
5. (see Liu and Noor [31]).

The main object this idea is to find sufficient conditions for certain normalized analytic function to satisfy:

,

and

where and are given p-valent functions with

**Theorem (2.2.1)**: Let q be univalent function in U, with q()=1, ,and suppose that

Re (1+ ) max {; – Re (} z U, (2.36)

where , 1 ,

(z) q(z) + ( ) z q(z) . (2.37)

where

(z)=(1-)

, (2.38)

then

q(z) , (2.39)

and q is the best dominant of (2.37) .

**proof** : If we consider the analytic function

(2.40)

by differentiating (2.40) logarithmically with respect to z , we deduce that

(2.41)

From (2.41), by using the identity (2.34) , we have

(1-) +=

hence the subordination (2.37) is equivalent to

(2.42)

An application of Lemma 1.3.2,with α=1 and leads to (2.39) .

Taking in Theorem (2.2.1), where -1the condition (2.36) becomes

Re

It is easy to check that the function is convex in U, and Since a convex domain symmetric with respect to the real axis ,hence

inf{ Re ( (2.44)

Then the inequality (2.43) is equivalent to

hence we obtain the following result .

**Corollary (2.2.1).** Let

max

If ,satisfies the subordinution

then

and q(z)= is the best dominant of (2.45) .

Taking A=1 and B= -1 in Corollary (2.2.1),we obtain the following Corollary.

**Corollary (2.2.2):** Let is given by (2.38) , satisfies the subordination :

, (2.46)

then

and q(z)= is the best dominant of (2.46) .

**Theorem (2.2.2**): Let q(z) be univalent in U ,with q()=1 and q(z) for all zU,let ,n,m with n+m. Let and suppose that and satisfy the next conditions :

and

(2,48)

If

– p ) (2.49)

then

and q is the best dominant of (2.49) .

**Proof :** According to (2.47),we consider the analytic function

S(z)=, (zU). (2.50)

By logarithmically differentiating of (2.50) yields

– p ) .

In order to prove our result we will use Lemma 1.3.1. In this Lemma consider

()= and ()= ,

then is analytic in and () is analytic in .Also ,if we let

(z)=z q(z) (q(z)) = ,

and

h(z)=(q(z))+(z) = + , (zU) .

from the assumption (2.48),we see that (z) starlike function in U,and also have :

Re( = Re (1+) (zU) .

Now,by Lemma 1.3.1, we derive the subordination (2.49), implies S(z)q(z), and the function q(z) is the best dominant of (2.49) .

Taking n=, m=1 , =1 and q(z)= in Theorem (2.2.2),it is easy to cheek that assumption (2.48) holds wherever -1AB1,hence we obtain the next result.

**Corollary (2.2.3)**: Let -1AB1 and ,let and suppose that

If

(2.51)

then

and is the best dominant of (2.51) . (The power is the principal one) .

Putting n=0 , m=p =1 , α= , v= (a,band in Theorem (2.2.2) ,then combining this to gather with Lemma 1.3.5 we obtain the next result due to Obradovis et al .[44,Theorem (2.2.1)] .

**Corollary (2.2.4)[41]**:Let a,b such that let and suppose that for all If

(2.52)

then

and is the best dominant (2.52) . (The power is the principal one) .

Again setting n=0, m=p =α =1 , and in Theorem (2.2.2) . and using Lemma 1.3.2 we obtain the next result .

**Corollary (2.2.5)**:Let -1 A B 1 with B0 , and suppose that Let such that for all z U and let . If

then

, (2.53)

and is the best dominant of (2.53) . (The power is the principal one) .

: Let q be univalent in U , with q(0)=1,let , and let n,m with n+m . Let  and suppose that and q satisfy the next two conditions :

(2.54)

and

Re(1+(z. (2.55)

If

, (2.56)

and

(2.57)

then

and q(z) is the best dominant of (2.57).

