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Nearly Maximal Ideals

A research

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By

Ahmed Jraeo Abd

Supervised by

Assist. Lectur. Adel Salim Tayyah

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Signature:

Name: Adel Salim Tayyah

Title: Assist Lecture

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Contents

Abstract

We define ideal weaker than maximal ideal, namely nearly maximal ideal. Some results and properties on nearly maximal ideal are discussed.

Introduction

Throughout our work all rings are monoid. The basic definitions and results on ring theory are mentioned in section1. In section2, we generalize the maximality of ideals by new concept as: a right ideal A is nearly maximal of R, if $B/A \subseteq^s R/A$ for every right ideal B satisfy $A \subseteq B \subsetneq R$. Counterexamples are given to explain that nearly maximality is dissimilar maximality. We also give, in section2, some results and characteristics about nearly maximal ideals describe their relationship with some rings and ideals.

1.Primary definitions and results in rings

Definition 1.1[1]. A set $R \neq \emptyset$ with (+) and (·) is called ring if the conditions below hold:

(1) $(R, +)$ forms an abelian group.
(2) $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ for all $x, y, z \in R$.
(3) $x \cdot (y + z) = x \cdot y + x \cdot z$ and $(y + z) \cdot x = y \cdot x + z \cdot x$ for all $x, y, z \in R$.

If R has identity element 1 where $x \cdot 1 = 1 \cdot x = x$ for all $x \in R$, then it is called monoid. Throughout our work every rings are monoid. A subset $I \neq \emptyset$ of R is called right ideal if I is ring and $x \cdot r \in I$, $\forall x \in I, \forall r \in R$. Easley to check that $R/I = \{r + I: \ r \in R\}$ with $(r_1 + I) +' (r_2 + I) = (r_1 + r_2) + I$ and $(r_1 + I) \cdot' (r_2 + I) = (r_1 \cdot r_2) + I$ forms ring. For any two ideals I_1 and I_2, the sum $I_1 + I_2$ is called direct sum if $I_1 \cap I_2 = 0$ and denoted by $I_1 \oplus I_2$, also I_1 is called summand.

Definition 1.2[2]. The right ideal I of a ring R is called:

(1) Maximal if $I \subsetneq K \subseteq R$ implies $K = R$ for all right ideal K and denote $I \subseteq^{max}$.
(2) Small in R if $I + K = R$ implies $K = R$ for all right ideal K and denote by $I \subseteq^{s} R$.
(3) Simple if $I \neq 0$ and has no nonzero ideal contained in it.
(4) Semisimple if any right ideal of R contained in I is summand of I.

Definition 1.3[2]. The ring is called division if its elements has inverse; equivalently, if it is simple.

Theorem 1.4[2, Corollary 3.1.14]. A is a maximal ideal in $R \leftrightarrow$ the ring R/A is division.

Theorem 1.5[3,Theorem 1.5, p.9]. If I and K be right ideals of R, then $(I + K)/K \cong I/(I \cap K)$.

Definition 1.6[2, Definition 9.1.2]. Jacobson radical of any ring R is defined as

$$J(R) = \bigcap_{A \subseteq^{max} R} A = \sum_{B \subseteq^{s} R} B$$

Theorem 1.7[2, Theorem 9.2.1]. If a right ideal I of R is semisimple, then $J(I) = 0$.

Proposition 1.8[1, Proposition 1.2.3]. Every ideal is contained in some maximal ideals.

Remark 1.9[2,Theorem 9.1.4(b)]. For any ring R, $J(R/J(R)) = 0$.

2. Results

Definition 2.1. We say the right ideal A is nearly maximal of R, if $B/A \subseteq^s R/A$ for every right ideal B satisfay $A \subseteq B \subsetneq R$.

Example 2.2.

(1) Every maximal right ideal is nearly maximal.

(2) The all ideals in the ring $\mathbb{Z}_{12}$ are $< 2 >$, $< 3 >$, $< 4 >$, $< 6 >$. The only maximal ideals are $< 2 >$ and $< 3 >$. We have $< 4 > \subseteq < 2 > \subseteq \mathbb{Z}_{12}$. Since $< 2 >/< 4 > \cong < 6 >$ and $\mathbb{Z}_{12}/< 4 > \cong < 3 >$ by Theorem 1.5, and $< 6 > \subseteq^s < 3 >$. Thus $< 4 >$ is nearly maximal in $\mathbb{Z}_{12}$ but it is not maximal ideal.

(3) Assume that $6\mathbb{Z}$ is nearly maximal ideal in $\mathbb{Z}$. Since $6\mathbb{Z} \subseteq 2\mathbb{Z} \subseteq \mathbb{Z}$, then $2\mathbb{Z}/6\mathbb{Z} \subseteq^s \mathbb{Z}/6\mathbb{Z}$. But $\mathbb{Z}/6\mathbb{Z}$ is semisimple and hence 0 is the only small in $\mathbb{Z}/6\mathbb{Z}$ by Theorem 1.7. Thus $2\mathbb{Z}/6\mathbb{Z} \cong 0$, that is $6\mathbb{Z} = 2\mathbb{Z}$, a contradiction. Therefore $6\mathbb{Z}$ is not nearly maximal ideal.

Recall that, a ring R is called local if $R/J(R)$ is division ring; equivalently, $J(R)$ is a maximal right (or left) ideal (see [3, p. 96]).

