Bourbaki and Palais proper Actions On d-Algebra

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Abstract:
The main goal of this work is to create a general two types of proper D – space , namely, Bourbaki proper D - space and Palais proper D - space to explain the relation between Bourbaki proper and Palais proper D - space and to study some of examples and propositions of Bourbaki proper and Palais proper D - space.

Introduction

One of the very important concepts in topological d – algebras is the concept of d - algebra actions. This paper studies an important class of actions namely, Bourbaki Proper D – spaces.

Y. Imai and K. Iseki [4] and K. Iseki [5] introduced two classes of abstract algebras: namely, BCK-algebras and BCI-algebras. It is known that the class of BCK algebras is a proper subclass of the class of BCI-algebras. In [2], [3] Q. P. Hu and X. Li introduced a wide class of abstract algebras: BCH-algebras. They have shown that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. J. Neggers and H. S. Kim [6] introduced the notion of d-algebras which is another generalization of BCK-algebras, and investigated relations between d-algebras and BCK-algebras. They studied the various topologies in a manner analogous to the study of lattices. However, no attempts have been made to study the topological structures making the star operation of d – algebra continuous. Theories of topological groups, topological rings and topological modules are well known and still investigated by many mathematicians.

Even topological universal algebraic structures have been studied by some authors. In section one of this work, we include some of results which thin will needed in the next sections. In section two, we deal with the definitions, examples, remarks and propositions, of topological d – algebra..In section two, we deal with the definitions, examples, remarks, propositions, theorem and corollaries of proper function. Section three recalls the definition of Bourbaki proper D – space, gives a new type of Bourbaki proper D – space (to the best of our Knowledge), namely, proper D – space and studies some of its properties, where D – space is meant T_2 – space topological X on which a locally compact, non – compact, T_2 – topological d – algebra D acts continuously on the left. In section four, we give the definitions, propositions, theorems and Examples of a Palais proper D - space are given as well as the relation between Bourbaki proper and Palais proper D – space is studied.

2- Preliminaries

2.1 Definition[1]: *Let X and Y be spaces and $f: X\to Y$ be a function. Then f is called continuous function if $f^{-1}(A)$ is an open set in X for every open set A in Y.*

2.2 Proposition[1] : *Let $f:X\to Y$ be a function of spaces. Then f is a continuous function if and only if $f^{-1}(A)$ is a closed set in X for every closed set A in Y.*

2.3 Definition[1]: *(i) A function $f: X\to Y$ is called closed function if the image of each closed subset of X is an closed set in Y.*

(ii) A function $f: X\to Y$ is called open function if the image of each open subset of X is an open set in Y.

2.4 Definition [1]: *Let X and Y be spaces . Then a function $f: X \to Y$ is called a homeomorphism if:*

(i) f is bijective .

(ii) f is continuous .

(iii) f is closed (open).

2.5 Definition[1]: *Let $(\chi_g)_{g\in G}$ be a net in a space X , $x\in X$. Then :*

i) $(\chi_g)_{g\in G}$ converges to x (written $\chi_g\to x$) if $(\chi_g)_{g\in G}$ is eventually in every neighborhood of x . The point x is called an limit point of $(\chi_g)_{g\in G}$, and the notation "$\chi_g\to\infty$" is mean that $(\chi_g)_{g\in G}$ has no convergent subnet.

ii) $(\chi_g)_{g\in G}$ is said to have x as an cluster point [written $\chi_g \alpha\, x$] if $(\chi_g)_{g\in G}$ is frequently in every neighborhood of x .

2.6 Proposition[1]: *Let $(\chi_g)_{g\in G}$ be a net in a space (X, T) and x_o in X. Then $\chi_g\ \alpha\ x_o$ if and only if there exists a subnet $(\chi_{g_m})_{g_m\in G}$ of $(\chi_g)_{g\in G}$ such that $\chi_{g_m}\to x_o$.*

2.7 Remark[1]: *Let $(\chi_g)_{g\in G}$ be a net in a space (X, T) such that $\chi_g\ \alpha\ x$, $x\in X$ and let A be an open set in X which contains x. Then there exists a subnet $(\chi_{g_m})_{g_m\in G}$ of $(\chi_g)_{g\in G}$ in the set A such that $\chi_{d_m}\to x$.*

2.8 Definition[1]: *A subset A of space X is called compact set if every open cover of A has a finite sub cover. If A=X then X is called an compact space.*

2.9 Proposition[1]: *Let X be a space and F be an closed subset of X. Then $F\cap K$ is compact subset of F, for every compact set K in X.*

2.10 Proposition[1]: *Let Y be an open subspace of space X and $A\subseteq Y$. Then A is an compact set in Y if and only if A is an compact set in X.*

2.11 Definition[1]:

(i) A subset A of space X is called relative compact if $\overline{A}$ is compact.

(ii) A space X is called locally compact if every point in X has a relative compact neighborhood.