**Proof :** If we consider the function r(z) by

Differentiating (2.58) logarithmically with respect to z, we obtain

and

In order to prove our result we will use Lemma 1.3.1. In this Lemma consider

then is analytic in and is analytic in ,also if we let

and

from (2.55),we see that (z) is starlike function in U. we also have

by applying Lemma 1.3.1, the proof is completed .

Taking q(z)= in Theorem (2.2.3), where -1 and with reference to (2.44),the condition (2.55) becomes

Thus, for the particular case we obtain the following result.

**Corollary (2.2.6) :** Let -1 and with

Let and impose that

and let If

(2.59)

then

and is the best dominant of (2.59).

Taking and q(z)=in Theorem (2.2.3), we obtain the following result.

**Corollary( 2.2.7):** Let such that If

(2.60)

then

and is the best dominant of (2.60).

Here, too,we are trying to create sufficient conditions:

,

and

where are given p – valent functions in U with

**Theorem (2.2.4)** : Let q be convex in U with q(0)=1, with Re( . Let and suppose that If the function

,

is univalent in the unit disc U , and

where

then

and q is the best subordinant of (2.61) .

**Proof** : We know the function p(z) by

From the supposition of Theorem 2.2.4,the function p(z) is analytic in U. Differentiating (2.62) logarithmically with respect to z , we obtain

After several accounts , and using the identity (2.34) from (2.63) , we get

and now , by using Lemma 1.3.3 , we get the wanted result .

Taking in Theorem (2.2.4) , we obtain the following Corollary .

**Corollary (2.2.8)**. Let q be convex in U with q(0)=1, let , with Re(. Let and suppose that If the function

(2.64)

is univalent in U, and

where

then

and is the best subordinant of (2.64).

Using like cases to those of the proof of Theorem (2.2. 3), and then by applying Lemma 1.3.4, we obtain the following result .

**Theorem (2.2.5)**: Let q be convex in U with q()=1, let and let with n+m and and suppose that *f* satisfies the next conditions :

and

If the function X(z) given by (2.39) is univalent in U , and

then

and q(z) is the best subordinant of (2.65).Combining Theorem (.2.2.2 ) with Theorem (2.2.3) and Theorem (2.2.4) with Theorem (2.2.5),we obtain , respectively ,the following two sandwich results.

**Theorem (2.2.6)**: Let be two convex function in U with with Re(.Let and suppose that If the function

is univalent in the unit disc U , and

(2.66)

then

and are respectively, the best subordinant and the best dominant of (2.66).

**Theorem (2.2.7**) : Let be two convex function in U with let and let with n+m0 and and suppose that satisfies the next conditions :

and

If the function X(z) given by (3.24) is univalent in U, and

then

and are respectively, the best subordinant and the best dominant of (2.67).

**Chapter Three**

**On Differential Subordination Results For Meromorphic Univalent and Multivalent Functions with is Applications**

**Introduction**

This chapter is completely devoted to a study of differential subordination results for meromorphic univalent and multivalent functions with its applications, having Laurent series expansion containing positive and negative terms .Actually a differential subordination in the complex plane is the generalization of a differential inequality on the real line. The concept of differential subordination plays a very important role in functions of real variable. This concept also enables us to study the range of original function. In the theory of complex –valued function there are several differential applications in which a characterization of a function is determined from a differential condition. Miller and Mocanu [36] have contributed number of papers on differential subordination. The study of differential subordination stems out from textbooks by Duren[14], Goodman [18] and Pommerenke [45].

This chapter is divided into two sections. The first section is concerned with third-order differential subordination results of meromorphic univalent functions involving integral operator, such as, let if the functions satisfy the following condition:

and

.

Then

The second section deals with the applications of differential subordination of a class of meromorphic p-valent functions. We obtain some geometric properties, such as, coefficient inequality, growth and distortion bounds, closure theorem, radius of convexity, extreme points, convex linear combination and partial sums .

