



## Research paper

## Power losses in two-degrees-of-freedom planetary gear trains: A critical analysis of Radzimovsky's formulas

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## ABSTRACT

Two-degrees-of-freedom gearing are key elements of any planetary gear train. Thus, the full understanding of their basic mechanics and mechanical efficiency evaluation is not only of significant theoretical interest, but important in many industrial applications. Some previous investigations on the estimation of mechanical efficiency in two-degrees-of-freedom differential gearing rely upon the relationships deduced by Radzimovsky. However, to the best of authors' knowledge, a critical analysis on the validity conditions of these relationships is not available yet in technical literature. To fill the apparent gap, in this paper a broad physical interpretation of mechanical efficiency analysis of two-dof planetary gear trains is offered. The discussion allowed to outline the theoretical limits of the Radzimovsky formulas. The current approach provides three different results for the power loss of a two-input gear pair entity with the planet carrier as the output link. This leads to the conclusion that even for the same PGT and for the same input and output links the power loss has, for each valid sequence of angular velocities, a peculiar mathematical expression.

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## 1. Introduction

Modern hybrid gear drives can be considered as an assembly of two-dof planetary gear trains (PGT). As it will be herein discussed, the basic mechanics of a two-dof basic PGT may show not obvious features that limit the validity of classic formulas for computing its mechanical efficiency.

Apparently, the earliest contribution on the computation of mechanical efficiency in two dof PGT is reported in classic and often cited companion papers authored by Radzimovsky [1,2]. In this paper, through a critical analysis of Radzimovsky's formulas, their limitations will be outlined and explained.

A conventional approach for calculating the mechanical efficiency<sup>1</sup> of a two-dof GPE is based on the assumption that the gear train is a combination of two one-dof inversions of the gear train (e.g. [1–8]). However, as it will be herein shown, under certain circumstances the relationship among the powers flowing in a PGT cannot be treated as the sum of two separate powers flowing through the links of a gear drive having one of its driving links fixed.

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E-mail address: [anahed.juber@qu.edu.iq](mailto:anahed.juber@qu.edu.iq) (A. Hussein Juber).<sup>1</sup> In the present analysis we will assume members with constant velocities and meshing losses only. Friction among teeth will be also constant.

## Nomenclature

PGT:	Planetary gear train
One-dof:	One-degree-of-freedom
Two-dof:	Two-degrees-of-freedom
GPE:	Gear pair entity
link- $i$ and link- $j$ :	Meshing gears
link- $k$ :	Gear carrier
$L_{i(j-k)}$ :	Power loss of the GPE when link- $i$ is fixed and links $j$ and $k$ are input and output links, respectively.
$L_{(i,j-k)}$ :	Power loss of the GPE when operating with links $i$ and $j$ as input and $k$ as output
$P_x = T_x \omega_x$ :	Power through link- $x$ . If $P_x > 0$ , the link- $x$ is an input or driving link. Conversely, If $P_x < 0$ , the link- $x$ is an output or driven link.
$P_x^y = T_x(\omega_x - \omega_y)$ :	Potential or virtual power of link $x$ when link $y$ is considered fixed
$N_{j,i} = \pm \frac{Z_j}{Z_i}$ :	Planet gear ratio (+: internally meshing gears, -: externally meshing gears)
$T_x$ :	Torque on link $x$
$Z_x$ :	Number of teeth on gear $x$
$\omega_x$ :	Absolute angular velocity of link $x$
$\eta_E$ :	Planetary (Epicyclic) gear train mechanical efficiency
$\eta_{x(y-z)}$ :	Mechanical efficiency of the GPE when link $x$ is fixed and links $y$ and $z$ are driving and driven links, respectively.
$\eta_{(x,y-z)}$ :	Mechanical efficiency of the GPE when operating with links $x$ and $y$ as driving links and $z$ as driven.
$\eta_{(x-y,z)}$ :	Mechanical efficiency of the GPE when operating with links $y$ and $z$ as driven links and $x$ as driving link.

Pennestrì and Freudenstein [4], in their systematic method for mechanical efficiency analysis of planetary gear trains, combined the formulas which were used for estimating the efficiency of one-dof planetary gear train inversions deduced by Merritt [9] and Macmillan [10] with the Radzimovsky's formulas for two-dof differential gearing.

With regard to the efficiency of two-dof planetary gear trains, Pennestrì and Valentini [5] compared several gear train mechanical efficiency formulas. In the same reference was offered an interpretation of the analytical relations originally deduced by Radzimovsky [1,2] based on the general power balance equation of two parallel machine units.

Pennestrì et al. [6] expanded the formulas proposed by Pennestrì and Freudenstein [4], and applied them to power split transmissions for hybrid vehicles.

Mantriota and Pennestrì [11] proposed and experimentally validated load dependent losses model for epicyclic gear trains. Coaccioli and Pennestrì [8] evaluated the mechanical efficiency of an epicyclic gear drive in hybrid scooters based on graph algorithm.

The use of kinematic inversion in epicyclic gear train analysis has a well-established tradition. For instance, with reference to velocity analysis, in the tabular method the angular speeds of the links are obtained in two steps. First, the gear carrier (or arm) is assumed fixed. Then the relative velocity ratios are set equal to gear ratios, as in ordinary gear trains. The resulting system of equations is solved for the unknown angular velocities.

Merritt [9,12] has been the first to recognize the usefulness of kinematic inversion to obtain the mechanical efficiencies and torque ratios. He also compiled a table of torques and mechanical efficiencies for all the epicyclic arrangements of a basic train.

Macmillan also computes the algebraic expressions of efficiencies for all the basic gear train inversions [10]. Similar tables, with an increased number of entries, have been reported by Pollone [13], Müller [14], Maggiore [15], Monastero [16], Pennestrì et al. [4,6], and Esmail [17].

Macmillan observes that the torques acting on the links and power losses are independent of the observer's motion. The following quote is taken from Macmillan [18]

*...our analysis is based upon an important principle relating to torques and the power lost in friction; this is the fact that magnitudes of the torques acting upon the various members of the gear are quite independent of the motion of the observer who measures them. In addition, the power lost, being determined solely by the internal torques and the relative motions of the wheels within the gear, is also independent of the observer's motion.*

Early applications of kinematic inversion to obtain the mechanical efficiency of epicyclic gear trains are also due to Poppinga [19], Terplan [20], Kudlyavtzev [21], and Looman [22].