2.12 Proposition[1]: *Let X and Y be spaces and $f: X\to Y$ be a continuous function. Then an image f(A) is compact in Y for every A is compact in X.*

2.13 Definition[1]: *Let $f: X\to Y$ be a function of spaces. Then f is called an compact function if $f^{-1}(A)$ is a compact set in X for every compact set A in Y.*

2.14 Proposition[1]: *Let X, Y be a spaces and $f: X\to Y$ be compact function. If F is an closed subset of X and B is an open set in Y, then $f_{/F} : F\to B$ is compact.*

2.15 Definition[1]: *Let X and Y be two spaces. Then $f:X\to Y$ is called a proper function if:*

(i) f is continuous function.

(ii) $f\times I_Z: X\times Z\to Y\times Z$ is a closed function, for every space Z.

2.16 Proposition [1]: *Let X and Y be spaces and f: $X \to Y$ be a continuous function Then the following statements are equivalent:*

(i) f is a proper function.

(ii) f is a closed function and f^{-1} $(\{y\})$ is an compact set, for each $y \in Y$.

(iii) If $(\chi_g)_{g \in G}$ is a net in X and $y \in Y$ is a cluster point of f (χ_g), then there is a cluster point $x \in X$ of $(\chi_g)_{g \in G}$ such that f $(x) = y$.

2.17 Proposition[1]: *Let X, Y and Z be spaces, f: $X \to Y$ and g: $Y \to Z$ be two proper function. Then $gof: X \to Z$ is a proper function.*

2.18 Proposition[1]: *Let f_1: $X_1 \to Y_1$ and f_2: $X_2 \to Y_2$ be two function. Then $f_1 \times f_2$: $X_1 \times X_2 \to Y_1 \times Y_2$ is proper function if and only if f_1 and f_2 are proper functions.*

2.19 Proposition[1]: *Every proper function is closed.*

2.20 Proposition[1]: *Let f: $X \to P = \{w\}$ be a continuous function on a space X. Then f is a proper function if and only if X is a compact, where w is any point which does not belongs to X.*

2.21 Lemma[1]: *Every continuous function from a compact space into a Hausdorff space is closed.*

2.22 Remark[1]: *If X is a space, then the diagonal function Δ: $X \to X \times X$ such that $\Delta(x) = (x, x)$ is continuous.*

2.23 Proposition : *If X is a Hausdorff space, then the diagonal function Δ: $X \to X \times X$ is proper function.*

Proof: *Let(χ_g, χ_g) α (x_1, x_2), so there exists a subnet of (χ_g, χ_g) ,say itself, such that $(\chi_g, \chi_g) \longrightarrow (x_1, x_2)$,then $\chi_g \longrightarrow x_1$ and $\chi_g \longrightarrow x_2$, since X is a T_2 – space , then $x_1 = x_2$. Then there is $x_1 \in X$ such that $\chi_g \alpha\, x_1$ and $\Delta(x_1) = y$. Hence by Proposition (2.16.iii) Δ is a proper function.*

2.24 Proposition[1]: *Let f_1: $X \to Y_1$ and f_2: $X \to Y_2$ be two proper functions. If X is a Hausdorff space, then the function f: $X \to Y_1 \times Y_2$, $f(x) = (f_1(x), f_2(x))$ is a proper function.*

2.25 Proposition[1]: *Let X and Y be a spaces and f: $X \to Y$ be a continuous, one to one function. Then the following statements are equivalent:*

(i) f is a proper function.

(ii) f is a closed function.

(iii) f is a homeomorphism of X onto closed subset of Y.

2.26 Proposition[1]: *Let X and Y be a spaces, such that Y is a T_2 – space and f: $X \to Y$ be a continuous function. Then the following statements are equivalent:*

(i) f is a compact function.

(ii) f is a proper function.

(iii) If $(\chi_g)_{g \in G}$ is a net in X and $y \in Y$ is a cluster point of $f(\chi_g)$, then there is a cluster point $x \in X$ of $(\chi_g)_{g \in G}$ such that $f(x) = y$.

3 – Bourbaki Proper D – Space.

In this section, we examine the definition of topological d-algebra and some issues and examples related to the subject and we define the space Bourbaki proper D – space.

3.1Definition [8]: *A non-empty set D together with a binary operation $*$ and a zero element 0 is said to be a d – algebra if the following axioms are satisfied for all x, $y \in$ D*

1)) $x * x = 0$

2) $0 * x = 0$

3)$x * y = 0$ and $y * x = 0$ imply that $x = y$.

