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Linear Differential Equations of Higher Order and Dynamical systems

A Thesis

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Abstract

The aim of this Work is to study the existence and uniqueness of the solution of the higher order linear differential equations. Also we study some important Concepts in the Theory of dynamical Systems. Such as the flow of an autonomous equation, Orbits and invariant Sets.

INTRODUCTION

Differential equations are called partial differential equations (pde) or ordinary differential equations (ode) according to whether or not they contain partial derivatives. The order of a differential equation is the highest order derivative occurring. A solution (or particular solution) of a differential equation of order n consists of a function defined and n times differentiable on a domain D having the property that the functional equation obtained by substituting the function and its n derivatives into the differential equation holds for every point in D

Linear differential equations of order two and more .Many physical phenomena are expressed in terms of second order equation. The theoretical background of the second and nth order equations is common and hence, we study below the general higher order equations.

The class of nth order equations is divided mainly into two sub-classes: (i) differential equations with constant coefficients, and (ii) differential equations with variable coefficients. The equations in sub-class (i) can be shown to be related to the study of algebraic equations which can be solved in closed forms. However, the equations in sub-class (ii) quite often pose difficulties while obtaining closed form solutions.

The main problem of the study of higher order equations is the existence of solutions and then uniqueness of solutions. We consider homogeneous equations to begin with and then adopt necessary modifications to study non-homogeneous equations. Some known methods of solving certain class of equations in closed forms are also dealt with and suitable illustrations are given.

1.1 Higher Order Equations

A general *n*th order equation is o the form

$$F(t, x, x', x^{(n)}) = 0; x' = \frac{d}{dt}$$

Where F is a real or complex valued function defined on $1\times R^{n+1}$, I being an interval on a real line. In particular , a linear homogeneous equation of order *n* has the form

$$a_{0^{(1)x^{(n)}}} + a_{1^{(1)x^{(n-1)}}} + \dots\ a_{n^{(1)\,x}} = 0$$

Where $x_{0\neq 0\,,\,a_1}$, a_n are known real or complex valued functions of t defined on an interval I of a real line? The above equation can also be written as

$$x^n\ (\mathrm{t}) + b_{1^{(t)}}\ x^{(n-1)}\ (\mathrm{t}) + \dots + b_{n^{(t)}}\ x\ (\mathrm{t}) = 0 \qquad (1.1)$$

Where $b_{i^{(t)}} = \frac{a_{i^{(t)}}}{a_{0^{(t)}}}$; $a_{0^{(t)}} \neq 0$ for $t \in I$; i=1, 2, *n*.

The notation *L(x)* stands for a linear operator *L* operating on a function *x* which is differentiable *n* times on an interval *I* and *L(x) (t)* stands for the value of *L(x)* at time $\in I$. This notation is useful in the subsequent developments. The equation

$$L(x)\ (t) = 0,\ t \in I$$

Is a linear homogeneous differential equation of order *n,* while the non-homogeneous equations is of the type

$$L(x) = x^n + b_1(t)_{x^{(n-1)}} + b_n\ (t)\ x = h\ (t),\ t \in I \qquad (1.2)$$

Where b_1, b_n, h are given functions defined on *I*. In general, we assume that the functions b_1, b_n, h are continuous on *I*. This requirement is sufficient for the existence and uniqueness of solutions of the Equation (1.2)

In order to get a specific solution we need initial data in the form of initial conditions. For *n*th order equation, we assume *n* initial conditions. These conditions are given in the form

$$x(t_0)=\alpha_1,\ x'(t_0))=\alpha_2 \ldots.,\ x^{(n-1)}(t_0)=\alpha_n \qquad (1.3)$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are given constants . They can be real or complex. Here t_0 is the initial point belonging to the interval I. The Equation (1.2) together with the initial conditions (1.3) forms the initial value problem (IVP) for the *n*th order equation.

A second order linear IVP is of the form

$$\begin{cases} x'' + b1(t)x' + b2(t)x = h(t)\ , t \in I \\ x(t0) = \alpha 1\ , \qquad x'(t0) = \alpha 2 \end{cases}$$

While a third order linear IVP is

$$\begin{cases} x''' + b1(t)x'' + b2(t)x' + b3(t)x = h(t)\ , t \in I \\ x(t0) = \alpha 1\ , \qquad x'(t0) = \alpha 2,\ \ x''(t0) = \alpha 3. \end{cases}$$

There are several useful equation in applications which belong to higher order equations. For example

(i) $x'' + \lambda x = 0$, λ constant; equation of harmonic oscillator;

(ii) $x'' + ax' + bx = 0$, equation of damped oscillations;

(iii) $L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{I}{c} = 0$, equation of electrical network;

(iv) $g''(t) + 2\alpha g'(t) + \mu_0^2 g = 0$, equation employed in the detection of diabetes.

Since several equations of higher order represent almost accurately physical phenomena, it is desirable to study these equations systematically. In particular, many equations cannot be solved in a closed form, i.e. in terms of elementary functions. The only way out for the study of such equations is the use of analytical methods. There are several analytical tools developed which work satisfactorily.

Before we study the analytical methods, a mathematical model is discussed in the next section.

1.2 A Modeling Problem

Suppose that there are two living species which depend for their survival on a common source of food supply. This fact results into a competition in consuming the available food. The phenomenon is commonly noticed in the plant life having common supply of water, fertilizers and minerals.

Let $x(t)$ be the size of the population of the first species at a time t and $y(t)$ be the population of the second species at time t. Because of the dependence on a common food supply (which is limited), there starts the competition.

In the case of a single species x, the rate of its growth is assumed to be proportional to x, since the competition is absent. In this case, the mathematical equation governing the reduces to

$$x'(t)= ax(t),$$

and similarly the growth rate of the other species (in the absence of x) is

$$y'(t)=cy(t).$$

Assume that $a \geq 0$, $c \geq 0$ and t_0 is the initial time. We study the growth phenomena for $t \geq t_0 \geq 0$.

However, when the competition between two species begins, the growth rate of x, i.e. x', is retarded. The rate of retardation is naturally proportional to the size of the population of the y species al time t. Hence, the new set of growth equations are assumed to be of the form

$$\begin{aligned} x'(t) &= ax(t)-by(t), \\ y'(t) &= cy(t)-dx(t). \quad t \geq 0; \qquad (1.4) \end{aligned}$$

Here it is assumed that a, b, c, d are all non-negative constants. The negative terms $by(t)$ and $dx(t)$ in the Equations (1.4) represent the fall in the growth rate of the respective species. In general, these coefficients may also depend on time t. We have taken them as constants to simplify the problem. These coefficients can be determined on the basis of statistical information available for the two species. Although the problem stated above is describing the rate of growth of two species it is clear that it can be easily extended to n distinct species depending on common source of limited food supply.

Differentiating the equation in x in (1.4) w.r.t. t and using the second equation, we get

$$x''(t)= ax'(t)-by'(t)$$

$$=ax'(t)-b[cy(t) -dx(t)]$$

$$=ax'(t)+c\,[x'(t)-ax(t)]+bdx(t)$$

$$=(a+c)x'(t)+(bd-ac)x(t)$$

Hence, x satisfies a second order linear differential equation

$$x''(t)-(a+c)x'(t)+(ac-bd)x(t)=0$$

It will be shown subsequently this equation possesses a solution x of the form

$$x(t)=c_1e^{\alpha t}+c_2e^{\beta t}; \quad t\geq 0,$$

Where c_1 and c_2 are arbitrary constants which can be determined by the initial data given by the two initial conditions . It is known that the numbers α and β occurring in the solution are the roots of a quadratic equation called characteristic equation

$$r^2-(a+c)\,r+(ac-bd)=0.$$

The other solution y can be calculated easily. Let us consider a particular case. Let a=3,b=1, c=3, d=1. The quadratic equation is

$$r^2- 6r+8=0$$

Hence, $\alpha=2$, $\beta=4$. The solution is given by

$$x(t)=c_1e^{2t} + c_2e^{4t}.$$

Using this solution x and the equations (2.4), it follows that

$$y(t)=c_1e^{2t} - 3c_2e^{4t}.$$

Now suppose that initially, i.e. at t=0, $x(0)$=1000 and $y(0)$=1200. It than follows that

$$c_1 + c_2 = 1000,$$

$$c_1 - 3c_2 = 1200;$$

Yielding $c_1=1050$ and $c_2=-50.$

Hence $x(t)= 1050e^{2t} - 50e^{4t},\ \ y(t) = 1050e^{2t} + 150e^{4t}$

It is noted that $y(t)$ continues to increases; however, for some value if t ,$x(t)$ can become zero.