More recent applications of the kinematic inversion principle outlined by Macmillan to obtain power losses in epicyclic gearing are due to Yu and Beachley [23] and, Chen and Angeles [24]. In particular, Yu and Beachley [23] introduced the term *latent power* or *gearing power* to denote the power through a link assuming the observer rotating with the gear carrier.

Chen and Angeles [24] coined the term *virtual power* as the power measured in an arbitrary moving frame. Hence, latent power coincides with the virtual power passing through a gear mesh at the corresponding carrier frame. They proposed an algorithm, based on the detection of virtual power flow, to compute the power loss and the efficiency of a PGT. The choice of gear carrier as observer’s frame was usually privileged in many investigations. In fact, from such a frame the PGT appears as an ordinary gear train and evaluation of power losses is greatly simplified. Laus et al. [25] proposed a method to determine the efficiency of complex gear trains based on graph and screw theories. Esmail [17] resorted to kinematic inversion for his method of PGT efficiency analysis and coined the term *potential power* which corresponds to the power  $P_w^j = T_w(\omega_w - \omega_j)$  transmitted by link- $w$  in a moving reference frame at which link  $j$  appears fixed. Esmail also defined the *potential power ratio* as the ratio between the potential power  $P_w^j$  and the actual power  $P_w$  transmitted through link  $w$ . Jianying and Qingchun [7] performed efficiency analysis of 2K-H planetary gear transmissions.

Experimentally validated gear train efficiency analyses based on multibody dynamics approach are due to Pennestrì and Mantriota [11], Chen [26], Chen and Chen [27], and Mohammadpour et al. [28],

Thoughtful results on efficiency analysis of two-dof PGT have been reported in several investigations by Chen and Liang [29], Chen [30], Chen and Chen [31], Davies et al. [32], Esmail and Hassan [33], and Esmail [17] [34].

Reviews of contributions on graph-based PGT analysis are due to Belfiore and Pennestrì [35], and Xue et al. [36].

Although the analytical expressions adopted for the estimation of efficiency in two-dof differential gearing [4,5,7,9] are in agreement with those deduced by Radzimovsky [1,2], the relationships involved do not cover all possible gear train operating conditions [17]. In this paper a critical analysis of Radzimovsky’s formulas is proposed.

## 2. Kinematically driven and torque driven two degrees-of-freedom PGT

In two-dof PGTs we can distinguish between kinematically driven and torque driven transmissions.

To simplify the analytical treatment, in this section we assume absence of losses.<sup>2</sup> For a GPE block a system of three fundamental equations can be established. The system is composed of the Willis’ equation, the external torques equilibrium and power balance condition. There are six unknowns, namely the angular speeds  $\omega_i, \omega_j, \omega_k$  and external torques  $T_i, T_j, T_k$ .

There are two possibilities of obtaining a solution of the previous system:

- Assign two angular speeds and one external torque. In this case the system will be named *kinematically driven*.
- Assign two torques and one angular speed. In this case the system will be named *torque driven*.

For a driving (or input) link  $w$   $P_w = T_w\omega_w > 0$ . Conversely, for a driven (or output) link  $P_w = T_w\omega_w < 0$ .

We observe that links with prescribed speeds (or torques) are not necessarily driving links. Moreover, under a kinematic inversion, the role of driving or driven link may switch.

## 3. The Radzimovsky formulas

Since the original references [1,2] may not be readily available to readers, the reasoning of Radzimovsky is summarized in this section.

Fig. 1 shows two kinds of gear pair entities (GPEs); with externally and internally meshing gears. Each GPE consists of two engaging gears with their carrier. The three links forming the GPE can rotate with power flowing at all three links. External and internal GPEs are the building blocks of compound and complex planetary gear trains.

In the block representation of the external GPE shown at the right of Fig. 2(a), the power enters the GPE through links  $i$  and  $j$  and leaves it from link  $k$ .

The planetary gear train is considered as an assembly of two component trains with distinct efficiencies. Two working modes are considered:

- two driving members ( $i$  and  $j$ ) and one driven member ( $k$ );
- one driving member ( $i$ ) and two driven members ( $j$  and  $k$ ).

With reference to the first working mode (see Fig. 2(a)), the angular speed of shaft  $k$  has two components

$$\omega_k = \tau_{ki}\omega_i + \tau_{kj}\omega_j = \omega'_k + \omega''_k \tag{1}$$

with

$$\tau_{ki} = \frac{1}{1 - N_{j,i}} \tag{2a}$$

$$\tau_{kj} = \frac{N_{j,i}}{N_{j,i} - 1} \tag{2b}$$

depending on gear teeth ratio.

<sup>2</sup> We assume that the presence of losses does not change the algebraic sign of powers.

Similarly, the power  $P_k$  delivered on driven shaft  $k$  is considered as the sum of components  $P'_k$  and  $P''_k$  due to powers  $P_i$  and  $P_j$ , respectively:

$$|P_k| = P'_k + P''_k \quad (3)$$

with

$$P'_k = \eta_{j(i-k)} P_i \quad (4a)$$

$$P''_k = \eta_{i(j-k)} P_j \quad (4b)$$

Hence, the efficiency of the overall system is

$$\eta_{(i,j-k)} = \frac{P'_k + P''_k}{P_i + P_j} = \frac{|P_k|}{\frac{P'_k}{\eta_{j(i-k)}} + \frac{P''_k}{\eta_{i(j-k)}}} \quad (5)$$

Without any explanation, Radzimovsky assumes that  $P'_k$  and  $P''_k$  are proportional to  $\omega'_k$  and  $\omega''_k$ , respectively

$$P'_k = \frac{P_k}{\omega_k} \omega'_k \quad (6a)$$

$$P''_k = \frac{P_k}{\omega_k} \omega''_k \quad (6b)$$

Substituting Eq. (6) into Eq. (5) and taking into account Eq. (4), follows the first Radzimovsky efficiency formula (see Eq. (11) of [2])

$$\eta_{(i,j-k)} = \frac{\omega_k}{\frac{\tau_{ki}\omega_i}{\eta_{j(i-k)}} + \frac{\tau_{kj}\omega_j}{\eta_{i(j-k)}}} \quad (7)$$

For the second working mode (see Fig. 2(b)), we consider member  $i$  as driving member and  $j$  and  $k$  as driven members. The angular speed of shaft  $i$  can be expressed as a linear combination of the two driven shafts  $k$  and  $j$

$$\omega_i = \tau_{ik}\omega_k + \tau_{ij}\omega_j = \omega'_i + \omega''_i \quad (8)$$

with  $\tau_{ik} = (1 - N_{j,i})$  and  $\tau_{ij} = N_{j,i}$  depending on gear teeth ratio.