3.2 Definition [2]: An element e of D is called a left identity if $e*a=a$, a right identity if $a*e = a$ for all $a\in D$ and $a\neq e$. If e is both left and right identity then we called e is an identity element. Also we say that $(D,*)$ is d – algebra with identity element

3.3 Example:

i) Let D be any non – empty set and $P(D)$ is power set of D then $(P(D),-)$ is d – algebra and ϕ is right identity in $(P(D),-)$.

ii) let $D=\{0,a,b,c\}$ and define the binary operation $*$ on D by the following table:

$*$	0	a	b	c
0	0	0	0	0
a	0	0	b	c
b	0	b	0	a
c	0	c	a	0

Table (1)

Then the pair $(D,*)$ is d – algebra with identity element a.

3.4 Definition[2] : Let $(D,*)$ be a d – algebra and T be a topology on D. The triple $(D,*,T)$ is called a topological d – algebra (denoted by Td – algebra) if the binary operation $*$ is continuous.

3.5 Example:

i) Let $D=\{0,a,b,c\}$and $*$be define by the following table:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	b	0	b
c	c	c	c	0

Table (2)

It is clear that $(D,*)$ is d – algebra and $T=\{\phi,\{b\},\{c\},\{0,a\},\{b,c\},\{0,a,b\},\{0,a,c\},D\}$ is topology on D such that the triple $(D,*,T)$ is a topological d – algebra.

ii) Let R be a set of real number and $*$ is a binary operation which define by $a*b = a.(a-b)^2$ then $(R,*)$ is d – algebra and $(R,*,T)$ is Td – algebra where T is usual topology on R.

3.6 Definition[2]: A topological transformation d – algebra is a triple (D,X,φ) where D is a T_2 – topological d – algebra with left identity e, X is a T_2 – topological space and $\varphi : D\times X \to X$ is a continuous function such that:

(i) $\varphi(d_1,\varphi(d_2, x)) = \varphi(d_1d_2, x)$ for all $d_1 \neq d_2$ and $d_1, d_2\in D$, $x\in X$.

(ii) $\varphi(e, x) = x$ for all $x\in X$, where e is the left identity element of D.

We shall often use the notation $d.x$ for $\varphi(d,x)$ $d.(h,x)= (dh).x$ for $\varphi(d, \varphi(h,x))= \varphi(dh,x)$. Similarly for $H\subseteq D$ and $A\subseteq X$ we put $HA=\{ha / h\in H, a\in A\}$ for $\varphi(H, A)$.

3.7 Remark[2]:

(i) The function φ is called an action of D on X and the space X together with φ is called a D – space (or more precisely left D – space).

(ii) The subspace $\{d.x \,/\, d \in D\}$ is called the orbit (trajectory) of x under D, which denoted by Dx [or $\gamma(x)$], and for every $x \in X$ the stabilizer subgroup D_x of D at x is the set $\{d \in D / d.x = x\}$.

(ii) A set $A \subseteq X$ is said to be invariant under D if DA = A.

3.8 Definition: *A D – space X is called a Bourbaki proper D – space (proper D – space) if the function $\theta : D \times X \to X \times X$ which is defined by $\theta(d, x) = (x, d.x)$ is a proper function.*

3.9 Example: *Let $Z_3 = \{-1,0,1\}$ and $*$ is define by the table:*

*	0	-1	1
0	0	0	0
-1	0	0	-1
1	0	-1	0

Table(3)

Then $(Z_3,*)$ is d – algebra and $(Z_3,*,\tau)$ is Td – algebra where τ is discrete topology on Z_3 . The act on the topological space S^n [as a subspace of R^{n+1} with usual topology] as follows:

$0.\ (x_1, x_2, \dots, x_{n+1}) = (0,0, \dots ,0)$

$1.\ (x_1, x_2, \dots, x_{n+1}) = (x_1,x_2,\dots,x_{n+1})$

$-1.\ (x_1, x_2,\dots, x_{n+1}) = (-x_1, -x_2, \dots, -x_{n+1})$

Since Z_3 is compact, then by Proposition (2.20) the constant function $Z_3 \to P$ is a proper. Also the identity function is a proper, then by Proposition (2.18) the proper function of $Z_3 \times S^n$ into $P \times S^n$ is a proper.

Since $P \times S^n$ is homeomorphic to S^n Then by Proposition (2.25) the homeomorphism of $P \times S^n$ onto S^n is proper. Since $Z_3 \times S^n \to P \times S^n$ is continuous and open function. Then by Proposition (2.17) the composition $Z_3 \times S^n \to S^n$ is a proper. Let φ be the action of Z_3 on S^n. Then φ continuous, one to one function so φ is continuous function. Since S^n is T_2 – space, then by Lemma (2.21) φ is a closed. Then by Proposition (2.25) φ is a proper function. Thus by Proposition (2.24) $Z_3 \times S^n \to S^n \times S^n$ is a proper D – space.