**3.1 On Third-order Differential Subordination Results of Meromorphic Univalent Functions** **Involving Integral Operator**

Let H(U) be in the class of functions which are analytic in the open unit disk.

For and let

,

and also let =[1,1]. Let W denote the class of the functions of the form

which are analytic and meromorphic univalent in the punctured unit disk:

A.Y. Lashin[27] introduced and investigated the integral operator which is defined as follows:

For given by (3.1) , we have

from (3.3) we note that

In recent years, several authors obtained many interesting results for the theory of second-order differential subordination and superordination for example ([3,4,7,26,56]), thus the aim of this idea to investigate extension to the third -order differential subordination. The first authors investigated the third order, Ponnusamy[46] published in 1992. In 2011, Antonio and Miller [6] extended the theory of the second-order differential subordination in the open unit disk introduced by Miller and Mocanu [36] to the third-order case. The determined properties of the function that satisfy the following third –order differential subordination.

Recently, the only a few of authors discussed the third -order differential subordination and superordination for analytic functions in U for example ([1,9,22,60,61]). We will now recall the basic concept in the theory of the third -order differential subordination due to Antonio and Miller [5], which are required in our next investigations.

**Definition (3.1.1).[6,p.440]:** Let and the function h(z) be univalent in U. If the function (z) is analytic in U and satisfies the following third -order differential subordination

then (z) is called the solution of the differential subordination. A univalent function q(z) is called the a dominant of the solutions of the differential subordination or more simply a dominant if for all p(z) satisfying (3.6). A dominant that satisfies for all dominates q(z) of (3.6) is said to be the best dominant.

**Definition (3.1.2)[6,p.441]:** Let denote the set of the function q that are analytic and univalent on the set , where , is such that min for . Further let the subclass of for which q(0)=a be denoted by (a) and (1)=Q1.

**Definition (3.1.3)[6,p.449].** Let Ω be a set in , The class of admissible functions consists of these functions that satisfy the following admissibility condition.

whenever ,

and where

We first define the following class of allowable functions, which are wanted in proving the differential subordination theorem involving the integral operator Defined by (3.3).

**Definition (3.1.4):** Let Ω be a set in . The class of admissible functions consists of those functions that satisfy the following admissibility condition,

wherever

and

where

**Theorem (3.1.1):** Let , if the functions satisfy the following condition :

and

. (3.8)

Then

**Proof:** Define the analytic function in U by

The differentiating (3.9) with respect to z using (3.3), we have

further computations show that

and

Define the transformation from  4 to by

,

and

.

Let

.

The proof will make use of Theorem (1.3.6), using equations (3.9) to (3.12) and from (3.13) we obtain

Hence (3.9) becomes

Note that

and

.

Thus, the admissibility condition for in Definition 3.1.4 is equivalent to the admissibility condition for as given in Definition 3.1.3, with n = 2. Therefore, by using (3.7) and Theorem 1.3.6.

We have,

The following result is an extension of Theorem 3.1.1 to the case, where the behavior of q(z) on is not known.

**Theorem (3.1.2)**: Let the set and let the function q be univalent U with q() = 1. Let for some where. If the function and satisfy the following conditions:

and

,

then

**Proof:** By using Theorem 3.1.1, yields

This outcome follows easily from the subordination property .

If is a simply connected domin, then for some conformal mapping h(z) of U on Ω. In this case, the class is written as the following two results are immediate consequence of Theorem 3.1.1 and Theorem 3.1.2.

**Corollary (3.1.1):** Let [h,q] . If the function and satisfy the condition (3.7) and:

then

**Corollary (3.1.2):** Let and let the function q be univalent in U with q() = 1. Suppose also that for some . If the function and satisfy the condition:

And

then

The following results yields the best dominant of the differential subordination (3.16).

**Theorem (3.1.3)**: Let the function h be univalent in unit open disk U. Also, let the function and be given by (3.13). Suppose that the differential equation

has a solution q(z) with q()= 1, which satisfies the condition (3.7). If the function satisfies the condition (3.16) and the function:

is analytic in U

and q(z) is the best dominant.