The input power  $P_i$  has two components  $P'_i$  and  $P''_i$  transmitted to shafts  $k$  and  $j$ , respectively. Radzimovsky assumes again that these power components are proportional to angular speed components

$$P'_i = \frac{P_i}{\omega_i} \omega'_i \quad (9a)$$

$$P''_i = \frac{P_i}{\omega_i} \omega''_i \quad (9b)$$

Hence, the powers delivered at the driven shafts are

$$|P_k| = \eta_{(j,i-k)} P'_i \quad (10a)$$

$$|P_j| = \eta_{(i,j-k)} P''_i \quad (10b)$$

The efficiency of the overall system is

$$\eta_{(i-k,j)} = \frac{|P_k| + |P_j|}{P'_i + P''_i} \quad (11)$$

Substituting equalities (10) into Eq. (11) and taking into account Eqs. (8) and (9), follows the second Radzimovsky formula (see Eq. (25) of [2])

$$\eta_{(i-k,j)} = \frac{\eta_{j(i-k)}\tau_{ik}\omega_k + \eta_{k(i-j)}\tau_{ij}\omega_j}{\tau_{ik}\omega_k + \tau_{ij}\omega_j} \quad (12)$$

The mechanical efficiency, for the working mode shown in Fig. 2(a), can be written as:

$$\eta_E = \frac{|T_k\omega_k|}{T_i\omega_i + T_j\omega_j} \quad (13)$$

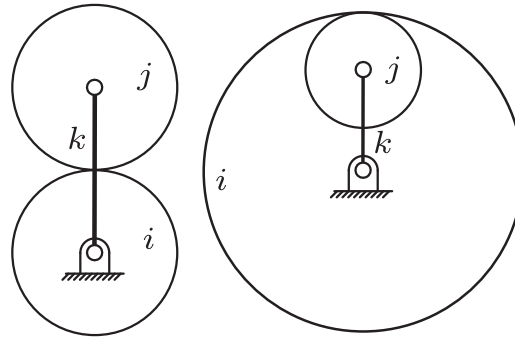


Fig. 1. A GPE (a) with externally meshing gears and (b) with internally meshing gears.

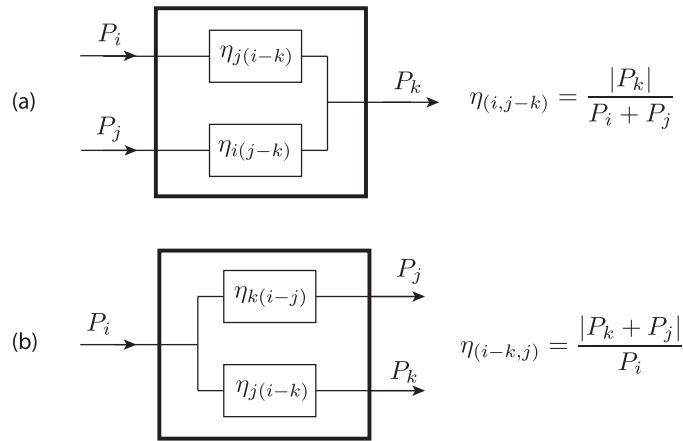


Fig. 2. Block representation of a 2 dof GPE: possible working modes.

Kasuba and Radzimovsky presented also an experimental validation of the formulas [37].

Pennestrì and Valentini [5] recognized formula (7) as the efficiency of a general mechanical system with two units connected in parallel (see Fig. 2(a)). For the case under consideration, the efficiencies of the two power paths are  $\eta_{j(i-k)}$  and  $\eta_{i(j-k)}$ , respectively.

In fact, expressed the gear unit overall mechanical efficiency as:

$$\eta_E = \frac{\eta_{j(i-k)}T_i\omega_i + \eta_{i(j-k)}T_j\omega_j}{T_i\omega_i + T_j\omega_j} \tag{14}$$

the algebraic form that follows is numerically equivalent to the first Radzimovsky formula (7).

Similarly, for the working mode shown in Fig. 2(b), the overall efficiency can be written as:

$$\eta_E = \frac{\eta_{k(i-j)}\omega'_i + \eta_{j(i-k)}\omega''_i}{\omega_i} \tag{15}$$

which is equivalent to the second Radzimovsky formula (12).

#### 4. Nomograph

Let us focus our attention on the two input GPE shown in Fig. 2(a). In order to visualize the angular speeds, torques, powers, and power flow direction, it is convenient to construct a nomograph for this case. Esmail [38,39] give a detailed description for drawing nomographs.

A nomograph of any GPE consists of three parallel scales, graduated for the angular velocities of the three links  $i$ ,  $j$ , and  $k$  such that an intersecting straight line enables related angular velocities to be read off directly.

With reference to the first working mode, Fig. 3 shows the nomograph drawn for the GPE shown in Fig. 1, with the axes labeled with the corresponding speeds;  $\omega_i$ ,  $\omega_j$  and  $\omega_k$ . The spacing of the lines is shown here as  $N_{j,i} = -\frac{Z_j}{Z_i}$  and 1. Herein, the minus (plus) sign is used for externally (internally) meshing gears  $i$  and  $j$  whereas  $Z_j$  and  $Z_i$  are number of teeth on gears  $j$  and  $i$ , respectively. With reference to the second working mode, Fig. 4 shows the nomograph drawn for the GPE with

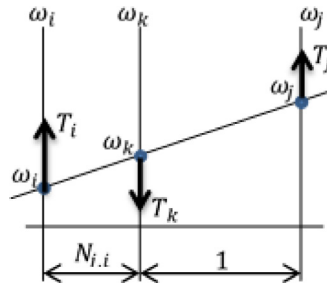


Fig. 3. Nomograph for the GPE shown in Fig. 1(a).