$x(t)$=0 implies that $e_{2t} = \frac{1050}{50} = 21$

i. e. $t=\frac{1}{2}\log_e 21$ would be the time at which the x species will become extinct.

1.3 Linear Independence

The concept of linear dependence and independence has a species role in the study of differential equations. Consider real or complex valued functions defined on an interval I of the real line.

Definition 1.1 Two real or complex valued functions $x_1(t)$ and $x_2(t)$ defined on an interval I are said to be linearly dependent on I if there exist two constants c_1 and c_2, at least one of them not zero, such that

$$c_1x_1(t) + c_2x_2(t) = 0, \quad t \in I.$$

Functions $x_1(t)$ and $x_2(t)$ are said to be linearly independent on I if they are not linearly dependent on I.

This definition implies that in case two functions $x_1(t)$ and $x_2(t)$ are linearly independent and in addition

$$c_1x_1(t) + c_2x_2(t) = 0, \quad t \in I$$

then c_1 and c_2 are necessarily both zero. Further, if two functions are linearly dependent on I, then one of them is a constant multiple of the other. The constants c_1 and c_2 may be real or complex.

The functions $x_1(t) = \sin t$, $x_2(t) = \cos t$, $-\infty < t < \infty$ are linearly independent.

For $$c_1 \sin t + c_2 \cos t = 0; \quad t \in (-\infty, \infty).$$

So also $$c_1 \cos t - c_2 \sin t = 0$$

Solving these two equations in c_1 and c_2 we observe that $c_1=0$ and $c_2=0$.

Example 1.1 (i) The functions $e^{i\alpha t}$, $\sin \alpha t$, $\cos \alpha t$; $-\infty < t < \infty$, α being a real number, are linearly dependent there since

$$e^{i\alpha t} - \cos \alpha t - i \sin \alpha t = 0.$$

(ii) The functions 1, 1+con 2t, $\sin^2 t$; $-\infty < t < \infty$, are linearly dependent since

$$c_1 + c_2(1+\cos 2t) + c_3 \sin^2 t = 0$$

holds true for the choice of $c_1 = 1$, $c_2 = -\frac{1}{2}$, $c_3 = -1$.

(iii) The functions 1, t, t^2,....,t^n, $-\infty < t < \infty$, are linearly independent there. For, let there be constants $c_1, \ldots, c_n$ such that

$$c_1 + c_2 t + \ldots + c_n t^n = 0.$$

Differentiating this relation w.r.t. t successively on $-\infty < t < \infty$ yields

$$c_2 + 2c_3 t + 3c_4 t^2 + \ldots + nc_n t^{n-1} = 0,$$

$$2c_3 + 6c_4 t + \ldots + n(n-1)c_n t^{n-2} = 0,$$

$$\vdots$$

$$n!\, c_n = 0.$$

The last relation proves that $c_n = 0$. Using this fact in the last but one relation it is seen that $c_{n-1} = 0$.Continuing this process we prove that $c_{n-2} = 0, \ldots, c_2 = 0$ and $c_1 = 0$.

(iv) The functions $x_1(t)=t^2, x_2(t)=t|t|$ are linearly independent on $-\infty< t< \infty$. For

$$c_1t^2+c_2t|t|=0$$

Implies $c_1t^2+c_2t^2=0$ for $t\geq 0$

and $c_1t^2-c_2t^2=0$ for $t<0$.

Should hold simultaneously .This is possible only when $c_1+c_2=0$ and $c_1-c_2=0$, i.e. $c_1=0$ and $c_2=0$.

Remark In the above example the two functions are linearly dependent on the intervals $-\infty< t\leq \infty$ and $0\leq t< \infty$. However, they are linearly independent on the interval $-\infty< t< \infty$. Thus, the set of functions may be linearly independent on an interval but they may not be so on a proper part of such interval. However, if a set of functions are linearly dependent on an interval I, they continue to remain linearly dependent on any sub-interval of I.

1.4 Equations with Constant Coefficients

Let us consider a linear differential equation of order n

$$L(x) = x^{(n)}+b_1x^{(n-1)}+\dots+b_nx=0, \quad t\in I \tag{1.5}$$

Where the coefficients $b_1,\dots,b_n$ are constants, real or complex. While considering the first order equation

$$x' + ax=0; \quad t\in I \ (a \text{ is a constant}) \ , x'=\frac{d}{dt}$$

we find that it has a solution

$$x(t) = ce^{at}, \quad t\in I;$$

Where c is an arbitrary constant. This conclusion provides us a clue that we may look for an exponential solution of higher order equation. The following example illustrates the procedure.

Example 1.2 Consider a second order equation

$$L(x)=x''+b_1x'+b_2x=0, -\infty< t< \infty$$

where b_1 and b_2 are real constants. Assume that $x(t)=e^{\lambda t}$ is a solution.

Then, $$L(e^{\lambda t})=(\lambda^2+b_1\lambda+b2)e^{\lambda t}=0.$$

Since $e^{\lambda t}\neq 0$ for any finite value of t and λ, it follows that

$$p(\lambda)=\lambda^2+b_1\lambda+b_2=0.$$

Hence, $L(e^{\lambda t})=0$ if and only if λ is a root of the quadratic equation $p(\lambda)=0$.This equation has two roots (say) λ_1 and λ_2. There are three possibilities:

(i) λ_1,λ_2 are real and $\lambda_1\neq\lambda_2$;

(ii) λ_1,λ_2 are real and $\lambda_1=\lambda_2$;

(iii) λ_1 and λ_2 are complex. In this case the roots are complex conjugates (because b_1 and b_2 are real).

Let us consider the three cases one after the other.

(i) When $\lambda_1\neq\lambda_2$, clearly

$$L(e^{\lambda 1t})=0,\ L(e^{\lambda 2t})=0, \qquad -\infty< t<\infty.$$

Denote the two solution by $\phi_1(t)$ and $\phi_2(t)$. Then we have

$$\phi_1(t)=e^{\lambda 1t} \text{ and } \phi_2(t)=e^{\lambda 2t}, -\infty< t<\infty.$$

It is further observed that these solutions are linearly independent on the real line. For, if

$$c_1+c_2e^{(\lambda 2-\lambda 1)t}=0$$

then we have

$$c_1+c_2e^{(\lambda 2-\lambda 1)t}=0; \qquad -\infty< t<\infty.$$

Since $e^{(\lambda 2-\lambda 1)t}\neq 0$, it follows that $c_1=0$ and hence $c_2=0$. One can note, at this point, that solutions of the given equation are related to the algebraic equation $p(\lambda)=0$.

(ii) When $\lambda_1=\lambda_2$, we can conclude that $\phi_1(t) = e^{\lambda_1 t}$ is one of the solutions. Let us search another solution ϕ_2 so that $\phi_1$are linearly independent. Since the roots are equal

$$P(\lambda) = \lambda^2 + b_1\lambda + b_2 = (\lambda-\lambda_1)^2.$$

Hence, it follows that $b_1= -2\lambda_1$ and $b_2=\lambda_1^2$. The given equation takes the form

$$L(x) = x'' - 2\lambda_1 x' + \lambda_1^2 x = 0.$$

Let us verify if $\phi_2(t) = te^{\lambda_1 t}$ is a solution. We have

$$L(te^{\lambda_1 t}) = (te^{\lambda_1 t})'' - 2\lambda_1(te^{\lambda_1 t})' + \lambda_1^2(te^{\lambda_1 t})$$

$$= [(t\lambda_1 + 1)\lambda_1 + \lambda_1]e^{\lambda_1 t} - 2\lambda_1(t\lambda_1 + 1)e^{\lambda_1 t} + t\lambda_1^2 e^{\lambda_1 t} = 0.$$

Hence, the two Solution are

$$\phi_1(t) = e^{\lambda_1 t} \text{ and } \phi_2(t) = te^{\lambda_1 t}; \quad -\infty < t < \infty.$$

The solutions ϕ_1 and ϕ_2 are linearly independent for

$$c_1 e^{\lambda_1 t} + c_2 te^{\lambda_1 t} = 0; \quad -\infty < t < \infty.$$

Implies that (because $e^{\lambda_1 t} \neq 0$), $c_1+c_2 t=0$. This is possible on $-\infty < t < \infty$ when $c_1=0$ and $c_2 = 0$.