**Proof:** In view of Theorem 3.1.1, we deduce that q is a dominant of (3.16). Since q satisfies (3.18), it is also a solution of (3.16) and therefore q will be dominated by all dominates. Hence q is the best dominant.

In view of Definition 3.1.4 in the special case the class of a dimissble function . denoted by is expressed as follows :

**Definition (3.1.5):** Let Ω be a set in and . The class of a dimissble functions consists of these functions such that

wherever

**Corollary (3.1.3):** Let ], if the function is satisfies

and

then

.

Specially if then we denote by Corollary 3.1.3. can now be written as below.

**Corollary (3.1.4):** Let if the function is satisfies the following condition

and

then

.

**Corollary (3.1.5):** Let . If the function is satisfies

,

and

then

**Proof:** Let where .To apply Corollary 3.1.3, we must show that that is, that (3.19) is satisfied. In fact it follows easily , because of

,

whenever . The required result now follows the Corollary 3.1.3.

**Definition (3.1.6) :** Let Ω be a set in , > , and . The class of admissible function [Ω , q] consists of those function that satisfy the following admissibility condition :

whenever

and

where { z U ,

**Theorem (3.1.4) :** Let If the function and satisfy the following conditions :

And

then

**Proof .** Let

Then, from (3.3) and (3.23) , we get

After a simple computation, we have

and

Define the transformation from by :

and

Let

The proof will make use of Theorem 1.3.6 . using equations (3.22) to (3.25)and from (3.26) , we obtain

Hence (3.21) becomes

Note that

and

Thus, the admissibility condition for in Definition 3.1.6 is equivalent to the admissibility condition for as given in Definition 3.1.3, with n=2. Therefore,by using (3.20) and Theorem 1.2.7,we have

If is asimply connected domain , then Ω=h(U) for some conformal mapping h(z) of U on to Ω q`. In this case , the class is written as The following results is an immediate consequence of Theorem 3.1.4.

**Theorem (3.1.5)** : Let . If the function and satisfy condition :

and

(3.27)

then

In the special case, when q(z)=1+Mz,M > 0 ,and in view of Definition 3.1.6 the class of admissible functions is denoted by is described below.

**Definition (3.1.7) :** Let Ω be a set in The class of admissible function consists of those functions such that

whenever ,for all and

**Corollary (3.1.6) :** Let If the function satisfies

and

then

In the special case, when Ω=q(U)={:the class is simply denoted and Corollary 3.1.6 has the following form

**Corollary (3.1.7) :** Let If the function satisfies the following condition :

and

then

**Corollary (3.1.8):** Let . If the function satisfies

and

then

**Proof .** By taking

and Ω=h(U),where

Using Corollary 3.1.6, we need to show that Since

whenever for all The proof is complete .

**3.2 Applications of Differential Subordination of a Class of Meromorphic p-valent Functions**

Letdenote the class of functions of the form

(3.28)

which are analytic and p-valent in the punctured unit disk Jun-Kim Srivastara [23] defined an integral operator for as follows

(3.29)

If is of the form (3.28), then

In particular, when p=1 we have:

.

**Definition(3.2.1)**: A function is said to be in the classof functions of the form (3.28) which satisfies the condition

where non zero complex number

We can re-write the condition (3.32) as

. (3.32)

In the following theorem, we give a sufficient and necessary condition to be the function in the class

**Theorem (3.2.1)**: Let be given by (3.28). Then if and only if

The results is sharp for the function given by (3.34)

**Proof**: Assuming that the inequality (3.34) holds true and Then, we have

=

= , by hypothesis.

Hence, by the Maximum Modulus Theorem, we have

Conversely, suppose that Then from (3.32), we have

Since Re(z) for all z(zwe have

Re(.

We choose the value of z on the real axis and z , we get

. Sharpness of the result follows by setting

**Corollary (3.2.1):** Let . Then

,.