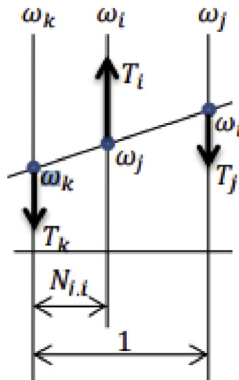


Fig. 4. Nomograph for the GPE shown in Fig. 1(b).

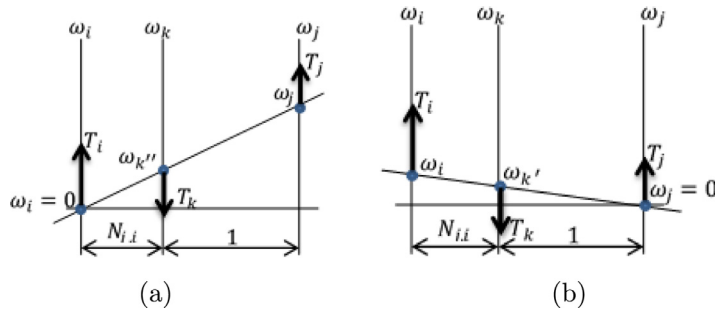


Fig. 5. Nomographs for the two one-dof paths of the power flow through the PGT.

internally meshing gears shown in Fig. 1(b). The axes are labeled with the corresponding speeds  $\omega_i$ ,  $\omega_j$  and  $\omega_k$ . The spacing of the lines is shown here as  $N_{j,i} = \frac{z_j}{z_i}$  and 1. To perform the power flow analysis, torque vectors are drawn according to methodology discussed by Esmail [10]. Downward torque vectors are considered to be negative. Power flowing out of the system is also considered to be negative.

**5. Radzimovsky’s approach (first working mode)**

Since it was not possible to calculate directly the efficiency of the two-dof planetary gear train, the power had been assumed to be fed in the system in two different one-dof paths. With reference to the first working mode, the first and second paths were followed by the input power from link  $i$  and link  $j$ , as shown in Fig. 5(a) and 5(b), respectively.

Therefore, the efficiency of the two-dof planetary gear train was calculated indirectly by using two one-dof trains as follows: The meshing losses for the two one-dof paths of the power flow can be written as:

$$L_{j(i-k)} = (\eta_{j(i-k)} - 1)T_i\omega_i \tag{16a}$$

$$L_{i(j-k)} = (\eta_{i(j-k)} - 1)T_j\omega_j \tag{16b}$$

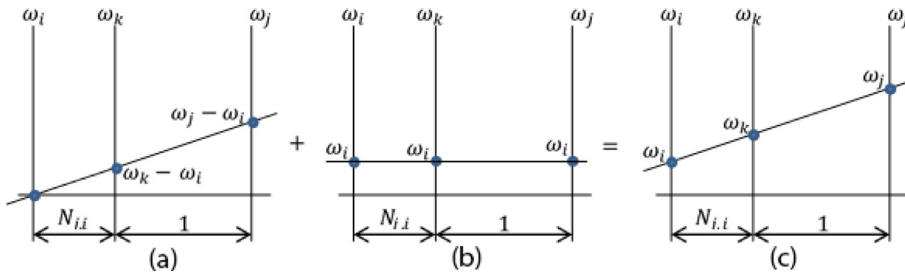


Fig. 6. (a) First partial velocity, (b) second partial velocity, and (c) the total velocity of the links of the external GPE shown in Fig. 1.

By adding Eq. (16), the total power losses in the fixed reference frame can be written as follows:

$$L_{(i,j-k)} = (\eta_{j(i-k)} - 1)T_i\omega_i + (\eta_{i(j-k)} - 1)T_j\omega_j \tag{17}$$

By substituting these losses in the overall efficiency expression of the two-dof PGT with driving links  $i$  and  $j$ , we get the formula

$$\eta_{(i,j-k)} = \frac{T_i\omega_i + T_j\omega_j + L_{(i,j-k)}}{T_i\omega_i + T_j\omega_j} = \frac{\eta_{j(i-k)}T_i\omega_i + \eta_{i(j-k)}T_j\omega_j}{T_i\omega_i + T_j\omega_j}$$

which coincides with that originally proposed by Radzimovsky and previously expressed by Eq. (14).

The first operating condition for which Radzimovsky’s equation fails to correctly estimate the power losses is that when

$$\omega_i = \omega_j = \omega_k \neq 0 \tag{18}$$

Referring to Eq. (17), it is worth emphasizing that even when there is no relative motion between the meshing gears, the estimated power losses by Radzimovsky’s formula do not vanish. In fact, when the gear train rotates as a rigid coupling, the power is transmitted without losses, and the power meshing losses are zero, i.e.

$$L_{(i,j-k)} = 0 \tag{19}$$

### 6. Potential power approach (first working mode)

When the angular velocity ratio between members  $i$  and  $j$  is unit, the theoretical efficiency becomes 100%. To satisfy this explicit kinematic condition, we introduced the potential power approach to calculation the power losses. In a two-dof planetary gear train the absolute angular velocity of a generic link can be considered as a superposition of two partial motions. The first partial motion is the rotation of a link  $u$  (or  $w$ ) relative to a second link  $v$  and is herein called the **relative velocity**. The first partial velocity of link  $v$  relative to itself is zero, while the absolute velocity of link  $v$  is not necessarily equal to zero. The second partial motion is an equal rotation of all links of the planetary gear train that is equal to the velocity of link  $v$  and is herein called the **coupling velocity**. No relative motion occurs and the gear train rotates as a rigid coupling. If the two partial motions are superimposed, the absolute velocity of each link is obtained as the algebraic sum of its two partial velocities, that is

$$\omega_u = (\omega_u - \omega_v) + \omega_v \tag{20a}$$

$$\omega_w = (\omega_w - \omega_v) + \omega_v \tag{20b}$$

$$\omega_v = (\omega_v - \omega_v) + \omega_v \tag{20c}$$

For the case shown in Fig. 3, let  $u \equiv k$ ,  $w \equiv j$ , and  $v \equiv i$ , then the first partial velocity, second partial velocity, and the total velocity of the links of the GPE shown in Fig. 1 can be represented as in Fig. 6.