3.10 Lemma: *If X is a D – space then the function $\theta : D \times X \to X \times X$ which is defined by $\theta(d, x) = (x, d.x)$ is a continuous function and $\theta^{-1}(\{(x, y)\})$ is closed in $D \times X$ for every $(x, y) \in X \times X$.*

Proof: *Since: $\theta : D \times X \xrightarrow{I_D \times \Delta} D \times X \times X \xrightarrow{\varphi \times I_X} X \times X \xrightarrow{\overset{f}{\cong}} X \times X$, where φ is action of D on X. Then $\theta = f \circ \varphi \times I_X \circ I_D \times \Delta$ is continuous function and $\theta^{-1}(\{(x, y)\})$ is closed in $D \times X$ for every $(x, y) \in X \times X$.*

3.11Theorem: *Let X be a proper D – space and let H be a closed subset of D. If Y is an open subset of X which is invariant under H, then Y is a proper H – space.*

Proof: *Since X is a proper D – space, then the function $\theta : D \times X \to X \times X$ which is defined by $\theta(d, x) = (x,d.x)$ is a proper function. [To prove that $\omega : H \times Y \to Y \times Y$ is a proper function which is defined by $\omega(h, y) = \theta(h,y)$ for each $(h, y) \in H \times Y$].*

(1) By Lemma (3.10) $\theta : D\times X\to X\times X$ is continuous, then $\omega : H\times Y\to Y\times Y$ is continuous.

(2) Let $(h_g, y_g)_{g\in G}$ be a net in $H\times Y$ such that $\omega((h_g, y_g))\ \alpha\ (x, y)$ for some $(x, y)\in Y\times Y$. Then $(y_g, h_g y_g)\ \alpha\ (x, y)$ in $Y\times Y$. Let A be an open subset of $X\times X$ such that $(x, y)\in A$. Since Y is open in X, then $Y\times Y$ is an open set in $X\times X$. Then $A\cap(Y\times Y)$ is an open set in $X\times X$. But $(x, y)\in A\cap(Y\times Y)$ and $(y_g, h_g y_g)\ \alpha\ (x, y)$, thus $(y_g, h_g y_g)$ is frequently in $A\cap(Y\times Y)$ and then $(y_g, h_g y_g)$ is frequently in A, thus $(y_g, h_g y_g)\ \alpha\ (x, y)$ in $X\times X$. since $\theta : D\times X\to X\times X$ is an proper function, then by Proposition (2.16) there exists $(h, x_1)\in D\times X$ such that $(h_g, y_g)\ \alpha\ (h, x_1)$ and $\theta((h, x_1))=(x, y)$, hence $(x_1, h x_1)=(x,y)$. Thus $x_1=x$ and therefore $h_g\ \alpha\ h$. Since $(h_g)_{g\in G}$ is a net in H , and H is closed . Then there exists $(h, x)\in H\times Y$ such that $\omega(h, x) =\theta(h,x) = (x, y)$. Then from (1), (2) and by Proposition (2.16) the function $\omega : H\times Y\to Y\times Y$ is a proper function. Hence Y is a proper H–space.

3.12 Corollary: *Let X be a proper D–space and Y be an open subset of X which is invariant under D. Then Y is a proper D–space.*

3.13 Corollary: *Let X be a proper D–space and let H be closed subset of D. Then X is a proper H – space.*

3.14 Proposition: *Let X be a proper D–space, $x\in X$ and $T=\{x\}\times X$. Then the function $\theta_T: \theta^{-1}(T) \to T$ is a proper function, where $\theta : D\times X\to X\times X$ such that $\theta(g, x) = (x, g.x)$, $\forall(g, x)\in D\times X$.*

Proof: *Since X is a T_2 – space, then $\{x\}$ is closed set in X . Thus $\{x\}$ is closed set in X. So each $D\times\{x\}$ and $\{x\}\times X$ are closed in $D\times X$ and $X\times X$ (respectively). Now, Let F be a closed set in $\theta^{-1}(T) = D\times\{x\}$, Since $F=F\cap(D\times \{x\})$, then F is closed in $D\times X$ and $\theta_T(F) =\theta(F)\cap(\{x\}\times X)$, therefore $\theta(F)$ is closed in $X\times X$ then by Proposition (2.19) $\theta_T(F)$ is closed in $X\times X$. But $\theta_T(F)\subseteq\{x\}\times X$, then there exists a subset V of X such that $\theta_T(F) = \{x\}\times V$. Since $\theta_T(F)$ is closed in $X\times X$, so $\{x\}\times V$ is an closed set in $\{x\}\times X$, hence $\theta_T(F) = \{x\}\times V$ is a closed set in $T= \{x\}\times X$ there for $\theta_T: \theta^{-1}(T)\to T$ is closed. Now, let $(x, y)\in\{x\}\times X$. Since θ is proper function, then by Proposition (2.26) θ is a compact function. Then $\theta^{-1}(\{(x, y)\})$ is compact in $D\times X$. Then by $\theta_T^{-1}(\{(x, y)\})$ is compact set in $D\times\{x\} = \theta^{-1}(T)$. Thus by Proposition (2.26) θ_T is a proper function.*

Let X be a D – space and A , B be two subset of X . We mean by ((A, B)) the set $\{d\in D / dA\cap B\neq\phi\}$.