In the following theorems, we obtain thegrowth and the distortion theorems for the functions in the class

**Theorem (3.2.2):** If the function defined by (3.28) is in the class then for , we have:

where equality holds true for the function

**Proof**: Since Then from (3.33)

we conclude that

Thus for

or

and

or

On using (3.38) and (3.39) inequality (3.35) follows.

**Theorem (3.2.3):** If (then

The result is sharp for the function is given by (3.36).

**Proof:** The proof is similar to that of Theorem (3.2.2).

In the next theorem, we obtain extreme points for the class

**Theorem (3.2.4):** Let , (3.40)

for . Then if and only if it can be expressed in the form:

**Proof :** Let

Then

Using Theorem (3.2.1), we easily get .

Conversely, let . From Theorem (3.2.1), we have

for

Setting

for

Then

This completes the proof.

In the following theorem, we obtain theradius of convexity for the class

**Theorem (3.2.5):** Let the function (z) defined by (3.28) in the class Then is meromorphically p-valent convex of order in the disk where

, (3.42)

The result is sharp for the function given by (3.34).

**Proof:** A function meromorphic p-valent convex of order if

We must show that

for (3.43)

We have .

Thus, (3.43) will be satisfied if

(3.44)

Since we have

Hence, (3.44) will be true if

,

or equivalently

which follows the result.

**Theorem (3.2.6):** The class is closed under convex linear combinations.

**Proof:** Let and be the arbitrary elements of . Then for every t and . we show that . Thus we have

.

Hence

.

This completes the proof.

**Theorem (3.2.7):** Let the functions defined by

,

be in the class for every k = (1,2,3,…). Then the function defined by

also belong to the class where = **proof:** Since , it follows the Theorem (3.2.1) that

,

for every k = 1,2,3,…,. Hence

Then .

**Theorem (3.2.8):** Let be given by(3.28). We define the partial sums as follows :

Also suppose that

Then, we have

and

Each of the bounds in and is the best possible for .

**Proof**: We can see from (3.46) that

Therefore, we have:

By setting

and applying we find that

which readily yields the assertion (3.47) if , we take

Then which shows that the bound in (3,47) is the best possible for

Similarly, if we put

and make use of (3.52), we have

which leads us to the assertion (3.48). The bound(3.49) is sharp for each with the function given by (3.52). The proof of the theorem is complete.

**References**

[1] J. S. Abdul Rahman, A.H. S. Mushtaq and Al.H. F. Mohammed, Third- order differential subordination and superordination results for meromorphically univalent functions involving linear operator, European Jorunal of Scientific Research(EJSR), 132(1)(2015),57-65.

[2] R. M. Ali, V. Ravichandran, M.H. Khan, K.G. Subramanian, Differential sandwich theorems for certain analytic functions, Far East J. Math. Sci., 15 (2004) 87–94.

[3] R. M. Ali, V. Ravichandran and N. Seenivasagan, Subordination and superordination of the Liu-Srivastava linear operator on meromorphic functions, Bull. Malays. Math. Sci. Soc.,31 (2008), 193-207.

[4] R. M. Ali, V. Ravichandran and N. Seenivasagan, Differential subordination and superordination of analytic functions defined by the multiplier transformation, Math. Inequal. Appl.,12 (2009), 123-139.

[5] K. AL-Shaqsi ; Strong Differential Subordinations Obtained with New Integral Operator Defined by Polylogarithm Function ,Int. J . Math. Math. Sci.,Volume 2014.

[6] J. A. Antonino and S. S. Miller, Third- order differential inequalities and subordinations in the complex plane, Complex Var. Elliptic Equ. ,56 (2011), 439-454.

[7] M. K. Aouf and T. M. Seoudy , Subordination and superordination of a certain integral operator on merormorphic functions, Comput. Math. Appl, 59 (2010), 3669-3678.

