**On Differential Sandwich Results For Analytic Functions**

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**Abstract:** In this paper ,we obtain some subordination and superordination results involving the integral operator .Also,we get Differential sandwich results for classes of univalent functions in the unit disk.

**Keywords:**Analytic function, univalent function, differential subordination , superordination.

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**1-Introduction :**

Let H=H(U) be the class of analytic functions in the open unit disk For n a positive integer and Let H[a, n] be the subclass of H consisting of functions of the form:

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| --- |
| (a ). (1.1) |

Also, let A be the subclass of H consisting of functions of the form:

|  |
| --- |
| (1.2) |

Let . The function is said to be subordinate to , or is said to be subordinate to , if there exists a Schwarz function *w* analytic in U with and |*w*(z)| < 1 ( ), such that , In such a case we write . If is univalent in , then if and only if and

Let : If p and are univalent functions in U and if p satisfies the second-order differential superordination.

|  |
| --- |
| (1.3) |

then p is called a solution of the differential superordination (1.3). ( If *f* is subordinate to , then is superordinate to *f* ) . An analytic function q is called a subordinant of (1.3) , if for all the functions p satisfying (1.3).

An univalent subordinant that satisfies for all the subordinants q of (1.3) is called the best subordinant. Miller and Mocanu [5] have obtained conditions on the functions and for which the following implication holds :

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| --- |
| (1.4) |

For  ,Al-shaqsi [2] defined the following integral operator:

(1.5)

We also note that the operator   defined by ([1.5](https://www.hindawi.com/journals/ijmms/2014/260198/#EEq1.1)) can be expressed by the series expansion as follows :

|  |
| --- |
| . (1.6) |

Moreover, from ([1.6](https://www.hindawi.com/journals/ijmms/2014/260198/#EEq1.2)), it follows that

(1.7)

Ali et al.[1] obtained sufficient conditions for certain normalized analytic functions to satisfy

where and are given univalent functions in U with . Also, Tuneski [9] obtained sufficient conditions for starlikeness of in terms of the quantity .Recently, Shanmugam et al.[7,8], Goyal et al .[4] also obtained sandwich results for certain classes of analytic functions.

The main object of the present paper is to find sufficient conditions for certain normalized analytic functions f to satisfy:

,

and

,

where q1 and q2 are given univalent functions in U with q1(0)= q2(0)= 1.

**2-Preliminaries :**

In order to prove our subordination and superordination result , we need the following definition and lemmas.

**Definition 2.1 [5] :** Denote by Q the set of all functions that are analytic and injective on where

(2.1)

and are such that (ξ) ≠0 for ξ ∈∂U \ E(*f*).

**Lemma 2.1 [5] :** Let q be univalent in the unit disk U and let θ and be analytic in a domain D containing q(U) with when Set

Suppose that

1. is starlike univalent in ,
2. Re for .

If is analytic in withand

(2.2)

then and is the best dominant of (2.2).

**Lemma 2.2 [6]:** Let q be convex univalent in function in U and let with

If is analytic in , and

(2.3)

then and is the best dominant of (2.3).

**Lemma 2.3 [6]:** Let q be convex univalent in U and let , further assume that Re . If Q and is univalent in U, then

(2.4)

which implies that and q is the best subordinant of (2.4).

**Lemma 2.4 [3]:** Let q be convex univalent in the unit disk U and let be analytic in domain D containing q . Suppose that

1. Re
2. Q.

If

is univalent in U and

, (2.5)

then and is the best subordination of (2.5).

**3- Subordination Results :**

**Theorem 3.1 :** Let q be convex univalent in U with

Re . (3.1)

If the subordination

(3.2)

then

(3.3)

and is the best dominant of (3.2).

**Proof** : Define the function p by

(3.4)

Differentiating (3.4) with respect to z logarithmically, we get

(3.5)

Now , in veiw of (1.7), we obtain the following subordination

therefore ,

The subordination (3.2) from the hypothesis becomes

An application of Lemma 2.2 with and

Putting in Theorem 3.1 ,we obtain the following

**Corollary 3.1** : Let

Re .

If satisfies the subordination

then

and is the best dominant.

**Theorem 3.2** : Let q be convex univalent in U with and assume that q satisfies

Re , (3.6)

where and .