The two partial motions transmit their associated partial powers by two different principles as will be subsequently shown. In the relative partial motion, the power is transmitted from the driving link to the driven link solely through a one-dof planetary gear train in which link  $v$  is the fixed link. This relative partial power shall be called the potential power of the link in a moving reference frame in which link  $v$  is relatively fixed. Thus, the potential power represents a partial power which is transmitted by links  $u$  and  $w$  of the planetary gear train and can be obtained for each of these links by calculating the product of their velocities relative to link  $v$  and their torques. Within the gear train, the potential-power flows from the link that transmits the positive potential power to the link transmitting the negative potential power.

When the planetary gear train operates solely in the second partial motion it behaves like a rigid coupling. The three connected shafts rotate with the same angular velocity  $\omega_v$  and the power is transmitted through the links of the gear train without losses.

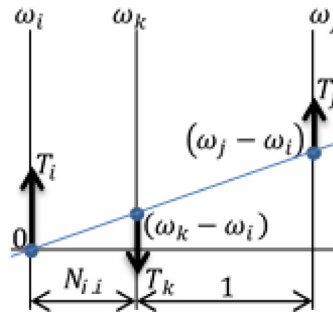


Fig. 7. Nomograph for power flow through the GPE as observed by an observer located on link *i*.

Thus, the meshing losses for the two-dof PGT can be computed considering the potential power of the one-dof PGT in the moving reference frame in which link *v* is relatively fixed.

In the potential power method, the coupling-link (the link with zero relative velocity or the relatively “grounded” link) is selected such that the sign of the relative velocities of the other two links remains the same as the sign of their original velocity. To achieve this the coupler-link should be the one with the lowest absolute angular velocity. We take special note of the advantage of our choice of system. An observer located on the coupling-link would observe that the driving and driven links remain unchanged.

In case of setting a constant link (e.g. the carrier as in the previous works) as a coupler-link, then a need arises to know which of the remaining two links becomes the driving link in the new moving reference frame. The driving link is essential in calculating the power losses. *The fact that a link is the driver, in the fixed reference frame or in a certain moving reference frame, does not necessarily mean that it will maintain the same function in another moving reference frame.*

For the case represented in Fig. 7, we have

1.  $\omega_j > \omega_k > \omega_i > 0$
2.  $T_j\omega_j > 0, T_j > 0$  and link *j* is a driving link.
3.  $T_k\omega_k < 0, T_k < 0$  and link *k* is a driven link.

**Case 1.** If link-*i* is the coupler-link, since  $\omega_j > \omega_i$ , then  $\omega_j - \omega_i > 0$ . Similarly, if  $\omega_k > \omega_i$ , then  $\omega_k - \omega_i > 0$ . The relative partial velocities  $(\omega_j - \omega_i)$ , and  $(\omega_k - \omega_i)$  of links *j* and *k* relative to link *i* are still larger than zero (positive). Since  $T_j > 0$  and  $(\omega_j - \omega_i) > 0$ , consequently  $T_j(\omega_j - \omega_i) > 0$  and link *j* is still a driving link. Similarly, since  $T_k < 0$ , and  $(\omega_k - \omega_i) > 0$ , then  $T_k(\omega_k - \omega_i) < 0$  and link-*k* is still a driven link. In summary, links *j* and *k* maintain their original role of driving and driven links, respectively, also under a kinematic inversion.

**Case 2.** If the carrier is the coupling-link, then, since  $\omega_j > \omega_k$ , we have  $(\omega_j - \omega_k) > 0$ . Similarly, for  $\omega_k > \omega_i$ , we have  $(\omega_i - \omega_k) < 0$ . Since  $T_j > 0$ , and  $(\omega_j - \omega_k) > 0$ , then  $T_j(\omega_j - \omega_k) > 0$  and link-*j* is still a driving link. Since  $T_i > 0$ , and  $(\omega_i - \omega_k) < 0$ , then  $T_i(\omega_i - \omega_k) < 0$  and link-*i*, which was a driving link in the absolute motion, becomes a driven link under a kinematic inversion.

**Case 3.** If link-*j* is the coupling-link, then since  $\omega_j > \omega_i$ , we have  $(\omega_i - \omega_j) < 0$ . Similarly, for  $\omega_j > \omega_k$ , we have  $(\omega_k - \omega_j) > 0$ . Since  $T_i > 0$ , and  $(\omega_i - \omega_j) < 0$ , consequently  $T_i(\omega_i - \omega_j) < 0$ , and link-*i* becomes a driven link under a kinematic inversion. Since  $T_k < 0$ , and  $(\omega_k - \omega_j) < 0$ , then  $T_k(\omega_k - \omega_j) > 0$  and link-*k*, which was a driven link in the absolute motion, becomes a driving link under a kinematic inversion.

The losses in link-*i* moving reference frame, shown in Fig. 7, can be computed from the potential power as follows

$$L_{(i,j-k)} = (\eta_{i(j-k)} - 1)T_j(\omega_j - \omega_i) \tag{21}$$

The term  $T_j(\omega_j - \omega_i)$  is the potential power which the PGE would have if link-*i* were relatively fixed, but is only a part of link-*j* input power  $T_j\omega_j$  due to the difference in magnitude between  $\omega_j$  and  $\omega_i$  where both velocities are positive.