[8] M. K. Aouf and A. O. Mostafa, Sandwich theorems for analytic functions defined by convolution, Acta Univ. Apulensis Math. Inform., 21(2010), 7-20.

[9] A. A. Attiya, O. S. Kwon, P. J. Hyang and N. E. Cho, An Integrodifferential operator for meromorphic functions associated with the Hurwitz - Lerch Zeta function , Filomat 30, 7 (2016), 2045-2057.

[10] L. de Branges, A proof of the Bieberbach Conjecture, Acta Math. ,154(1985), 137-152.

[11] T. Bulboac, Classes of first order differential superordinations, Demonstratio Math., 35(2) (2002), 287-292.

[12] T. Bulboaca, Differential Subordinations and Superordinations, Recent Results, House of Scientific Book Publ., Cluj-Napoca,( 2005).

[13] N. E. Cho, O.H. Kwon and H.M. Srivastava, Inclusion and argument properties for certain subclass of multivalent functions associated with a family of linear operators, J. Math. Anal. Appl., 292 (2014), 470–483.

[14] P. L. Duren,Univalent Functions, in: Grundlehren der Mathematischen Wissenschaften, Band 259, Springer-Verlag, New York, Berlin, Heidelberg and Tokyo, (1983).

[15] P. R. Garabedian and M. A. Schiffer, A proof of the Bieberbach conjecture for the fourth coefficient , J. Rational Mech. Anal., 4(1955),427-465.

[16] I. Graham and G .kohr, Geometric function theory in one and higher dimensions .New York: Marcel Dekker I Nc.,(2003).

[17] R . M. Goel and N.S. Sohi, A new criterion for p-valent functions, Proc. Amer. Math. Soc., 78 (1980) 353–357.

[18] A. W. Goodman, Univalent Functions, Vol. I , II, Mariner, Tampa, FL, (1983).

[19] S. P. Goyal,P.Goswami and H. Silverman, Subordination and superordination results for a class of analytic multivalent functions,Int.J.Math.Math. Sci., Article ID 561638,(2008),1-12.

[20] W. K. Hayman, Multivalent Functions, Second Edition, Printed in Great Britain at the Univeristy press, Cambridge, 1994.

[21] Yu. E. Hohlov, Operators and operations in the class of univalent functions, Izv, Vvssh. Ucebn. Zaved. Math., 10(1987), 83–89.

[22] R. W. Ibrahim, M. Z. Ahmad and H. F. Al-Janaby, Third-order differential subordination and superordination involving a fractional operator, Open Math. , 13 (2015), 706-728.

[23] I . B. Jung, Y.C. Kim and H. M Srivastava , The Hardy space of analytic functions associated with certain one parameter families of integral operations , J. Math.Anal Appl.,176(1993),138-197.

[24] W. Kaplan, Close -to -convex schlicht functions, Michigan Math. J., 1(1952), (1953),169-185.

[25] R. Kargar, A. Bilavi , S. Abdolahi and S. Maroufi, A class of multivalent analytic functions defined by a new linear operator, J. Math. Comp. Sci., 8 (2014), 326–334.

[26] S. Kavitha, S. Sivasubramanian and R. Jayasankar, Differential subordination and superordination results for Cho-Kwon-Srivastava operator, Comput. Math. Appl, 64(2012) 1789-1803.

[27] A. Y. Lashin, On certain subclass of meromorphic functions associated with certain integral operators, Comput. Math. Appl., 59 (2010), 524—531.

[28] E. Lindelöf, Mémoire sur certaines inégalities dans la théorien des functions dansle voisinage dún quelques propriétiés nouvells dices functions dans le voisinage dún point singulier esse-ntel, Ann. Soc. Sci. Finn., 35(7)(1909), 1-35.

[29] J. E. Littlewood, On inequalities in the theory of functions, Proc. London Math. Soc. , 23(2)(1925), 481-519.

[30] J. E. Littlewood, Lectures on the theory of functions, Oxford Uinve-rsity Press, Oxford and London, (1944).