Proposition 2.3. A right ideal A of a ring R is nearly maximal iff R/A is local ring.

Proof. ($\Rightarrow$) Let K/A any maximal ideal in R/A. Since A is nearly maximal, then $K/A \subseteq^s R/A$. Therefore, $K/A \subseteq J(R/A)$, but $J(R/A) \subseteq K/A$, and so $K/A = J(R/A)$. Hence $J(R/A)$ is maximal and this means that R/A is local.

($\Leftarrow$) Let R/A is local ring , then $J(R/A)$ is maximal ideal. Now, let K any right ideal such that $I \subseteq K \subsetneq R$. Thus by Proposition 1.8, there is a maximal ideal N/A contained K/A. But $J(R/A)$ is the only maximal ideal, thus $K/A \subseteq J(R/A)$, and so $K/A \subseteq^s R/A$. Hence A is nearly maximal in R. ■

Corollary 2.4. For a ring R and a right ideal A, the following conditions are corresponding:

(1) If A is nearly maximal, then it is maximal.
(2) If R/A is local ring, then it is division ring.

Proof. (1)⇒(2) Let R/A is local ring, then A is nearly maximal by Proposition 2.3, and so A is maximal by hypothesis. Thus R/A is division ring by Theorem 1.4.

(2)⇒(1) Let A is a nearly maximal right ideal, then R/A is local ring by Proposition 2.3, and hence R/A is division ring. Therefore A is maximal ideal by Theorem 1.4. ■

Proposition 2.5. Let A be a proper ideal in R, then the following are corresponding:

(1) A is a nearly maximal ideal.
(2) If m is not generator element in R and $m \notin A$, then $< m + A > \subseteq^s R/A$.

Proof. (1)⇒(2) Assume that A is nearly maximal right ideal. Let $m \in R$ where m is not generator element and $m \notin A$. Since $A \subseteq mR + A \subsetneq R$, thus $< m + A > \subseteq^s R/A$.

(2)⇒(1) Suppose that A is not nearly maximal right ideal, thus there is right ideal B such that $A \subsetneq B \subsetneq R$ and B/A is not small in R/A. Now, let $b \in B$ and $b \notin A$. By hypothesis, $< b + A > \subseteq^s R/A$. Thus $\sum_{\forall b \in B} < b + A > \subseteq J(R/A)$, and hence $B/A \subseteq J(R/A)$, so $B/A \subseteq^s R/A$, and this a contradiction. Thus A is nearly maximal. ■

Proposition 2.6. Every proper right ideal contains nearly maximal right ideal is also nearly maximal.

Proof. Let A be a nearly maximal right ideal of a ring R and B any right ideal with $A \subseteq B$. If K any right ideal where $A \subseteq B \subseteq K \subsetneq R$, then $K/A \subseteq^s R/A$. Define the map $\alpha: R/A \longrightarrow R/B$ by $\alpha(r + A) = r + B$. Now we will prove α is epimorphism, $\alpha\big((r + A) + (s + A)\big) = \alpha\big((r + s) + A\big) = (r + s) + B =$ $(r + B) + (s + B) = \alpha(r + A) + \alpha(s + A)$ and $\alpha\big((r + A) \cdot (s + A)\big) = \alpha\big((r \cdot s) + A\big) = (r \cdot s) + B = (r + B) \cdot (s + B) = \alpha(r + A) \cdot \alpha(s + A)$ for all $r, s \in R$, and clear that α is onto. Let $(K/B) + (L/B) = R/B$, then $(B/A) + (K/A) + (L/A) = R/A$ where $\ker(\alpha) = B/A$, since $K/A \subseteq^s R/A$, then $(B/A) + (L/A) = R/A$, so $\alpha(B/A) + \alpha(L/A) = \alpha(R/A)$. Hence $L/B = R/B$, thus $K/B \subseteq^s R/B$, and this means B is nearly maximal right ideal. ■

Proposition 2.7. For a ring R which has at less two proper ideals. If every right ideals are maximal, then R is semisimple.

Proof. Let A and B any two proper right ideals with $A \neq B$. Thus A and B are maximal ideals in R. Now, if $A \cap B \neq 0$, then $A \cap B$ is maximal ideal by hypothesis, and so $A \cap B = A$, that is $A \subseteq B$. Since A and B are maximal ideals, then $A = B$, and this a contradiction with $A \neq B$. Therefore $A \cap B = 0$. If $A = A + B$, then $B \subseteq A$, and hence $A = B$, a contradiction. So $A \subsetneq A + B$. Thus the maximality of A implies that $A + B = R$, and hence $A \bigoplus B = R$. Therefore R is semisimple ring. ■

The converse of above result is not necessary correct for example $\mathbb{Z}_6 \bigoplus \mathbb{Z}_6$ is semisimple but the ideal $\mathbb{Z}_6$ is not maximal.

Theorem 2.8. The ring R is local $\longleftrightarrow J(R)$ is nearly maximal as right (or left) ideal.

Proof. ($\Rightarrow$) By definitions of nearly maximal ideal and local ring.

(⇐) Let $J(R)$ be a nearly maximal right ideal and let A any maximal ideal, thus $J(R) \subseteq A \subsetneq R$. Therefore, $A/J(R) \subseteq^{s} R/J(R)$, but $J(R/J(R)) = 0$ by Remark 1.9, so $A/J(R) = 0$, that is $A = J(R)$. Therefore $J(R) \subseteq^{max} R$ means that R is local. ■

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