From now on, we will use D – space, which satisfies the property if (X,T) and (Y,T′) be two space and $\forall\chi_g\to x$, $y_g\to y$ in X and Y, respectively, then $(\chi_g, y_g) \to (x,y)$.

3.15 Proposition: *Let X be a D – space. If for every $x, y\in X$ there exists an open set A_x of X contains x and an open set A_y of X contains y such that $K=((A_x, A_y))$ is relatively compact in D, then X is a proper D – space.*

Proof: *We prove that $\theta : D\times X \to X\times X$, $\theta(d, x) = (x, dx)$ is a proper function. Let $(d_g, \chi_g)_{g\in G}$ be a net in $D\times X$ such that $\theta((d_g, \chi_g))=(\chi_g, d_g\chi_g)\ \alpha\ (x, y)$, where $(x, y)\in X\times X$. Now, since $x, y\in X$, then there exists an open set A_x contains x and an open set A_y contains y such that the set $K = ((A_x, A_y))$ is relatively compact in D. Thus $A_x\times A_y$ is an open set in $X\times X$ and $(x, y)\in A_x\times A_y$, so there exists a sub net $(\chi_{g_m}, d_{g_m}\chi_{g_m})_{g\in G}$ of $(\chi_g, d_g\chi_g)$ in $A_x\times A_y$ and $(\chi_{g_m}, d_{g_m}\chi_{g_m})\to(x, y)$, hence $\chi_{g_m}\to x$ and $d_{g_m}\chi_{g_m}\to y$. Since $\chi_{g_m}\in A_x$, and $d_{g_m}\chi_{g_m}\in A_y$, Then $d_{g_m}. A_x\cap A_y\neq\phi$, $\forall g_m$, so $d_{g_m}\in K$, but K is relatively compact in*

D, then by Proposition (2.6) (d_{g_m}) has a limit point, say $t \in D$. Since $\chi_{g_m} \to x$, then (d_g, χ_g)$\to$ (t,x),so θ ((d_g, χ_g))$\to$ θ ((t,x)) ,i.e. , $(\chi_{g_m}, d_{g_m}\chi_{g_m}) \to$ (x, tx), thus $d_{g_m}\chi_{g_m} \to$ tx but $d_{g_m}\chi_{g_m} \to y$ and since X is a Hausdorff space, then tx = y. But $(\chi_{g_m}, g_{g_m}\chi_{g_m})_{g\in G}$ is a sub net of (χ_g, d_g χ_g) and (d_{g_m} , χ_{g_m})$\to$ (t, x), then (d_g, χ_g) α (t, x), thus θ ((t, x)) = (x, y).Then by Proposition (2.16) we have θ is a proper function. Hence X is a proper D – space.

3.16 Corollary: *Let X be a D–space such that D is discrete space. If for every x, $y \in X$ there is an open set A_x in X contains x and an open set A_y in X contains y such that the set K = ((A_x, A_y)) is finite, then X is a proper D–space.*

Let X be a D–space and $x \in X$. The set J(x)={$y \in X$: there is a net $(d_g)_{g\in G}$ in D and there is a net $(\chi_g)_{g\in G}$ in X with $d_g \to \infty$ and $\chi_g \to x$ such that $d_g x \to y$} is called first prolongation limit set of x,[3]. J(x) is a good tool to discover about the Bourbaki proper D–space.

3.17 Proposition: *Let X be a D–space. Then X is a Bourbaki proper D–space if and only if J(x)=ϕ for each $x \in X$.*

Proof: $\Rightarrow$ Suppose that $y \in J(x)$, then there is a net $(d_g)_{g\in G}$ in D with $d_g \to \infty$ and there is a net $(\chi_g)_{g\in G}$ in X with $\chi_g \to x$ such that $d_g\chi_g \to y$, so θ ((d_g,χ_g))= (χ_g, $d_g\chi_g$) $\to$ (x, y). But X is a Bourbaki proper, then by Proposition (2.26) there is(d, x_1) $\in d\times X$ such that (d_g,χ_g) α (d, x_1). Thus $(d_g)_{g\in G}$ has a sub net (say itself), such that $d_g \to d$, which is contradiction, thus J(x)=ϕ.

$\Leftarrow$ Let $(d_g,\chi_g)_{g\in G}$ be a net in $D\times X$ and (x, y)$\in X\times X$ such that θ((d_g,χ_g))=(χ_g, $d_g\chi_g$) α (x, y), so $(\chi_g, d_g\chi_g)_{g\in G}$ has a sub net, say itself, such that (χ_g, $d_g\chi_g$) $\to$ (x, y), then $\chi_g \to x$ and $d_g\chi_g \to y$. Suppose that $d_g \to \infty$ then $y \in J(x)$, which is contradiction . Then there is $d \in D$ such that $d_g \to d$, then (d_g, χ_g) $\to$ (d, x) and θ (d, x) = (x, y). Thus by Lemma (3.10) and Proposition (2.16) X is a Bourbaki proper D–space.