[31] J. L. Liu and K.I. Noor, Some properties of Noor integral operator, J. Natur. Geom., 21 (2002), 81–90.

[32] Löwner, Üntersuchungen über schilicht Konforme Abbildungen des einheitschkreises, I. Math. Ann., 89(1923), 103-121.

[33] W. Ma and D. Minda, A unified treatment of some special classes of univalent functions,in Proceedings of the conference on Complex an- alysis (Tianjin, 1992), 157-169, Internat. Press ,Cambridge,MA.

[34] J. E. Miller, Convex meromorphic mapping and related functions, proc. Amer. Math. Soc.,25(1970),220-228.

[35] S. S. Miller and P.T. Mocanu, Differential subordinations and univ- alent function, Michigan Math. J. ,28 (1981), 157-171.

[36] S. S. Miller and P. T. Mocanu , Differential subordinations: Theory and Applications,in: Series on Monographs and Textbooks in Pure and Applied Mathematics, Vol.225, Marcel Dekker, Incorporated, New York and Basel, (2000).

[37] S. S. Miller and P. T. Mocanu, Subordinations of differential super- ordinations, Complex Variables, 48(10)(2003),815-826.

[38] A . O. Mostafa and M. K. Aouf, Sandwich theorems for certain sub- classes of analytic functions defined by family of linear operat-ors, J. Appl. Anal. , 15 ,n0.2, (2009), 269-280.

[39] G . Murugusundaramoorthy and N. Magesh, Differential subordinations and superordinations for analytic functions defined by the Dziok-Srivastava linear operator, J. Inequal. Pure Appl. Math., 7, no.4, (2006).

[40] K. Noshiro, On the theory of schlicht functions, J. Fas., Hokkaido Univ., 2(1934-32),129-155.

[41] M. Obradovic, M.K. Aouf, S. Owa, On some results for starlike functions of complex order, Publ. Inst. Math. (Beograd) (N.S.), 46 (60) (1989), 79–85.

[42] S. Owa, and H. M. Srivastava, Univalent and starlike generalized hypergeometric function ,Cand.J.Math.,37(5) (1987).1057-1077.

[43] R. Pederson and M. A. Schiffer, A proof of the Bieberbach conjecture for the fifth coefficient , Arch. Rational Mech. Anal., 45(1972),161-193.

[44] R. Pederson, A proof of the Bieberbach conjecture for the sixth coefficient , Arch. Rational Mech. Anal., 31(1968/1969),331-351.

[45] Ch. Pommerenke, Univalent Functions, Vanderhoeck and Ruprecht, Gottingen, (1975).

[46] S. Ponnusamy, P. O. Juneja , Third-order differential inequalities in the complex plane, Current Topics in Analytic Function Theory , World Scientific , Singapore , London , (1992).

[47] M. S. Robertson, On the theory of univalent functions, Ann. Math., 37(1936), 374-408.

[48] W. Rogosinski, On subordinations functions, Proc. Combridge Philes Soc., 35(1939), 1-26.

[49] W. Rogosinski, On the coefficients of subordinations, Proc. London Math. Soc., 48(2)(1945), 48-82.

[50] W. C. Royster, On the univalence of a certain integral, Michigan Math. J., 12 (1965), 385–387.

[51]S . Ruscheweyh, Neighbourhood of univalent function , Proc. Amer. Math. Soc., 81(1981),521-527.

[52] H. Saitoh, A liner operator and its applications of first order differential subordinations, Math. Japon., 44 (1996), 31-38.

[53]A. Schild and H.Silrerman ,Convolutions of univalent function with negative coefficients, Ann.Uni. Mariae-Curiae-Sklodowsk a set .A .,(1975).

[54] T. N. Shanmugam, V. Ravichandran and S. Sivasubramanian, Differential sandwich theorems for some subclasses of analytic functions, Aust. J. Math. Anal. Appl., 3 (1) (2006), 1-11.