The losses estimated by Radzimovsky’s approach are given by Eq. (17) as

$$L_{(i,j-k)} = (\eta_{j(i-k)} - 1)T_i\omega_i + (\eta_{i(j-k)} - 1)T_j\omega_j$$

Comparing Eq. (21) with Eq. (17), we first recognized that the second term on the right side of Eq. (17) is just a part of the total power losses estimated by Radzimovsky’s formula. This part alone is greater than the total losses estimated by the method based on the potential or virtual power (see Eq. (21)), i.e., since  $\omega_j > (\omega_j - \omega_i)$  and  $T_j > 0$ , then  $T_j\omega_j > T_j(\omega_j - \omega_i)$ . By multiplying the last inequality by  $(\eta_{i(j-k)} - 1)$ , we get:

$$(\eta_{i(j-k)} - 1)T_j\omega_j < (\eta_{i(j-k)} - 1)T_j(\omega_j - \omega_i) \tag{22}$$

The right side of this inequality represents the total losses computed by the potential power method (Eq. (21)) while the left side, which is a fraction of the total power losses, is estimated by Radzimovsky’s formula (17). It is clear that, for the case under consideration, Radzimovsky’s formula overestimates the power losses.



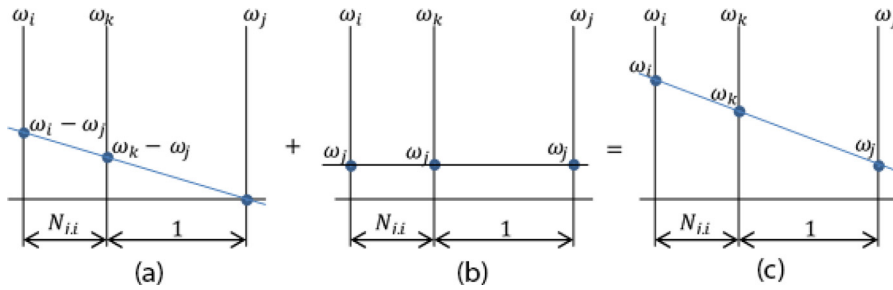


Fig. 8. (a) first partial velocity, (b) second partial velocity, and (c) the total velocity of the links of the external GPE shown in Fig. 1 (case 3).

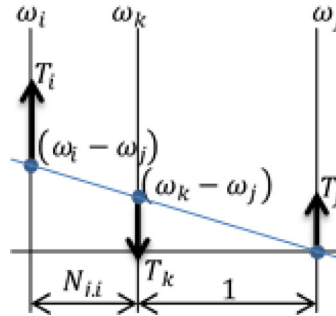


Fig. 9. Nomograph for power flow through the GPE as observed by an observer located on link-*j*.

A critical observation from (21) is that power loss is a function of the potential power rather than the actual powers. The fact that the relative partial velocity is always proportional to the power losses is an important key to the solution of two-dof efficiency problems.

**7. Influence of the operating conditions on power losses (first working mode)**

Since the power losses depend on relative partial velocities, it is generally necessary to solve for velocities before the power loss calculations can be made. For the clarity of the concepts outlined in the preceding sections, the step for solving for the velocities has been delayed. In the present paper the nomograph method was used to solve for the relation between the angular velocities of the links of the GPE. For the GPE shown in Fig. 1, we can distinguish the following three operating conditions:

1.  $\omega_j > \omega_k > \omega_i > 0$ , shown in Fig. 3.
2.  $\omega_j = \omega_k = \omega_i \neq 0$  and,
3.  $\omega_i > \omega_k > \omega_j > 0$ , shown in Fig. 8 (c).

The losses in link-*j* moving reference frame, shown in Fig. 9, can be calculated from the potential power as follows:

$$L_{(i,j-k)} = (\eta_{j(i-k)} - 1)T_i(\omega_i - \omega_j) \tag{23}$$

This leads to the important observation that even for the same PGT and for the same driving and driven links, we cannot write down a proper expression for the power losses unless the kinematic conditions on angular velocities is considered. In contrast to Radzimovsky's approach, which gives only one equation, the current approach gives three different results for the power losses of the GPE with links *i* and *j* as the driving links and link *k* as the driven link. In summary, these are:

$$L_{(i,j-k)} = (\eta_{i(j-k)} - 1)T_j(\omega_j - \omega_i) \quad \omega_j > \omega_k > \omega_i > 0 \tag{24a}$$

$$L_{(i,j-k)} = 0 \quad \omega_j = \omega_k = \omega_i > 0 \tag{24b}$$

$$L_{(i,j-k)} = (\eta_{j(i-k)} - 1)T_i(\omega_i - \omega_j) \quad \omega_i > \omega_k > \omega_j > 0 \tag{24c}$$

**8. Radzimovsky's approach (second working mode)**

With reference to the second working mode, the first and second paths were followed by the driven power from link-*k* and link-*j*, as shown in Fig. 10(a) and 10(b), respectively. The efficiency of the two-dof planetary gear train was calculated indirectly by using two one-dof gear trains as follows:

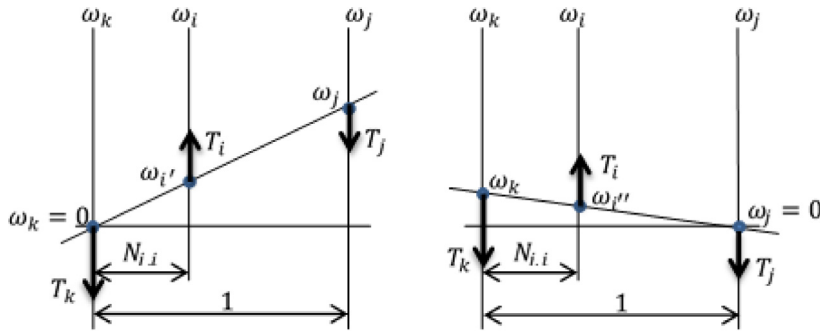


Fig. 10. Nomographs for the two one-dof paths of the power flow through the PGT.