Before considering the case (iii) we seek more information about the nature of solution ϕ_1 and ϕ_2. Firstly, if ϕ_1 is a solution, then $c\phi_1$ (where c an arbitrary constant, real or complex) is also a solution.

For $$L(c\phi_1) = cL(\phi_1) = 0.$$

Consider a function ϕ defined by

$$\phi(t) = c_1\phi_1(t) + c_2\phi_2(t); \quad -\infty < t < \infty$$

Where c1 and c2 constants. We note that ϕ is also a solution since

$$'L(\phi) = L(c_1\phi_1 + c_2\phi_2)$$

$$= (c_1\phi_1 + c_2\phi_2)'' + b_1(c_1\phi_1 + c_2\phi_2)' + b_2(c_1\phi_1 + c_2\phi_2)$$

$$=c_1(\phi''_1+b_1\phi'_1+b_2\phi)+ c_2(\phi''_1+b_1\phi'_1+b_2\phi)$$

$$=c_1L(\phi_1) + c_2\, L(\phi_2)=0\ .$$

Hence, $\phi(t) =c_1\phi_1(t) + c_2\phi_2(t)$ is also a solution for any choice of constants c_1 and c_2. We can say that $\{c_1\phi_1+c_2\phi_2,\ c_1,c_2$ are arbitrary constants$\}$ represents a family of solution of the given second order equation. This conclusion arrived at by employing a theorem proved subsequently in this chapter. Solution $\phi(t) =c_1\phi_1(t) + c_2\phi_2(t)$ is called a general solution of the equation. The general $-\infty< t< \infty$. We now consider case (iii).

(iii) let the roots λ_1 and λ_2 be complex. It is known that they occur in conjugate pairs. Suppose that $\lambda_1=\alpha+i\beta$. Then $\lambda_2=\alpha-i\beta$. The solutions corresponding to these roots are

$$e^{\lambda 1t} = e^{(\alpha+i\beta)t}=e^{\alpha t}[\cos\beta t+i\sin\beta t]$$

and $$e^{\lambda 2t} = e^{(\alpha-i\beta)t}=e^{\alpha t}[\cos\beta t-i\sin\beta t]$$

The general solution ϕ is then given by

$$\phi(t)=c_1e^{\lambda 1t} +c_2e^{\lambda 2t}$$

$$=c_1e^{\alpha t}[\cos\beta t+i\sin\beta t] +c_2e^{\alpha t}[\cos\beta t-i\sin\beta t]$$

$$=[(c_1+c_2) \cos\beta t +i(c_1-c_2) \sin\beta t]\,e^{\alpha t}$$

Let $$k_1=c_1+c_2 \text{ and } k_2 =i(c_1-c_2)$$

Hence $$\phi(t)= k_1e^{\alpha t}\cos\beta t +k_2e^{\alpha t} \sin\beta t; \qquad -\infty< t< \infty.$$

In this case we choose $\phi_1(t)= e^{\alpha t}\cos\beta t$ when $k_1=1$, $k_2=0$ and $\phi_2(t)= e^{\alpha t}\sin\beta t$ when $k_1=0$, $k_2=1$. Further, it can be shown that these two solutions are linearly independent. We sum up the conclusions from the three cases discussed above. The general solution ϕ of the equation

$$x''+b_1x'+b_2x'=0, \qquad -\infty< t< \infty$$

is associated with the quadratic equation (with real b_1 and b_2)

$$p(\lambda)\lambda^2+b_1\lambda+b_2 =0$$

having roots λ_1 and λ_2 and is given by

$$\lambda_1= (-b_1 + \sqrt{b_1^2 - 4b_2)}\ / 2$$

$$\lambda_2=(-b_1 - \sqrt{b_1^2 - 4b_2)}\ / 2$$

$$\alpha=-b / 2, \quad \beta=\sqrt{4b_2 - b_1^2)/2}\ .$$

where c1 and c2 are arbitrary constants and

$$\phi(t)=c_1e^{\lambda 1t} + c_2e^{\lambda 2t} \qquad \text{when } b_1^2>4b_2;$$

$$= (c_1+c_2t)e^{\lambda 1t} \qquad \text{when } b_1^2= 4b_2;$$

$$=[c_1 \cos\beta t+c_2 \sin\beta t\ ]e^{\alpha t} \qquad \text{when } \ b_1^2< 4b_2;$$

Remark Solving a differential equation means finding the general solution.

1.5 *n*th Order Equations

We now recall Equation (1.5) having constant coefficients. The discussion in the earlier part of this section provides us a clue to the search of a solution of Equation (1.5).It suggests that (1.5) can have solution of the form $e^{\lambda t}$ for a suitable choice of λ. Let $x(t)= e^{\lambda t}$. Substituting in (1.5), we see

$$L(e^{\lambda t})=(e^{\lambda t})^{(n)} + b_1(e^{\lambda t})^{(n-1)} +...+b_n(e^{\lambda t})$$

$$=(\lambda^n +b_1\lambda^{n-1}+....+b_n)\ e^{\lambda t}; \quad t\in I.$$

Clearly $e^{\lambda t}$ is a solution of (1.5) if $L(e^{\lambda t})=0$.In the above expression $e^{\lambda t} \neq 0$ for $t\in I$.

Hence, $$p(\lambda)=\lambda^n +b_1\lambda^{n-1}+...+b_n=0 \qquad (1.6)$$

If we choose λ such that $p(\lambda)=0$, then $e^{\lambda t}$ is a solution of the Equation(1.5).The polynomial Equation(1.6) is an *n*th order algebraic equation in λ. The Equation (1.6) is called the characteristic equation of (1.5).

The roots of the Equation $p(\lambda)=0$ are called characteristic roots. Since Equation (1.6) is of order n, it has n roots. They may be distinct, repeated, real or complex. Theoretically, Equation (1.6) can be solved. We have now the following theorem.

Theorem 1.1 If λ is a root of the characteristic Equation (1.6), then $e^{\lambda t}$ is a solution of the Equation (1.5) for $t \in I$.

In case the n characteristic roots of (1.6) namely $\lambda_1,..,\lambda_n$ are distinct from each other it follows from the above theorem that corresponding to each root there is a solution. So we get n solutions

$$\phi_1(t) = e^{\lambda 1t},\ldots,\phi_n(t)= e^{\lambda nt}; \quad t \in I.$$

Further, we can also show that their linear combination, namely, $c_1\phi_1 + \ldots + c_n\phi_n$ is also a solution of (1.5).

Example 1.3 Solve

$$x'''+6x''+11x'+6x=0; \quad -\infty< t< \infty.$$

The characteristic equation is

$$p(\lambda)= \lambda^3 +6\lambda^2+11\lambda+6= (\lambda+1)(\lambda+2)(\lambda+3)=0.$$

The characteristic roots are $\lambda_1=-1$, $\lambda_2=-2$, $\lambda_3=-3$. Hence, $\phi_1(t)=e^{-1}$, $\phi_2(t)=e^{-2t}$, $\phi_3(t)=e^{-3t}$.

The general solution is given by

$$\phi(t)= c_1e^{-t} +c_2e^{-2t} +c_3e^{-3t}; \quad -\infty< t< \infty.$$

Another possibility for the characteristic roots of $p(\lambda)=0$ is that some of them may be repeated. In the case of second order equation, it is observed that if the roots are repeated, the solution are $e^{\lambda t}$ and $te^{\lambda t}$. Based on these conclusions, it is natural to expect similar behavior for higher order equations.

Suppose that λ_1 is a root, repeated m_1 ($m_1 \leq n$) times, of the equation

$$p(\lambda) = \lambda^n + b_1\lambda^{n-1} + \ldots + b_n = 0.$$

So $p(\lambda)$ can be written in the form

$$p(\lambda)=(\lambda-\lambda_1)^{m1}q(\lambda)$$

Where $q(\lambda)$ is a polynomial of degree $n\text{-}m_1$ and $q(\lambda_1)\neq 0$.