[55] T. N. Shanmugam, S. Shivasubramanian and H. Silverman, On sandwich theorems for some classes of analytic functions, Int. J. Math. Math. Sci., Article ID 29684 (2006), 1– 13.

[56] T. N. Shanmugam, S. Sivasubramanian, H. M. Srivastava, Differential sandwich theorems for certain subclasses of analytic functions involving multiplier transformations, Integral Transforms Spec. Funct., 17 (12) (2006) ,889–899.

[57] T. N. Shanmugam, V. Ravichandran, M. Darus, S. Sivasubramanian, Differential sandwich theorems for same subclasses of analytic functions involving a linear operator, Acta Math. Univ. Comenian., 74 (2) (2007), 287–294.

[58] H. M. Srivastava and S. Owa, Some applications of the generalized hypergeomtric function involving certain subclasses of analytic functions, Publ. Math . Debrecen, 34(1987), 299-306.

[59] H. M. Srivastava and M. K. Aouf, A certain fractional derivative operator and its applications to a new class of analytic and multivalent functions with negative coefficients, Int. J. Math. Anal. Appl., 171 (1992), 1-13.

[60] H. Tang and E. Deniz, Third-order differential subordination results for analytic funct- ions involving the generalized Bessel functions, Acta Math.Sci. 34B(6),(2014),1707-1719.

[61] H. Tang, H. M.Srivastava, S. Li and L. Ma, Third - order differential subordination and superordination results for meromorphically multivalent functions associcted with the Liu-Srivastava operator, Abstract and Applied Analysis, (2014), Article ID 792175, 1-11 .

[62] N. Tuneski, On certain sufficient conditions for starlikeness, Int. J. Math. Math. Sci., 23(8) (2000), 521-527.

[63]Z. Wang, C. Gao, M. Liao, On certain generalized class of non-Bazilevic functions, Acta Math. Acad. Proc. Nyircg. New Series, 21 (2) (2005), 147–154.

[64]S. E. Warschawski, On the higher derivatives at the boundary in conformal mapping ,Trans. Amer. Math. Soc.,38, no.2, (1935), 310-340.

المستخلص

الغرض من هذه الرسالة هو دراسة بعض النتائج حول التبعية التفاضلية في نظرية الدالة احادية التكافؤ والمتعددة التكافؤ, بما تضمنته من نتائج حول الساندوج التفاضلية للدوال التحليلية،و نتائج التبعية والتبعية العليا لصنف من الدوال احادية التكافؤ. حصلنا على نتائج مرتبطة بالتبعية والتبعية العليا لصنف من الدوال متعددة التكافؤ في قرص الوحدة المفتوح. تعاملنا ايضاً مع نتائج التبعية التفاضلية من الرتبة الثالثة لدوال احادية التكافؤ الميرومورفية والمعرفة بواسطة مؤثر تكاملي. اعطينا أيضا بعض تطبيقات التبعية التفاضلية لصنف من الدوال متعددة التكافؤ الميرومورفية حيث حصلنا على بعض النتائج ،مثل، حدود المعامل ، حدود التشوية والنمو، مبرهنة الإنغلاق ، نصف قطر التحدب ، النقاط المتطرفة ، التركيب الخطي المحدب، والمجاميع الجزئية .

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**جمهورية العراق**

**وزارة التعليم العالي والبحث العلمي**

**جامعة القادسية / كلية علوم الحاسوب والرياضيات**

**قسم الرياضيات**

**دراسة بعض النتائج حول التبعية التفاضلية في نظرية الدالة احادية التكافؤ والمتعددة التكافؤ**

**رسـالة**

**مقدمة إلى مجلس كلية علوم الحاسوب والرياضيات في جامعة القادسية كجزء من متطلبات نيل درجة ماجستير علوم في الرياضيات**

**من قبل**

**سارة عبدالحميد جواد**

**بأشراف**

**أ.د. وقاص غالب عطشان**

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