3.18 Proposition: *Let X be a proper D–space, θ (d, x)=(x, d.x) $\forall$(d, x)$\in D\times X$ with the action $\varphi : D\times X \to X$,$\varphi$ (d, x) = d.x, $\forall$ (d,x) $\in D\times X$. Then for each $x \in X$, the function φ_x:D$\to$X, which is defined by: $\varphi_x(d)=\varphi(d,x)$ is a proper function.*

Proof: Let T={x}$\times$X $\subseteq$ X$\times$ X, then by Proposition (3.14) θ_T: $\theta^{-1}(T) \to T$ is a proper function. But:

$$\varphi_x = D \overset{f}{\cong} D\times\{x\} \xrightarrow{\theta_T} \{x\}\times X \overset{h}{\cong} X,$$ such that f and h are homeomorphisms.

Now:

i) Let A be an open set in X, then $h^{-1}(A)$ is a open set in {x}$\times$X. Since θ_T is continuous, then $\theta_T^{-1}(h^{-1}(A))$ is open in D$\times${x}. Since f is homeomorphism, then $f^{-1}(\theta_T^{-1}(h^{-1}(A)))$ is open in D. Then $\varphi_x(A) = f^{-1} o \theta_T^{-1} o\, h^{-1}(A)$ is open in D. Thus φ_x: D$\to$X is continuous.

ii) Let F is closed in D, then h(F) is closed in {x}$\times$X. Since θ_T: D$\times${x}$\to$ {x}$\times$X is a proper function, then by Proposition (2.19) $\theta_T(h(F))$ is closed, then $\varphi_x(F)=f(\theta_T(h(F)))$ is closed in X. Then φ_x: D$\to$X is closed.

iii) Let $y \in X$, then $h^{-1}(\{y\})$ = {(x, y)} such that $x \in X$, since X is T_2 – space, then {(x, y)} is a closed set in {x}$\times$X. Since θ_T is a continuous function, then θ_T^{-1} ({(x, y)}) is closed in D$\times${x}, so by Proposition (2.16) $\theta_T^{-1}(h^{-1}(\{y\})) = \theta_T^{-1}\{(x, y)\}$ is compact. Since f is homeomorphism, so it is clear that $f^{-1}(\theta_T^{-1}\{(x, y)\})$ is compact in D. Then $f^{-1}(\theta_T^{-1}($ h^{-1}

$(\{y\}))) = \varphi_x^{-1}(\{y\})$ is compact in D. Then by (i),(ii),(iii) and Proposition (2.16) φ_x is proper function.

3.19 Proposition: *Let X be a D–space and $\theta : D\times X \to X\times X$ be a function which is defined by $\theta(d, x) = (x, dx)$, $\forall(d, x)\in D\times X$. Then the following statements are equivalent:*

(i) X is a proper D–space.

(ii) $\theta^{-1}(\{(x, y)\})$ is a compact set, $\forall(x, y)\in X\times X$ and for all $x, y\in X$ and for all $U\in N(((x, y)))$, $\exists V_x\in N(x)$ and $V_y\in N(y)$ such that $((V_x, V_y))\subseteq U$.

(iii) $\theta^{-1}(\{(x, y)\})$ is a compact set, $\forall(x, y)\in X\times X$ and for all $x, y\in X$ and for all $U\in N(\theta^{-1}(\{(x, y)\}))$, $\exists V\in N((x, y))$ such that $\theta^{-1}(V)\subseteq U$.

Proof: *i)→iii) Let $x, y\in X$ and U be an open neighborhood of $\theta^{-1}(x, y)$. Since θ is a proper function, then by Proposition (2.19) θ is a closed function, so $V= (X\times X)\backslash\theta((D\times X)\backslash U)$ is an open neighborhood of (x, y) with $\theta^{-1}(V)\subseteq U$. So by Proposition (2.16) $\theta^{-1}(\{(x, y)\})$ is a compact set $\forall(x, y)\in X\times X$. Hence (iii), holds.*

iii)→i) Let F be a closed subset of $D\times X$ and let $(x,y)\in X\times X\setminus\theta(F)$, since $(D\times X)\setminus F$ is an open neighborhood of $\theta^{-1}(x, y)$, then by (iii) there is a neighborhood V of (x, y) such that $\theta^{-1}(V)\subseteq(D\times X)\backslash F$. Hence $V\cap\theta(F) = \phi$, so $(x, y)\notin\overline{\theta(F)}$, then $\overline{\theta(F)}=\theta(F)$. Hence θ is a closed function, since $\theta^{-1}(\{(x, y)\})$ is a compact set for every $(x, y)\in X\times X$, therefore by Proposition (2.16) θ is a proper function. Hence X is a proper D–space.