Further, observe that

$$p'(\lambda) = m_1(\lambda-\lambda_1)^{m1\text{-}1}q(\lambda) + (\lambda-\lambda_1)^{m1}q'(\lambda);$$

$$p^{(m1\text{-}1)}(\lambda) = m_1(m_1\text{-}1)\ldots 2(\lambda-\lambda_1)q(\lambda)$$

$$+\text{terms containing higher powers of } (\lambda-\lambda_1)$$

and $p^{(m1)}(\lambda)$ $m_1!$ $q(\lambda)$ +terms containing higher powers of $(\lambda-\lambda_1)$.

Hence, it follows that

$$p(\lambda_1)=0,\ p'(\lambda_1)=0,\ldots p^{(m1\text{-}1)}(\lambda_1)=0,\ \text{but } p^{(m1)}(\lambda_1)\neq 0.$$

We use these relations below. Observe that

$$L(e^{\lambda t})= (\lambda^n + b_1\lambda^{n\text{-}1}+\ldots+b_n)e^{\lambda t}=p(\lambda)e^{\lambda t}.$$

Differentiate $L(e^{\lambda t})$ w.r.t. λ. Note that here L is a linear operator. We have

$$\frac{\partial}{\partial\lambda}L\left(e^{\lambda t}\right)=L\left(\frac{\partial}{\partial\lambda}\ e^{\lambda t}\right)=L(te^{\lambda t})$$

and $$\frac{\partial}{\partial\lambda}L\left(e^{\lambda t}\right)= [p'(\lambda) +tp(\lambda)]e^{\lambda t}.$$

Hence, $$L\left(te^{\lambda t}\right) = [p'(\lambda) + tp(\lambda)]\ e^{\lambda t}.$$

At $\lambda=\lambda_1$, $p'(\lambda_1)=0$, $p(\lambda_1)=0$. Hence it follows that

$$L\left(te^{\lambda t}\right) = 0.$$

We conclude that $te^{\lambda t}$ is a solution of the Equation (1.5). Further, it is observed that

$$\frac{\partial^2}{\partial\lambda^2}L\left(e^{\lambda t}\right)=L\left(\frac{\partial^2}{\partial\lambda^2}\ e^{\lambda t}\right)=L(t^2e^{\lambda t})$$

$$=[p''(\lambda) +2\,tp'(\lambda) +t^2p(\lambda)]\ e^{\lambda t}\ .$$

At $\lambda=\lambda_1$

$$L(t^2e^{\lambda 1t}) = [p''(\lambda_1) + 2tp'(\lambda_1) + t^2p(\lambda_1)]\, e^{\lambda 1t} = 0.$$

This statement provides a conclusion that $t^2e^{\lambda t}$ is also a solution of the Equation (1.5). We continue this process of differentiation. Observe that

$$\frac{\partial^{m1-1}}{\partial\lambda^{m1-1}} L\left(e^{\lambda t}\right) = L\left(\frac{\partial^{m1-1}}{\partial\lambda^{m1-1}}\, e^{\lambda t}\right) = L(t^{m1-1}e^{\lambda t})$$

$$= [p^{(m1-1)}(\lambda) + (m_1-1)p^{(m1-2)}(\lambda)\,t + \frac{(m_1-1)(m_1-2)}{2!}\, p(\lambda)\,t^2 + \ldots + \mathrm{p}(\lambda)\,t^{m1-1}]\, e^{\lambda t}$$

Hence, for $\lambda=\lambda_1$, we get

$$L(t^{m1-1}e^{\lambda 1t}) = 0.$$

Yielding yet another solution $t^{m1-1}e^{\lambda 1t}$ of (1.5) .

It is to be observed that

$$\frac{\partial^{m1}}{\partial\lambda^{m1}} L\left(e^{\lambda t}\right) = L\left(\frac{\partial^{m1}}{\partial\lambda^{m1}}\, e^{\lambda t}\right) = L(t^{m1}e^{\lambda t})$$

$$= [p^{(m)}(\lambda) + \text{terms of lower order derivation}]$$

For $\lambda=\lambda_1$, $\quad \mathrm{p}^{(m)}(\lambda_1)\neq 0.$

Hence $\quad \mathrm{L}(t^{m1}\mathrm{e}^{\lambda t}) \not\equiv 0$

and $t^{m1}\mathrm{e}^{\lambda t}$ is, therefore, not a solution of (1.5).

Summing up the discussion above it is concluded that when a root λ_1 of $p(\lambda) = 0$ is repeated m_1 times, it is seen that we get m_1 solutions

$$\phi_1(t) = \mathrm{e}^{\lambda 1t},\ \phi_2(t) = t\mathrm{e}^{\lambda 1t}, \ldots, \phi_m(t) = t^{m1-1}\, e^{\lambda 1t}$$

Defined on an interval I. The story covered so far is of the root λ_1 repeated m_1 times. There may be other root λ_2 repeated m_2 times. The procedure has then to be repeated. We present this discussion in the form of a theorem.

Theorem 1.2 Suppose that $\lambda_1,\ldots,\lambda_s$ are s roots of the characteristic equation $p(\lambda)=0$ given in (1.6) repeated $m_1,\ldots,m_s$ times respectively, where $m_1+\ldots+m_s=n$. Then the n functions

$$e^{\lambda 1t}, te^{\lambda 1t},\ldots, t^{m1-1}e^{\lambda 1t}; \quad (m_1 \text{ function})$$

$$. \ , \ ., \ \ldots, \quad . \qquad (1.7)$$

$$e^{\lambda st}, te^{\lambda st}, \ldots, t^{ms-1}e^{\lambda st} \quad (m_s \text{ function })$$

are all solutions of the Equation (1.5) existing on the interval I. Further, if these functions are denoted by $\phi_1,\ldots,\phi_n$ then their linear combination, say ϕ, given by

$$\phi=c_1\phi_1+\ldots+c_n\phi_n$$

Where $c_1,\ldots, c_n$ are arbitrary constants, is also a solution of (1.5).

The set of n solutions given in (1.7) are, in fact, linearly independent. We prove this fact in a Subsequent theorem in this chapter.

In the case of second order equations, we observed that the roots of a characteristic equation can be complex and that they occur in conjugate pairs. Let $\lambda=\alpha+i\beta$ be a complex root. It was shown that $e^{\alpha t}\cos\beta t$ and $e^{\alpha t}\sin\beta t$ are two linearly independent solutions.

In the case of nth order equations among s distinct roots $\lambda_1,\ldots,\lambda_s$, it may happen that one or more of them may be complex. The following result takes care of such a possibility.

1.6 Equations with Variable Coefficients

A linear differentiation equation of order n with variable coefficients has the form

$$a_0(t)x^{(n)}+a_1(t)x^{(n-1)}+\ldots+a_n(t)x=h_1(t)$$

Where $a_0, a_1,\ldots,a_n, h_1$ are real or complex valued functions defined on an interval I of the real line R. Below, We assume that $a_0(t)\neq 0$ for any $t\in I$. With this assumption the above equation becomes

$$L(x)(t) \equiv x^{(n)}(t) + b_1(t)x^{(n-1)}(t) + \ldots + b_n(t)x(t) = h(t), \; t \in I. \quad (1.8)$$

Where $b_i = \frac{a_i}{a_0}$, $=1, \ldots, n$, and $h = \frac{h_1}{a_0}$ for $t \in I$.

Here L denotes an operator on a function x which is differentiable n times on I.

In case $h(t) \equiv 0$, $t \in I$, in (1.8), we have

$$L(x)(t) = x^{(n)}(t) + b_1(t)x^{(n-1)}(t) + \ldots + b_n(t)x(t) = 0, \; t \in I.$$
(1.9)

Which is a linear homogeneous equation . The Equation (1.8), $L(x)(t) = h(t)$; $h(t) \neq 0$ is linear non-homogeneous.