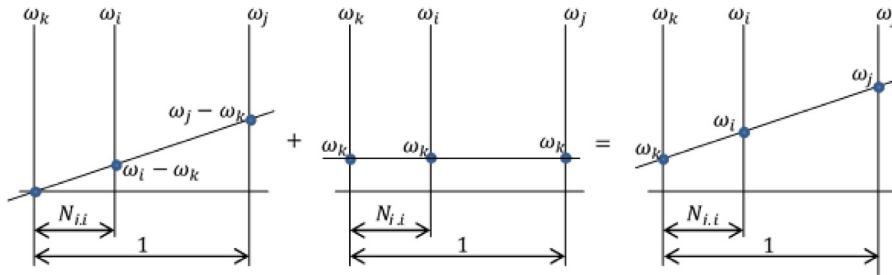


Fig. 11. (a) first partial velocity , (b) second partial velocity, and (c) the total velocity of the links of the external GPE shown in Fig. 1(b).

The meshing losses for the two one-dof paths of the power flow can be written as:

$$L_{k(i-j)} = (\eta_{k(i-j)} - 1)T_i\omega'_i \tag{25a}$$

$$L_{j(i-k)} = (\eta_{j(i-k)} - 1)T_i\omega''_i \tag{25b}$$

By adding Eqs. (25), the total power losses in the fixed reference frame can be expressed as follows:

$$L_{(i-j,k)} = (\eta_{k(i-j)} - 1)T_i\omega'_i + (\eta_{j(i-k)} - 1)T_i\omega''_i \tag{26}$$

By substituting these losses in the overall efficiency expression of a two-dof GPE with two driven links  $j$  and  $k$ , we get the formula

$$\eta_E = \frac{T_i\omega_i + L_{(i-j,k)}}{T_i\omega_i} = \frac{\eta_{k(i-j)}\omega'_i + \eta_{j(i-k)}\omega''_i}{\omega_i}$$

which coincides with that originally formulated by Radzimovsky and previously expressed by Eq. (15) or Eq. (5).

As before, the first operating condition for which the first Radzimovsky formula fails to correctly estimate the power losses is that when

$$\omega_i = \omega_j = \omega_k > 0 \tag{27}$$

Referring to Eq. (26), it is worth emphasizing that even when there is no relative motion between the meshing gears, the estimated power losses by Radzimovsky's formula do not vanish.

In fact, when the gear rotates as a rigid coupling, the power is transmitted without losses, and the power meshing losses are zero:

$$L_{i-j,k} = 0 \tag{28}$$

**9. Potential power approach (second working mode)**

The first, second and total link velocities of the GPE are shown, respectively, in Fig. 11(a)–(c). The meshing losses for the two-dof PGT can be computed considering the potential power of the one-dof PGT in the moving reference frame in which link- $k$  is relatively fixed. We take special note of the advantage of our choice system. The relative velocities  $(\omega_i - \omega_k)$ , and  $(\omega_j - \omega_k)$  of links  $i$  and  $j$  relative to link- $k$  are still greater than zero. An observer located on link- $k$  would observe that the driving and driven links remain unchanged. Link- $i$  is a driving link in both fixed and moving reference frames while link- $j$  is

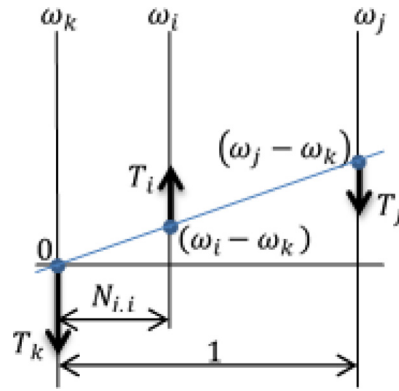


Fig. 12. Nomograph for power flow through the GPE shown in Fig. 1(b) as observed by an observer located on link-k.

a driven link. The losses in link-k moving reference frame, shown in Fig. 12, can be calculated from the potential power as follows:

$$L_{(i-j,k)} = (\eta_{k(i-j)} - 1)T_i(\omega_i - \omega_k) \tag{29}$$

Comparing Eq. (29) with Eq. (26), we first recognized that the second term on the right side of Eq. (26) is just part of the total power losses estimated by Radzimovsky’s formula. This part alone is greater than the losses estimated by the potential or virtual power approach. Considered that

$$\omega_i = \omega'_i + \omega''_i \tag{30}$$

and<sup>3</sup>  $\omega_k > \omega''_i$  or  $(\omega_k - \omega''_i) > 0$ . Hence, by adding  $(\omega_k - \omega''_i)$  to the right side of Eq. (30), a fortiori it is

$$\omega_i < \omega'_i + \omega''_i + (\omega_k - \omega''_i) \tag{31}$$

By simplifying and arranging the terms of Eq. (31), we obtain:

$$\omega'_i > (\omega_i - \omega_k) \tag{32}$$

Since  $\omega'_i > (\omega_i - \omega_k)$  and  $T_i > 0$ , then  $T_i\omega'_i > T_i(\omega_i - \omega_k)$ , and

$$(\eta_{k(i-j)} - 1)T_i\omega'_i < (\eta_{k(i-j)} - 1)T_i(\omega_i - \omega_k) \tag{33}$$

The right side of the last inequality is the total loss calculated by the potential power method (Eq. (29)) while the left side is only part of the total power losses estimated by Radzimovsky’s formula (Eq. (26)). It is clear that for the case under consideration Radzimovsky’s formula overestimates the power losses.

### 10. Influence of the operating conditions on power losses (second working mode)

For the GPE shown in Fig. 1(b), we can distinguish the following three operating conditions:

1.  $\omega_j > \omega_i > \omega_k > 0$ , shown in Fig. 11
2.  $\omega_i = \omega_k = \omega_j > 0$  and,
3.  $\omega_k > \omega_i > \omega_j > 0$  shown in Fig. 13(c)

The losses in link-j moving reference frame, shown in Fig. 14, can be calculated from the potential power as follows:

$$L_{(i-j,k)} = (\eta_{(j,i-k)} - 1)T_i(\omega_i - \omega_j) \tag{34}$$

In contrast to Radzimovsky’s approach, which gives only one equation, the current approach gives three different results for the power losses of the GPE with links k and j as the driven links and link-i as the input link. In summary, these are:

$$L_{(i-j,k)} = (\eta_{k(i-j)} - 1)T_i(\omega_i - \omega_k) \quad \text{for } \omega_j > \omega_i > \omega_k > 0 \tag{35a}$$

$$L_{(i-j,k)} = 0 \quad \text{for } \omega_i = \omega_k = \omega_j > 0 \tag{35b}$$

$$L_{(i-j,k)} = (\eta_{j(i-k)} - 1)T_i(\omega_i - \omega_j) \quad \text{for } \omega_k > \omega_i > \omega_j > 0 \tag{35c}$$

<sup>3</sup> Since  $\omega_i = \omega'_i + \omega''_i$ , with  $\omega'_i = N_{j,i}\omega_j$  and  $\omega''_i = \omega_k(1 - N_{j,i})$ , for  $0 < N_{j,i} < 1$ , it is  $\omega_k > \omega''_i$ .