ii)→iii) Let $x, y\in X$ and let U be a neighborhood of $\theta^{-1}(\{(x, y)\}) =((x , y))\times\{x\}$. Since $\theta^{-1}(\{(x, y)\})$ is compact, then there are neighborhood U' of ((x, y)) and W of {x} such that $U'\times W\subseteq U$, so by (ii) there are neighborhood V_x of x and V_y of y such that $((V_x, V_y))\subseteq U'$. But $\theta^{-1}((V_x\cap W)\times V_y)\subseteq U'\times W\subseteq U$. Hence (iii), hold.

iii) → ii) Let $x, y\in X$ and $U\in N((x, y))$. Then $U\times X\in N((x, y))\times\{x\}$. Thus $U\times X\in N\,\theta^{-1}(x, y)$ so by (iii) there exists $V\in N(x, y)$ such that $\theta^{-1}(V)\subseteq U\times X$. Then there are neighborhood V_x of x and V_y of y such that $\theta^{-1}(V_x\times V_y)\subseteq U\times X$. Hence (ii), holds.

3.20 Corollary: *Let X be a proper D–space, choose a point $x\in X$ and let U be neighborhood of the stabilizer D_x of x , then x has a neighborhood V such that U contains the stabilizer of all points in V.*

Proof: *Since U is neighborhood of the stabilizer D_x of x, then $U\in N(D_x)$. Since $D_x = ((x, x))$, then $U\in N(((x, x)))$. So by Proposition (3.20)there exist $V_x\in N(x, x)$ such that $((V_x, V_x))\subseteq U$. Let $y\in V_x$, then $D_y\subseteq((V_x, V_x))\subseteq U$.*

4 –Palais proper D - space:

From now on, in this section by D–space we mean a completely regular topological T_2 – space X on which a locally compact, non – compact, T_2 – topological d - algebra D continuously on the left (always in the sense of Palais proper D–space). in

4.1 Definition:

Let X be a D–space. A subset A of X is said to be thin relative to a subset B of X if the set $((A, B)) = \{d\in D / dA\cap B\neq\phi\}$ has a neighborhood whose closure is compact in D. If A is thin relative to itself, then it is called thin.

4.2 Remark:

The thin sets have the following properties:

(i) If A and B are relative thin and $K_1\subseteq A$ and $K_2\subseteq B$, then K_1 and K_2 are relatively thin.

(ii) Let X be a D–space and K_1, K_2 be a compact subset of X, then $((K_1, K_2))$ is closed in D.
(iii) If K_1 and K_2 are compact subset of D–space X such that K_1 and K_2 are relatively thin, then $((K_1, K_2))$ is a compact subset of D.

4.3 Definition:

A subset S of a D–space X is a small subset of X if each point of X has neighborhood which thin relative to S.

4.4 Theorem:

Let X be a D–space. Then:
(i) Each small neighborhood of a point x contains a thin neighborhood of x.
(ii) A subset of a small set is small.
(iii) A finite union of a small sets is small.
(iv) If S is a small subset of X and K is a compact subset of X then K is thin relative to S.

Proof:

i) Let S is a small neighborhood of x. Then there is a neighborhood U of x which is thin relative to S. Then ((U, S)) has neighborhood whose closure is compact. Let V= U∩S, then V is neighborhood of x and $((V, V)) \subseteq ((U, S))$, therefore V is V is thin neighborhood of x.

ii) Let S be a small set and $K \subseteq S$. Let $x \in X$, then there exists a neighborhood U of x, which is thin relative to S. Then $((U, K)) \subseteq ((U, S))$, thus ((U, K)) has neighborhood whose closure is compact. Then K is small.

iii) Let $\{S_i\}_{i=1}^{n}$ be a finite collection of small sets and $y \in X$. Then for each i there is neighborhood K_i of y such that the set $((S_i , K_i))$ has neighborhood whose closure is compact. Then $\bigcup_{i=1}^{n} ((S_i , K_i))$ has neighborhood whose closure is compact. But $((\bigcup_{i=1}^{n} S_i, \bigcap_{i=1}^{n} K_i)) \subseteq \bigcup_{i=1}^{n} ((S_i , K_i))$, thus $\bigcup_{i=1}^{n} S_i$ is a small set.

iv) Let S be a small set and K be compact. Then there is a neighborhood U_k of K, $\forall k \in K$, such that U_k is thin relative to S. Since $K \subseteq \bigcup_{k \in K} U_k$ .i.e., $\{U_k\}_{k \in K}$ is open cover of K , which is compact, so there is a finite sub cover $\{U_{k_i}\}_{i=i}^{n}$ of $\{U_k\}_{k \in K}$, since $((U_{k_i} ,S))$ has neighborhood whose closure is compact , thus $((\bigcup_{i=1}^{n} U_{k_i}, S))$ so is . But $((K,S)) \subseteq ((\bigcup_{i=1}^{n} U_{k_i}, S))$ therefore K is thin relative to S.