In general, to determine a specific solution of the Equation (1.8), we need n conditions. The initial value problem for the nth order equation is

$$L(x)(t) = h(t) \quad (t \in I)$$

With initial conditions

$$x(t_0) = \alpha_0, \quad x'(t_0) = \alpha_1, \ldots, x^{(n-1)}(t_0) = \alpha_{n-1}$$

Where $\alpha_0, \alpha_1, \ldots, \alpha_{n-1}$ are constants and $t_0 \in I$.

The first aspect of these equations which needs immediate consideration is the existence of solutions of the initial value problems.

We state the existence and uniqueness theorem for the nth order Equation (1.9).

Theorem 1.3 Let $b_1, \ldots, b_n$ be continuous functions on an interval I which contains a point t_0. Let $\alpha 0, \alpha_1, \ldots, \alpha_{n-1}$ be any n constants. Then there exists a unique solution ϕ on I of the nth order Equation (1.9) satisfying the initial conditions

$$\phi(t_0) = \alpha_0, \; \phi'(t_0) = \alpha_1, \ldots, \phi^{(n-1)}(t_0) = \alpha_{n-1}.$$

Suppose that $\phi(t), \ldots, \phi_n(t)$ are n solutions of $L(x)(t) = 0$ given in (1.9).Let $c1, \ldots, c_n$ be n arbitrary constants. Then it is seen that

$$L(c_1\phi_1 + \ldots + c_n\phi_n) = c_1L(\phi_1) + \ldots + c_nL(\phi_n) = 0.$$

This relation holds because L is a linear operator and that $L(\phi_i) = 0$, $i=1,\ldots,n$. In case the n solutions $\phi_1,\ldots, \phi_n$ are linearly independent on I , then the relation

$$c_1\phi_1 + \ldots + c_n\phi_n = 0, \quad t \in I$$

Where $c_1,\ldots, c_n$ are constants, implies that

$$c_1=0,\ldots, c_n=0$$

Example 1.4 Prove that there are three linearly independent solutions of the third order equation

$$x''' + b_1(t)x'' + b_2(t)x' + b_3(t)x = 0, \ t \in I \qquad (1.10)$$

When b_1 , b_2 and b_3 are functions defined and continuous on an interval I.

Applying Theorem 1.3, we claim that there exist solutions $\phi_1(t)$, $\phi_2(t)$ and $\phi_3(t)$ of (1.10) such that for $t_0 \in I$

$$\phi_1(t_0) = 1, \ \phi'_1(t_0)=0. \ \phi''_1(t_0)=0;$$

$$\phi_2(t_0) = 1, \ \phi'_2(t_0)=0. \ \phi''_2(t_0)=0; \qquad (1.11)$$

and $$\phi_3(t_0) = 1, \ \phi'_3(t_0)=0. \ \phi''_3(t_0)=1.$$

We show that the solutions ϕ_1 , ϕ_2 and ϕ_3 are linearly independent. Let

$$c_1\phi_1(t) + c_2\phi_2(t) + c_3\phi_3(t) = 0, \quad t \in I \qquad (1.12)$$

For some constants c_1, c_2 and c_3 . At $t=t_0$, from (1.12), we have

$$c_1\phi_1(t_0) + c_2\phi_2(t_0) + c_3\phi_3(t_0) = 0,$$

$$c_1\phi'_1(t_0) + c_2\phi'_2(t_0) + c_3\phi'_3(t_0) = 0,$$

$$c_1\phi''_1(t_0) + c_2\phi''_2(t_0) + c_3\phi''_3(t_0) = 0.$$

Using the initial data given in (1.11), it is seen that

$$c_1.\ 1 + c_2\ .0 + c_3\ .0 = 0,$$

$$c_1.\,0+c_2\,.1+c_3\,.0=0,$$

$$c_1.\,0+c_2\,.0+c_3\,.1=0,$$

Yielding $c_1=c_2=c_3=0$. The claim that ϕ_1, ϕ_2 and ϕ_3 are linearly independent is established.

1.7 Wronskian

Suppose that $\phi_1,\ldots,\phi_n$ are n real or complex valued functions defined on an interval I and each having derivatives of order n.For $t\in I$, define the determinant

$$w(t)=w(\phi_1,\ldots,\phi_n)(t)=\begin{vmatrix} \phi_1(t) & \phi_2(t) & \ldots & \phi_n(t) \\ \phi'_1(t) & \phi'_2(t) & \ldots & \phi'_n(t) \\ \phi_{1^{(n-1)}}(t) & \phi_{2^{(n-1)}}(t) & \ldots & \phi_{n^{(n-1)}}(t) \end{vmatrix}. \quad (1.13)$$

The function $w(t)$ is called the Wronskian of n-functions $\phi_1,\ldots,\phi_n$.

Theorem 1.4 (Abel's formula) Let the function $b_1,\ldots,b_n$ in Equation (1.9) be defined and continuous on an interval I. Let $\phi_1,\ldots,\phi_n$ be n linearly independent solutions of (1.9) existing on I containing a point t_0. Then

$$w(t)=\exp\left[-\int_{t_0}^{t} b_1(s)ds\right]w(t_0);\quad t_0,\ t\in I.$$

Proof Differentiate $w(t)$ to get

$$w'=\begin{vmatrix} \phi'_1 & \ldots & \phi'_n \\ \phi'_1 & \ldots & \phi'_n \\ \phi''_1 & \ldots & \phi''_n \\ . & \ldots & . \\ \phi_1^{(n-1)} & \ldots & \phi_n^{(n-1)} \end{vmatrix}+\begin{vmatrix} \phi_1 & \ldots & \phi_n \\ \phi''_1 & \ldots & \phi''_n \\ \phi''_1 & \ldots & \phi''_n \\ . & \ldots & . \\ \phi_1^{(n-1)} & \ldots & \phi_n^{(n-1)} \end{vmatrix}++\begin{vmatrix} \phi_1 & \ldots & \phi_n \\ \phi'_1 & \ldots & \phi'_n \\ \phi''_1 & \ldots & \phi''_n \\ . & \ldots & . \\ \phi_1^{(n)} & \ldots & \phi_n^{(n)} \end{vmatrix}$$

While doing so, we differentiate elements in each row keeping other rows fixed in the process. Since there are n rows in the determinant, derivative $w'(t)$ turns out to be the sum of n determinants of which the first (n -1) determinants are zero since each of them has two identical rows. Only the last determinant has all rows distinct. Now, since ϕ_1 ,…,ϕ_n are solutions of (1.9), we have

$$L(\phi_i) = \phi_i^n + b_1\phi_i^{(n-1)} + \ldots + b_n\phi_i = 0; \quad \text{for } I = 1,\ldots,n.$$

It follows that

$$\phi_i^n = - b_1\phi_i^{(n-1)} - \ldots - b_n\phi_i. \tag{1.14}$$

Observe that the last row of the last determinant contains the terms ϕ_i^n ,..., ϕ_n^n which can be replaced by the relation (1.14). The right side of (1.14) is the sum of functions $\phi_i,\ldots,\phi_i^{(n-1)}$ with some multiples. Note that these functions also appear in the upper rows. Hence, using the usual operations on the determinant, we obtain

$$w' = \begin{vmatrix} \phi'_1 & \ldots & \phi'_n \\ \phi'_1 & \ldots & \phi'_n \\ \phi''_1 & \ldots & \phi''_n \\ \phi_1^{(n-1)} & \ldots & \phi_n^{(n-1)} \\ . & \ldots & . \\ -b_1\,\phi_1^{(n-1)} & \ldots & -b_1\,\phi_n^{(n-1)} \end{vmatrix} = -b_1 w$$

We have shown that the function w satisfies a first order linear homogeneous equation

$$w' + b_1 w = 0; \quad t \in I$$

Whose solution is

$$w(t) = \exp\left[-\int_{t_0}^{t} b_1(s)ds\right] w(t_0); \quad t_0,\ t \in I.$$

The *proof* is complete.

It is clear from this relation that if $w(t_0) \neq 0$, then $w(t) \neq 0$ for $t \in I$. Hence, it is enough to show that Wronskian $w(t) \neq 0$ only at just one point of I. This criterion yields the linear independence of n solutions of (1.9).