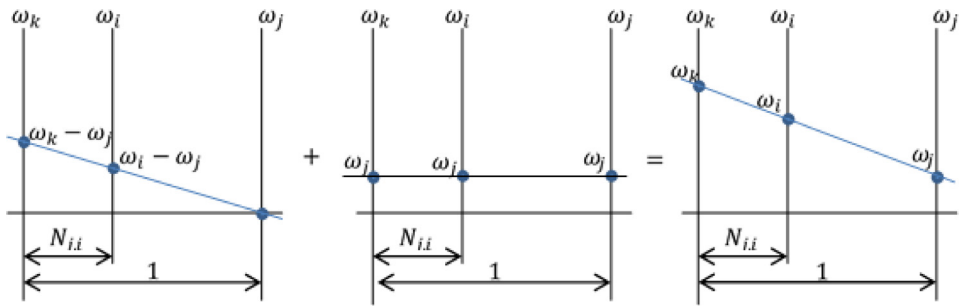


Fig. 13. (a) First partial velocity, (b) second partial velocity, and (c) the total velocity of the links of the GPE shown in Fig. 1(b) (case 3).

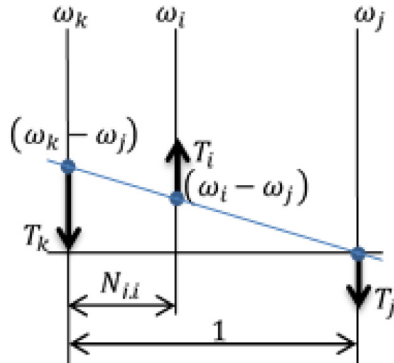


Fig. 14. Nomograph for power flow through the GPE shown in Fig. 1(b) as observed by an observer located on link-*j*.

**11. Numerical example**

To numerically appreciate the difference between the Radzimovsky’s formula and the present approach, the first working mode is considered.

Let us assume that  $N_{j,i} = -1.5$ ,  $\eta_{k(i-j)} = 0.900$ ,  $T_i = 1$ ,  $\omega_i = 8000$ ,  $\omega_k = 7980$ . Under these conditions, since  $P_k < 0$  and  $P_i > 0$ ,  $P_j > 0$ , link-*k* is the driven link and the remaining ones are driving links. Moreover, from Eq. (2), it is  $\tau_{ki} = 0.4$  and  $\tau_{kj} = 0.6$ .

Considered the current value of  $N_{j,i}$ ,<sup>4</sup> the entries No. 3a and No. 5a of Table 1 of reference [6] apply, respectively:

$$\eta_{j(i-k)} = \frac{(N_{j,i}\eta_{k(i-j)} - 1)}{N_{j,i} - 1} = 0.940 \tag{36}$$

$$\eta_{i(j-k)} = \frac{N_{j,i} - \eta_{k(i-j)}}{N_{j,i} - 1} = 0.960 \tag{37}$$

From Radzimovsky’s formula (7) follows the overall efficiency

$$\eta_{Radzimovsky} = \eta_{(i,j-k)} = \frac{7980}{\frac{0.4 \cdot 7980}{0.940} + \frac{0.6 \cdot 7966.667}{0.960}} = 0.952 \tag{38}$$

With the potential power approach (see formula (23))

$$\eta_{PotentialPower} = 1 - \frac{|L_{i,j-k}|}{P_i + P_j} = 1 - \frac{|(\eta_{i(j-k)} - 1)T_i(\omega_i - \omega_j)|}{P_i + P_j} = 0.999 \tag{39}$$

For the present case the difference is of the order of 5%, which is significant in many industrial applications.

**12. Conclusions**

The analytical expressions for the power loss of a typical gear pair entity with links *i* and *j* as the driving links and link-*k* as the driven link were derived.

<sup>4</sup> Because of difference in nomenclature, it is  $R = N_{j,i}$  and  $\eta_{ji}^0 = \eta_{k(i-j)}$ .

For the case under consideration, the results of our analysis can be summarized as follows:

1. When  $\omega_i = \omega_k = \omega_j \neq 0$ , Radzimovsky's formula fails to estimate the power losses correctly.
2. When  $\omega_j > \omega_k > \omega_i > 0$ , or  $\omega_i > \omega_k > \omega_j > 0$ , Radzimovsky's formula overestimates the power losses.
3. In contrast to Radzimovsky's approach, which provides only one equation, the current approach considers three different operating conditions for the estimation of power losses of the GPE. These conditions are distinguished by the inequalities of the angular velocities.
4. Thanks to a detailed analysis of the working conditions of a 2 dof planetary gearing, the kinematic conditions required for the validity of Radzimovsky's formula have been established.

A similar analysis was carried out for a two dof gear pair entity with one driving and two driven links. Also in this case the limitations of Radzimovsky's formula have been highlighted.

A graph-based-algorithm has been proposed for the estimation of the power losses in two d.o.f. planetary gear drives. The generic stationary motion of any link in planetary gear drive can be interpreted as a combination of a relative velocity of the link relative to a coupler link and the coupling velocity of the coupler link. The coupling velocity is present in some degree in all planetary gear trains and has the effect of transmitting the power through the links of the gear train without meshing losses. The potential power, due to the relative velocity, is the only cause of power meshing losses in two-dof PGTs and the power losses can be estimated in any moving reference frame.

### Supplementary material

Supplementary material associated with this article can be found, in the online version, at doi:[10.1016/j.mechmachtheory.2018.05.015](https://doi.org/10.1016/j.mechmachtheory.2018.05.015).

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