4.5 Definition:

A D–space X is said to be a Palais proper D–space if every point x in X has a neighborhood which is small set.

4.6 Examples: *i) Let $Z_3 = \{-1,0,1\}$ and $*$ is define by the table:*

*	0	-1	1
0	0	0	0
-1	-1	0	-1
1	1	1	0

Table (4)

Then Z_3, act on itself (as Z_3 with discrete topology) as follows:

$r_1.r_2 = r_1 * r_2 \quad \forall r_1, r_2 \in Z_3$.

It is clear that for each point $x \in Z_3$, there is a neighborhood which is small U of x, i.e, for any point y of Z_3, there exists a neighborhood V of y then $((U, V)) = \{r \in Z_3 / rU \cap V \neq \phi\} = Z_3$, then ((U, V)) has neighborhood whose closure is compact.

*(ii) R be locally compact topological d – algebra (as R with discrete topology and binary operation a*b = a(a-b)) acts on the completely regular Hausdorff space R^2 as follows:*

$r.(x_1, x_2) = (x_1, x_2)$, *for every* $r \in R - \{0\}$ *and* $(x_1, x_2) \in R^2$.

Clear R^2 is R – space. But $(0,0) \in R^2$ has no neighborhood which is a small. Since for any two neighborhoods U and V of (0,0) then ((U,V)) = R. Since R is not compact. Thus R^2 is not a Palais proper R – space.

4.7 Proposition:

Let X be a D–space . Then:

(i) If X is Palais proper D–space , then every compact subset of X is a small set.

(ii) If X is a Palais proper D–space and K is a compact subset of X , then ((K,K)) is a compact subset of D.

Proof:

i) Let A be a subset of X such that A is compact. Let $x \in X$, since X is a proper D–space then there is a neighborhood of x U which is small. Then for every $a \in A$ there exists a neighborhood U_a which is small , then $A \subseteq \bigcup_{a \in A} U_a$, since A is compact , then there exists $a_1, a_2, ..., a_n \in A$ such that $A \subseteq \bigcup_{i=1}^{n} U_{a_i}$, Thus by Theorem (4.3.iii.ii) A is a small set in X.

ii) Let X be a proper D–space and K is compact , then by (i) K is a small subset of X , and by Theorem (4.3.iv) K is thin , so ((K,K)) has neighborhood whose closure is compact. Then by Remark (4.2.iii) ((K,K)) is closed in D. Thus ((K,K)) is compact .

4.8 Proposition:

Let X be a D–space and y be a point in X . Then y has no small whenever $y \in J(x)$ for some point $x \in X$.

Proof:

Let $y \in J(x)$, then there is a net $(d_g)_{g \in G}$ in D with $d_g \to \infty$ and a net $(\chi_g)_{g \in G}$ in X with $\chi_g \to x$ such that $d_g\chi_g \to y$. Now, for each neighborhood S of y and every neighborhood U of x there is $g_o \in G$ such that $\chi_g \in U$ and $d_g\chi_g \in S$ for each $g \geq g_o$, thus $d_g \in ((U,S))$, but $d_g \to \infty$, thus ((U,S)) has no compact closure . i.e., S is not a small neighborhood.

4.9 Proposition: *Let X be a Palais proper D–space. Then $J(x) = \phi$ for each $x \in X$.*

Proof:

Suppose that there exists $x \in X$ such that $J(x) \neq \phi$, then there exists $y \in J(x)$. Thus there is a net $(d_g)_{g \in G}$ in D with $d_g \to \infty$ and a net $(\chi_g)_{g \in G}$ in X with $\chi_g \to x$ such that $d_g\chi_g \to y$. Since X be a proper D–space, then there is a small (thin) neighborhood U of *x. Thus there is $g_o \in G$ such that $d_g\chi_g \in U$ and $\chi_g \in U$ for each $g \geq g_o$, so $d_g \in ((U,U))$, which has a compact closure , therefore $(d_g)_{g\in G}$ must have a convergent subnet , which is a contradiction. Thus $J(x) = \phi$ for each $x \in X$.*

4.10 Proposition[3]: *Let X be a periodic D–space . Then $J(x) = \phi$ for each $x \in X$ if and only if every pair of points of X such that $Dx \neq Dy$ has relatively thin neighborhood.*

4.11 Proposition: *Let X be a periodic D–space and $Dx \neq Dy$ for every pair of points x,y of X. Then the definition of Palais proper D–space and the definition Bourbaki proper D–space are equivalent.*

Proof:

$\Rightarrow$) By Propositions (4.10) and (3.17).

$\Leftarrow$) let X be a Bourbaki proper D–space, then by Proposition (3.17) $J(x) = \phi$ for each $x \in X$. Let $x \in X$, we will show that x has a small neighborhood. by Proposition(4.10). Then there a small neighborhood U_x of x, Thus X is Palais proper D–space.

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