Example 1.5 Consider the second order equation

$$L(x) = x'' + b_1(t)x' + b_2(t)x = 0, \quad t \in I$$

Where b_1 and b_2 are continuous functions on I. Let u and v be any two twice-differentiable functions on I. Then it follows that

$$uL(\upsilon) - \upsilon L(u) = (u\upsilon'' - \upsilon u'') + b_1(u\upsilon' - \upsilon u').$$

Note that $w(u,\upsilon) = u\upsilon' - \upsilon u'$ and $w'(u,\upsilon) = u\upsilon'' - \upsilon u''$. Hence

$$uL(\upsilon) - \upsilon L(u) = w(u,\upsilon) + b_1 w(u,\upsilon).$$

In case u and υ are solution of the given equation, we have $L(u)=L(\upsilon)=0$ and $w' + b_1 w' = 0$

Yielding $$w(t) = w(t_0) = \exp\left[-\int_{t_0}^{t} b_1(s)ds\right], \quad t_0, t \in I.$$

(ii) Now consider the IVP

$$L(x) = x'' + b_1(t)x' + b_2(t)x = h(t), \quad t \in I$$

$x(t_0) = x'(t_0) = 0$; $t_0 \in I$. Let $x(t)$ be the solution of this IVP.

Let x1(t) and x2(t) be linearly independent solution of L(y)=0. Then we have

$$w'(x_1, x) + b_1(t)\, w(x_1, x) = x_1 L(x) - xL(x_1) = x_1 h(t)$$, since $L(x_1)=0$.

This is a first order equation in $w(x_1, x)$. Hence,

$$w(x_1, x)(t) = \exp\left[-\int_{t_0}^{t} b_1(s)ds\right] + \int_{t_0}^{t} \exp\left[\int_{t_0}^{t} b_1(u)du\right] x_1(s)\, h(s)\, ds.$$

We have used here the initial conditions $x(t_0) = x'(t_0) = 0$ and hence $w(x_1, x)(t_0) = 0$. It follows from Abel's formula that

$$w(x_1, x)(t) = w(x_1, x_2)(t) \int_{t_0}^{t} \frac{x_1(s)h(s)ds}{w(x_1, x_2)(s)}.$$

Similarly, we get

$$w(x_2, x)(t) = w(x_1, x_2)(t) \int_{t_0}^{t} \frac{x_2(s)h(s)ds}{w(x_1, x_2)(s)}.$$

Further,

$$[w(x_1, x_2)(t)]x(t) = [w(x_1, x)(t)]x_2(t) - [w(x_2, x)(t)]x_1(t)$$

$$= [w(x_1, x_2)(t)] \int_{t_0}^{t} \frac{[x_1(s)x_2(t) - x_2(s)x_1(t)]h(s)ds}{w(x_1, x_2)(s)}$$

Hence, $$x(t)=\int_{t_0}^{t}\frac{[x_1(s)x_2(t)-x_2(s)x_1(t)]h(s)ds}{w(x_1,x_2)(s)}.$$

Thus, we have obtained the solution of the non-homogeneous equation in terms of solution of corresponding homogeneous equation. We obtain the result by another method in the next section.

1.8 Variation of Parameters

The discussions that we had so far are mainly concerned with Equation (1.9). In this section, we take up for study the non-homogeneous linear equation, namely

$$L(x)=x^{(n)}+b_1(t)x^{(n-1)}+\ldots+b_n(t)x=h(t). \qquad (1.8)$$

Here we assume that $b_1,\ldots,b_n$, h are given real or complex valued continuous functions defined on an interval I. The corresponding homogeneous equation is

$$L(x)=x^{(n)}+b_1(t)x^{(n-1)}+\ldots.+b_n(t)x=0, \quad t\in I. \qquad (1.9)$$

A general result concerning the Equations (1.8) and (1.9) is the following.

Theorem 1.6 Let $\phi_1,\ldots,\phi_n$ be n linearly independent solution of the homogeneous Equation (1.9) existing on I. Let x_p denote any particular solution of the non-homogeneous Equation (1.8) existing on I. Then any solution x of (1.8) is given by

$$x(t)=x_p(t)+c_1\phi_1(t)+\ldots+c_n\phi_n(t); \quad t\in I. \qquad (1.15)$$

Where $c_1,\ldots,c_n$ are n arbitrary constants (real or complex).

Proof We use the method of verification. Clearly $L(\phi_i)(t)=0$ for $i=1,\ldots,n$. Since x_p is a solution of (1.8),

$$L(x_p)(t)=h(t); \quad t\in I.$$

Further

$$L(x)(t)=L(x_p+c_1\phi_1+\ldots+c_n\phi_n)(t); \quad t\in I.$$

$= L(x_p)(t) + c_1L(\phi_1)(t)+\ldots+ c_nL(\phi_n)(t) = h(t).$

Hence, $x(t)$ defined in (1.15) is a solution of (1.8). The proof is complete.

Observe that $x(t) - x_p(t) = c_1\phi_1(t) + \ldots + c_n\phi_n(t); \quad t \in I.$

This relation implies that the difference of two solutions of the Equation (1.8) is a solution of the homogeneous Equation (1.9).

Theorem 1.6 Let $\phi_1,\ldots,\phi_n$ be n linearly independent solutions of the Equation (1.9) existing on I. Let the real or complex valued function h be defined and continuous on I. Further, assume that $w(t)=w(\phi_1,\ldots,\phi_n)$ and $w_k(t)$ denotes the determinant $w(t)$ with kth column replaced by n elements 0,0,…,1. Then a particular solution $x_p(t)$ of (1.8) is given by

$$x_p(t)=\sum_{k=1}^{n} \phi_k(t) \int_{t_0}^{t} \frac{w_k(s)h(s)}{w(s)}\, ds; \quad t \in I. \tag{1.16}$$

Proof $\phi_1,\ldots,\phi_n$ are n linearly independent solutions of the Equation (1.9). Hence $c_1\phi_1 + \ldots + c_n\phi_n$ represents a general solution of (1.9). It is natural to expect that solution of(1.8) are related to the solution $c_1\phi_1 + \ldots + c_n\phi_n$. With this clue in mind, assume that

$$x_p(t) = c_1(t)\phi_1(t) + \cdots + c_n(t)\phi_n(t); \quad t \in I \tag{1.17}$$

Where $c_1,\ldots, c_n$ are functions of t to be determined.

Differentiate (1.17) to get

$$x'_p = \left(c_1\phi'_1 + \cdots + c_n\phi'_n\right) + [c'_1\phi_1 + \cdots + c'_n\phi_n]\,.$$

Assume that $$c'_1\phi_1 + \cdots + c'_n\phi_n=0,$$

Which yields

$$x''_p = \left(c_1\phi''_1 + \cdots + c_n\phi''_n\right) +[c'_1\phi'_1 + \cdots + c'_n\phi'_n].$$

Again assume that

$$c'_1\phi'_1 + \cdots + c'_n\phi'_n =0.$$

Continuing the procedure, we have

$$x_p^{(n-1)} = \left(c_1\phi_1^{(n-1)} + \cdots + c_n\phi_n^{(n-1)}\right) + [c'_1\phi^{(n-2)} + \cdots + c'_n\phi_n^{(n-2)}].$$

Assume that $$c'_1\phi^{(n-2)} + \cdots + c'_n\phi_n^{(n-2)} = 0.$$

Lastly, we get

$$x_p^{(n)} = \left(c_1\phi_1^{n} + \cdots + c_n\phi_n^{(n)}\right) + [c'_1\phi_1^{(n-1)} + \cdots + c'_n\phi_n^{(n-1)}].$$

Now, assume that

$$c'_1\phi_1^{(n-1)} + \cdots + c'_n\phi_n^{(n-1)} = h.$$

We verify now if, under these assumptions, $x_p(t)$ satisfies the Equation (1.8). Indeed it does, for

$$L(x_p) = L(c_1\phi_1 + \cdots + c_n\phi_n) = (c_1\phi_1 + \cdots + c_n\phi_n)^{(n)}$$
$$+ b_1(t)((c_1\phi_1 + \cdots + c_n\phi_n)^{(n-1)} + \ldots + b_n(t)(c_1\phi_1 + \cdots + c_n\phi_n)$$
$$= c_1(t)L(\phi_1) + \ldots + c_n(t)L(\phi_n) + h(t) = h(t)$$

Here $L(\phi_i)=0$, $i=1,\ldots,n$. While verifying the above steps, One has to note that $c_1,\ldots,c_n$ are functions of t. Hence $x_p(t)$ is a solution of (1.8).