Suppose that -is starlike univalent in U, if satisfies:

, (3.7)

where

, (3.8)

then

, (3.9)

and q(z)is the best dominant of (3.7).

**Proof**: Define the function p by

, (3.10)

by setting :

.

We see that is analytic in is analytic in and that .

Also, we get

and

It is clear that is starlike univalent in U ,

By a straightforword computation , we obtain

, (3.11)

where is given by (3.8).

From (3.7) and (3.11), we have

. (3.12)

Therefore , by Lemma 2.1, we get . By using (3.10) , we obtain the result .

Putting (-1 ) in Theorem 3.2 , we obtain the following corollary :

**Corollary3.2 :** Let -1 and

where and if satisfies

and is given by (3.8),

and is the best dominant.

**4-Superordination results :**

**Theorem 4.1:** Let q be convex univalent in U with

,

and ,

be univalent in U . If

, (4.1)

then

(4.2)

and q is the best subordinant of (4.1).

**Proof**: Define the function p by

. (4.3)

Differentiating (4.3) with respect to z logarithmically , we get

(4.4)

After some computations and using (1.7) , from (4.4), we obtain

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and now , by using Lemma 2.3, we get the desired result .

Putting in Theorem 4.1 , we obtain the following corollary :

**Corollary 4.1:** Let and Re

and

,

be univalent in U . If

,

then

and

is the best subordinant.

**Theorem 4.2:** Let q be convex univalent in U with

(4.5)

where z .

Suppose that is starlike univalent in U , let

,

and where is given by (3.8). If

then

and q is the best subordinant of

**Proof**: Define the function p by

, (4.8)

by setting

and

we see that is analytic in is analytic in and that *w* . Also we get

.

It is clear that is starlike univalent in U ,

By a straightforword computation ,we obtain

(4.9)

where is given by (3.8).

From (4.6) and (4.9) , we have

. (4.10)

Therefore , by Lemma 2.4, we get . By using (4.8), we obtain the result .

**5-Sandwich Results :**

Concluding the results of differential subordination and superordination we arrive at the following ''sandwich result''.

**Theorem 5.1** : Let q1 be convex univalent in U with q1(0)=1,Re {} and let q2 be univalent in U ,q2(0)=1 and , let

,

and

be univalent in U . If

,

and are respectively , the best subordinant and the best dominant .

**Theorem 5.2:** Let q1 be convex univalent in U with q1(0)=1, and satisfies (4.5), let q2 be univalent in U q2(0)=1, satisfies (3.6), let

And is univalent in U . Where is given by (3.8) . If

then

And are respectively , the best subordinant and the best dominant .

**References:**

1. R. M. Ali, V. Ravichandran, M.H. Khan and K.G. Subramanian, Dierential sandwich theorems for certain analytic functions, Far East J. Math. Sci., 15(1) (2004), 87-94.
2. K. AL-Shaqsi ; Strong Differential Subordinations Obtained with New Integral Operator Defined by Polylogarithm Function ,Int. J . Math. Math. Sci.,Volume 2014, Article ID 260198, 6pages.
3. T. Bulboaca, Classes of first order differential superordinations, Demonstratio Math., 35(2) (2002), 287-292.
4. S.P. Goyal,P.Goswami and H. Silverman, Subordination and superordination results for a class of analytic multivalent functions,Int.J.Math.Math. Sci., Article ID 561638,(2008),1-12.
5. S. S. Miller and P. T. Mocanu, Differential Subordination : Theory and Applications, Series on Monographs and Textbooks in Pure and Applied Mathematics (Vol. 225), Marcel Dekker Inc., New York and Basel, 2000.
6. S. S. Miller, P. T. Mocanu, Subordinates of di erential superordina-tions, Complex Variables,48(10)(2003),815-826.
7. T. N. Shanmugam, V. Ravichandran and S. Sivasubramanian, Differential sandwich theorems for some subclasses of analytic functions, Aust. J.Math. Anal. Appl., 3 (1) (2006), 1-11.
8. T. N. Shanmugam, S. Shivasubramanian and H. Silverman, On sandwich theorems for some classes of analytic functions, Int. J. Math. Math. Sci., Article ID 29684 (2006), 1– 13.
9. N. Tuneski, On certain sufficient conditions for starlikeness, Int. J. Math. Math. Sci., 23(8) (2000), 521-527.