**On Differential Sandwich Results For Analytic Functions**

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**Abstract:** In this paper ,we obtain some subordination and superordination results involving the integral operator .Also,we get Differential sandwich results for classes of univalent functions in the unit disk.

**Keywords:**Analytic function, univalent function, differential subordination , superordination.

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**1-Introduction :**

Let H=H(U) be the class of analytic functions in the open unit disk For n a positive integer and Let H[a, n] be the subclass of H consisting of functions of the form:

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| --- |
| (a ). (1.1) |

Also, let A be the subclass of H consisting of functions of the form:

|  |
| --- |
|  (1.2) |

Let . The function is said to be subordinate to , or is said to be subordinate to , if there exists a Schwarz function *w* analytic in U with and |*w*(z)| < 1 ( ), such that , In such a case we write . If is univalent in , then if and only if and

Let : If p and are univalent functions in U and if p satisfies the second-order differential superordination.

|  |
| --- |
|  (1.3) |

then p is called a solution of the differential superordination (1.3). ( If *f* is subordinate to , then is superordinate to *f* ) . An analytic function q is called a subordinant of (1.3) , if for all the functions p satisfying (1.3).

An univalent subordinant that satisfies for all the subordinants q of (1.3) is called the best subordinant. Miller and Mocanu [5] have obtained conditions on the functions and for which the following implication holds :

|  |
| --- |
|  (1.4) |

For  ,Al-shaqsi [2] defined the following integral operator:

 (1.5)

We also note that the operator   defined by ([1.5](https://www.hindawi.com/journals/ijmms/2014/260198/#EEq1.1)) can be expressed by the series expansion as follows :

|  |
| --- |
| . (1.6) |

Moreover, from ([1.6](https://www.hindawi.com/journals/ijmms/2014/260198/#EEq1.2)), it follows that

 (1.7)

Ali et al.[1] obtained sufficient conditions for certain normalized analytic functions to satisfy

where and are given univalent functions in U with . Also, Tuneski [9] obtained sufficient conditions for starlikeness of in terms of the quantity .Recently, Shanmugam et al.[7,8], Goyal et al .[4] also obtained sandwich results for certain classes of analytic functions.

The main object of the present paper is to find sufficient conditions for certain normalized analytic functions f to satisfy:

,

and

,

where q1 and q2 are given univalent functions in U with q1(0)= q2(0)= 1.

**2-Preliminaries :**

In order to prove our subordination and superordination result , we need the following definition and lemmas.

**Definition 2.1 [5] :** Denote by Q the set of all functions that are analytic and injective on where

 (2.1)

and are such that (ξ) ≠0 for ξ ∈∂U \ E(*f*).

**Lemma 2.1 [5] :** Let q be univalent in the unit disk U and let θ and be analytic in a domain D containing q(U) with when Set

Suppose that

1. is starlike univalent in ,
2. Re for .

If is analytic in withand

(2.2)

then and is the best dominant of (2.2).

**Lemma 2.2 [6]:** Let q be convex univalent in function in U and let with

If is analytic in , and

 (2.3)

then and is the best dominant of (2.3).

**Lemma 2.3 [6]:** Let q be convex univalent in U and let , further assume that Re . If Q and is univalent in U, then

 (2.4)

which implies that and q is the best subordinant of (2.4).

**Lemma 2.4 [3]:** Let q be convex univalent in the unit disk U and let be analytic in domain D containing q . Suppose that

1. Re
2. Q.

If

is univalent in U and

, (2.5)

then and is the best subordination of (2.5).

**3- Subordination Results :**

**Theorem 3.1 :** Let q be convex univalent in U with

Re . (3.1)

If the subordination

 (3.2)

then

 (3.3)

and is the best dominant of (3.2).

**Proof** : Define the function p by

 (3.4)

Differentiating (3.4) with respect to z logarithmically, we get

 (3.5)

Now , in veiw of (1.7), we obtain the following subordination

therefore ,

The subordination (3.2) from the hypothesis becomes

An application of Lemma 2.2 with and

Putting in Theorem 3.1 ,we obtain the following

**Corollary 3.1** : Let

Re .

If satisfies the subordination

then

and is the best dominant.

**Theorem 3.2** : Let q be convex univalent in U with and assume that q satisfies

Re , (3.6)

where and .

Suppose that -is starlike univalent in U, if satisfies:

 , (3.7)

where

, (3.8)

then

, (3.9)

and q(z)is the best dominant of (3.7).

**Proof**: Define the function p by

, (3.10)

by setting :

 .

We see that is analytic in is analytic in and that .

Also, we get

and

It is clear that is starlike univalent in U ,

By a straightforword computation , we obtain

, (3.11)

where is given by (3.8).

From (3.7) and (3.11), we have

. (3.12)

Therefore , by Lemma 2.1, we get . By using (3.10) , we obtain the result .

Putting (-1 ) in Theorem 3.2 , we obtain the following corollary :

**Corollary3.2 :** Let -1 and

where and if satisfies

and is given by (3.8),

and is the best dominant.

**4-Superordination results :**

**Theorem 4.1:** Let q be convex univalent in U with

,

 and ,

be univalent in U . If

, (4.1)

then

 (4.2)

and q is the best subordinant of (4.1).

**Proof**: Define the function p by

. (4.3)

Differentiating (4.3) with respect to z logarithmically , we get

 (4.4)

After some computations and using (1.7) , from (4.4), we obtain

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and now , by using Lemma 2.3, we get the desired result .

Putting in Theorem 4.1 , we obtain the following corollary :

**Corollary 4.1:** Let and Re

and

,

be univalent in U . If

 ,

then

and

 is the best subordinant.

**Theorem 4.2:** Let q be convex univalent in U with

 (4.5)

where z .

Suppose that is starlike univalent in U , let

,

and where is given by (3.8). If

then

and q is the best subordinant of

**Proof**: Define the function p by

, (4.8)

by setting

 and

we see that is analytic in is analytic in and that *w* . Also we get

.

It is clear that is starlike univalent in U ,

By a straightforword computation ,we obtain

 (4.9)

where is given by (3.8).

From (4.6) and (4.9) , we have

 . (4.10)

Therefore , by Lemma 2.4, we get . By using (4.8), we obtain the result .

**5-Sandwich Results :**

Concluding the results of differential subordination and superordination we arrive at the following ''sandwich result''.

**Theorem 5.1** : Let q1 be convex univalent in U with q1(0)=1,Re {} and let q2 be univalent in U ,q2(0)=1 and , let

 ,

and

be univalent in U . If

,

and are respectively , the best subordinant and the best dominant .

**Theorem 5.2:** Let q1 be convex univalent in U with q1(0)=1, and satisfies (4.5), let q2 be univalent in U q2(0)=1, satisfies (3.6), let

And is univalent in U . Where is given by (3.8) . If

then

And are respectively , the best subordinant and the best dominant .

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