Now, we need to determine the functions $c_1,\ldots,c_n$. The following conditions have been assumed

$$c'_1\phi_1 + \cdots + c'_n\phi_n = 0,$$
$$c'_1\phi'_1 + \cdots + c'_n\phi'_n = 0,$$
$$\vdots$$
$$c'_1\phi_1^{(n-1)} + \cdots + c'_n\phi_n^{(n-1)} = h.$$

This is a system of n simultaneous non-homogeneous algebraic equations in n unknowns, $c'_1\ldots,c'_n$ for a fixed t. Further, the determinant of the coefficients is set of solutions and is given by

$$c'_k(t) = \frac{w_k(t)h(t)}{w(t)}, \; k=1,\dots,n, \quad t \in I$$

Which leads to

$$c'_k(t) = \int_{t_0}^{t} \frac{w_k(s)h(s)}{w(s)} ds, \quad k=1,\dots,n$$

In view of (1.17) the formula (1.16) follows. The proof is complete.

Example 1.7 Consider the equation

$$L(x) = x'' - \frac{2}{t}x' + \frac{2}{t^2}x = t\sin t, \quad t\in[1,\infty).$$

It is easily verified that $\phi_1(t)=1$ and $\phi_2(t)=t^2$ are two linearly independent solutions of $L(x)(t)=0$ existing on $[1,\infty)$. Further, $w(\phi_1,\phi_2)(t)=t^2$. It follows that

$$c_1(t) = \int_1^t \frac{w_1(s)h(s)}{w(s)} ds = \int_1^t -\frac{s^3 \sin s}{s^2} ds$$

$$=-\int_1^t s\sin s\, ds = [t\cos t - \sin t] - [\cos 1 - \sin 1].$$

$$c_2(t) = \int_1^t \frac{w_2(s)h(s)}{w(s)} ds = \int_1^t -\frac{s^2 \sin s}{s^2} ds = -\cos t + \cos 1$$

The general solution $x(t)$ is then given by

$$x(t)=c_1(t)\phi_1(t)+c_2(t)\phi_2(t)+d_1\phi_1(t)+d_2\phi_2(t)$$

Where d_1 and d_2 are arbitrary constants

Hence $x(t) = [t\cos t - \sin t]t - \cos t.\, t^2 + (-\cos 1 + \sin 1 + d_1)t + (\cos 1 + d_2)t^2$

$$= -t\sin t + k_1 t + k_2\, t^2 \,,$$

2.1 Dynamical systems

You can think of a dynamical system as the time evolution of some physical system, such as the motion of a few planets under the influence of their respective gravitational forces. Usually you want to know the fate of the system for long times, for instance, will the planets eventually collide or will the system persist for all times?

For some systems (e.g., just two planets) these questions are relatively simple to answer since it turns out that the motion of the system is regular and converges, for example, to an equilibrium.

However, many interesting systems are not that regular! In fact, it turns out that for many systems even very close initial conditions might get spread far apart in short times. For example, you probably have heard about the motion of a butterfly which can produce a perturbance of the atmosphere resulting in a thunderstorm a few weeks later.

We begin with the definition: A dynamical system is a semigroup G with identity element e acting on a set M. That is, there is a map

$$T : G \times M \to M$$

$$(g, x)\ 7\!\to\ T_g(x) \qquad (2.1)$$

Such that

$$T_g \circ T_h = T_{g \circ h}\,,\ T_e = I. \qquad (2.2)$$

If G is a group, we will speak of an invertible dynamical system. We are mainly interested in discrete dynamical systems where

$$G = N_0 \text{ or } G = Z \qquad (2.3)$$

and in continuous dynamical systems where

$$G = R^+ \ or\ G = R. \qquad (2.4)$$

Of course this definition is quite abstract and so let us look at some examples first .

Example 2.1 The prototypical example of a discrete dynamical system is an iterated map. Let f map an interval I into itself and consider

$$T_n = f^n = f \circ f^{n-1} = \underbrace{f \circ \ldots \circ f}_{n \text{ times}}, \quad G = N_0. \qquad (2.5)$$

Clearly, if f is invertible, so is the dynamical system if we extend this definition for $n \in Z$ in the usual way. You might suspect that such a system is too simple to be of any interest. However, we will see that the contrary is the case and that such simple systems bear a rich mathematical structure with lots of unresolved problems.

Example 2.2 The prototypical example of a continuous dynamical system is the flow of an autonomous differential equation

$$T_t = \phi_t, \qquad G = \mathbb{R} , \tag{2.6}$$

Which we will consider in the following section .

2.1 The flow of an autonomous equation

Now we will have a closer look at the solutions of an autonomous system

$$x' = f(x), x(0) = x_0. \tag{2.7}$$

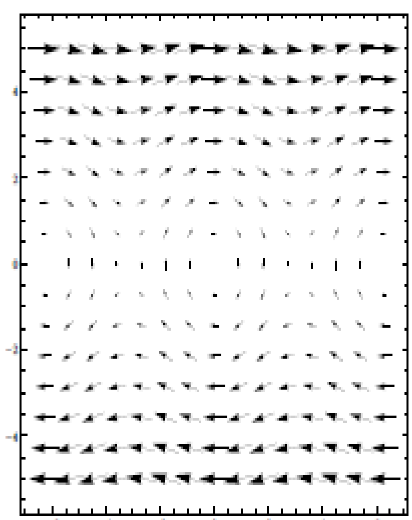
Throughout the rest of this book we will assume $f \in C^k (M, \mathbb{R}^n), k \geq 1$, where M is an open subset of $\mathbb{R}^n$.

Such a system can be regarded as a vector field on $\mathbb{R}^n$. Solutions are curves in $M \subseteq \mathbb{R}^n$ which are tangent to this vector field at each point. Hence to get a geometric idea of what the solutions look like, we can simply plot the corresponding vector field.

Example2.3 Using Mathematic the vector field of the mathematical pendulum, $f(x, y) = (y, -\sin(x))$, can be plotted as follows.

In [1]:= Vector Plot [{y, −Sin[x]}, {x, −2π, 2π}, {y, −5, 5}]

Out[1]=



In particular, solutions of the IVP (2.7) are also called integral curves or trajectories. We will say that φ is an integral curve at x_0 if it satisfies $\phi(0) = x_0$.

There is a (unique) maximal integral curve ϕ_x at every point x, defined on a maximal interval $I_x = (T-(x), T+(x))$.
Introducing the set

$$W = \cup_{X \in M} I_x \times \{x\} \subseteq R \times M \tag{2.8}$$

We define the flow of our differential equation to be the map

$$\Phi : W \to M, \qquad (t, x)\ 7\mapsto \phi(t, x), \tag{2.9}$$

Where $\phi(t, x)$ is the maximal integral curve at x. We will sometimes also use $\phi_x(t) = \phi(t, x)$ and $\phi_t(x) = \phi(t, x)$.

If $\phi(0)$ is the maximal integral curve at x, then $\phi(0 + s)$ is the maximal integral curve at $y = \phi(s)$ and in particular $I_x = s + I_y$. As a consequence, we note that for $x \in M$ and $s \in I_x$ we have

$$\phi(s + t, x) = \Phi(t, \Phi(s, x)) \tag{2.10}$$

For all $t \in I_{\phi}(s, x) = I_x - s$.

Theorem 2.1 Suppose $f \in C^k(M, R^n)$. For all $x \in M$ there exists an interval $I_x \subseteq R$ containing 0 and *a* corresponding unique maximal integral curve $\phi(0, x) \in C^k(I_x, M)$ at x. Moreover, the set W defined in (6.8) is open and $\phi \in C^k(W, M)$ is a (local) flow on M, that is, $\phi(0, x) = x$,

$$\phi(t + s, x) = \phi(t, \phi(s, x)), \qquad x \in M,\ s,\ t + s \in Ix. \tag{2.11}$$

Proof. It remains to show that W is open and $\phi \in C^k(W,M)$. Fix a point $(t_0, x_0) \in W$ (implying $t_0 \in Ix_0$) and set $\gamma = \phi\ x_0([0, t_0])$. There is an open neighborhood $(-\varepsilon(x), \varepsilon(x)) \times U(x)$ *of* $(0, x)$ around each point $x \in \gamma$ such that ϕ is defined and C^k on this neighborhood. Since γ is compact, finitely many of the neighborhoods $U(x)$ cover γ and hence we can find an $\varepsilon > 0$ and an open neighborhood U_0 of γ such that ϕ is defined on $(-\varepsilon, \varepsilon) \times U_0$. Next, pick $m \in N$ so large that $\frac{t_0}{m} < \varepsilon$ such that $K \in C^k(U0,M)$, where $K : U_0 \to M$, $K(x) = \phi\frac{t_0}{m}$. Furthermore, $K^j \in C^k(U_j, M)$ for any $0 \leq j \leq m$, where $U_j = K^{-j}(U_0) \subseteq U_0$ is open. Since $x_0 = K^{-j}(\phi(\frac{j}{m}t_0, x_0))$ we even have $x_0 \in U_j$, that is, U_j is nonempty. In particular,

$$\phi(t, x) = \phi(t - t_0, \phi(t_0, x)) = \phi(t - t_0, K^m(x))$$

is defined and C^k for all $(t, x) \in (t_0 - \varepsilon, t_0 + \varepsilon) \times U_m$.

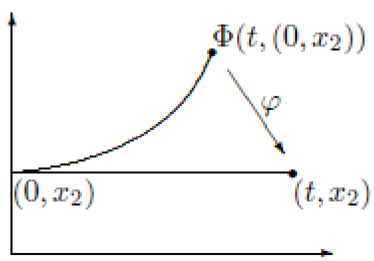
In particular, choosing $s = -t$ respectively $t = -s$ in (6.11) shows that $\phi t(0) = \phi(t, 0)$ is a local diffeomorphism with inverse $\phi_{-t}(0)$. Note also that if we replace $f \to -f$, then $\phi(t, x) \to \phi(-t, x)$.

Example 2.4 Let M = R and $f(x) = x^3$. Then $W = \{(t, x)|2tx^2 < 1\}$ and $\phi(t, x) = x \frac{x}{\sqrt{1-2x^2t}}$. $T-(x) = -\infty$ and $T+(x) = 1/(2x^2)$.
A point x_0 with $f(x_0) = 0$ is called a fixed point. Away from such points all vector fields look locally the same.
Lemma 2.1 (Straightening out of vector fields). Suppose $f(x_0) \neq 0$. Then there is a local coordinate transform $y = \phi(x)$ such that $x = f(x)$ is transformed to

$$y = (1, 0, \ldots, 0). \tag{6.12}$$

Proof. Abbreviate $\delta_1 = (1, 0, \ldots, 0)$. It is no restriction to assume $x_0 = 0$. After a linear transformation we see that it is also no restriction to assume
$f(0) = \delta_1$.

Consider all points starting on the plane $x_1 = 0$. Then the transform ϕ we are looking for should map the point $\phi(t, (0, x2, \ldots, xn))$ to
$(0, x_2, \ldots, x_n) + t(1, 0, \ldots, 0) = (t, x_2, \ldots, x_n)$.



Hence ϕ should be the inverse of

$$\psi((x_1, \ldots, x_n)) = \phi(x_1, (0, x_2, \ldots, x_n)),$$

Which is well defined in a neighborhood of 0 . The Jacobian determinant at 0 is given by

$$det\left(\frac{\partial\psi}{\partial x}\right) |_{x=0} = det\left(\frac{\partial\phi}{\partial t}\right), \left(\frac{\partial\phi}{\partial x_2}\right), \left(\frac{\partial\phi}{\partial x_n}\right) |_{t=0,x=0} = det\ \mathrm{I} = 1$$

since $\partial\phi/\partial x|_{t=0,x=0} = I_n$ and $\partial\phi/\partial t|_{t=0,x=0} = f(0) = \delta_1$ by assumption. So by the inverse function theorem we can assume that ψ is a local diffeomorphism and we can consider new coordinates $y = \psi-1(x)$. Since $(\partial\psi/\partial x)\delta_1 = \partial\psi/\partial x_1 = f(\psi(x))$ our system reads in the new coordinates

$$y = \left(\frac{\varphi\psi}{\varphi x}\right)^{-1} |_{y=\psi-1(x)}\ f(x) = \delta_1,$$

Which is the required form .

2.3 Orbits and invariant sets

The orbit of x is defined as

$$\gamma(x) = \phi\,(I_x \times \{x\}) \subseteq M. \qquad (6.15)$$

Note that $y \in \gamma(x)$ implies $y = \phi(t, x)$ and hence $\gamma(x) = \gamma(y)$ by (6.11). In particular, different orbits are disjoint (i.e., we have the following equivalence relation on M: $x \simeq y$ if $\gamma(x) = \gamma(y)$. If $\gamma(x) = \{x\}$, then x is called a fixed point (also singular, stationary, or equilibrium point) of ϕ. Otherwise x is called regular and $\phi(0\ , x) : I_x \to M$ is injective. Similarly we introduce the forward and backward orbits

$$\gamma\pm(x) = \_((0, T\pm(x), x). \qquad (6.16)$$

Clearly $\gamma(x) = \gamma_-(x) \cup \{x\} \cup \gamma_+(x)$. One says that $x \in M$ is a periodic point of ϕ if there is some $T > 0$ such that $\phi(T, x) = x$. The lower bound of such T is called the period, $T(x)$ of x, that is, $T(x) = \inf\{T > 0 | \phi(T, x) = x\}$. By continuity of ϕ we have $\phi(T(x), x) = x$ and by the flow property $\phi(t + T(x), x) = \phi(t, x)$. In particular, an orbit is called a periodic orbit if one (and hence all) point of the orbit is periodic.

It is not hard to see that x is periodic if and only if $\gamma+(x) \cap \gamma-(x)$ 6= $\emptyset$ and hence periodic orbits are also called closed orbits.

Hence we may classify the orbits of f as follows:

(i) fixed orbits (corresponding to a periodic point with period zero).

(ii) regular periodic orbits (corresponding to a periodic point with positive period).

(iii) non-closed orbits (not corresponding to a periodic point).

The quantity $T+(x) = \sup I_x$ (resp. $T-(x) = \inf I_x$) defined in the previous section is called the positive (resp. negative) lifetime of x. A point $x \in M$ is called σ complete, $\sigma \in \{\pm\}$, if $T_\sigma(x) = \sigma\infty$ and complete if it is both + and − complete (i.e., if Ix = R).

Lemma 2.2 (i). Arbitrary intersections and unions of σ invariant sets are σ invariant. Moreover, the closure of a σ invariant set is again σ invariant. (ii). If U and V are invariant, so is the complement U\V .

Proof. Only the last statement of (i) is nontrivial. Let U be σ invariant and recall that $x \in \overline{U}$ implies the existence of a sequence $x_n \in U$ with $x_n \to x$. Fix $t \in$

I_x. Then (since W is open) for N sufficiently large we have $t \in I_{x_n}$, $n \geq N$, and $\phi(t, x) = \lim_{n \to \infty} \phi(t, x_n) \in \overline{U}$.

Concerning (ii) let $x \in U \backslash V$. Then, if $\gamma(x) \cap V$ contains some point y, we must have $\gamma(y) = \gamma(x) \subseteq V$ contradicting our assumption $x \neq V$. Thus $\gamma(x) \subseteq U \backslash V$.

One of our main aims will be to describe the long-time asymptotics of solutions. For this we next introduce the set where an orbit eventually accumulates:

The $\omega\pm$-limit set of a point $x \in M$, $\omega\pm(x)$, is the set of those points $y \in M$ for which there exists a sequence $t_n \to \pm\infty$ with $\phi(t_n, x) \to y$.

Clearly, $\omega\pm(x)$ is empty unless x is $\pm$ complete. Observe, that $\omega\pm(x) = \omega\pm(y)$ if $y \in \gamma(x)$ (if $y = \phi(t, x)$ we have $\phi(t_n, y) = \phi(t_n, \phi(t, x)) = \phi(t_n + t, x)$). Hence $\omega\pm(x)$ depends only on the orbit $\gamma(x)$. Moreover .

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من قبل

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