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Some Commutativity Theorems For $*$ −Prime Rings With (σ, τ) −Dderivation

A Research

Submitted by

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بسم الله الرحمن الرحيم

(قُلْ هَلْ يَسْتَوِي الَّذِينَ يَعْلَمُونَ وَالَّذِينَ لا يَعْلَمُونَ إِنَّمَا يَتَذَكَّرُ أُوْلُوا الأَلْبَابِ)

صدق الله العظيم

سورة الزمر/ جزء من الايه ٩

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الى رسولنا الكريم محمد (صلى الله عليه واله وسلم) والى

آله الاطهار والى صاحب العصر والزمان (عليهم السلام)

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شكر وتقدير

الحمد لله المعلم بالقلم والشكر له على ما جاد وانعم والصلاة والسلام على نبيه وسيد المرسلين محمد(صلى الله عليه واله وسلم)

وبعد... فلا يسعنا ونحن نختم هذا العمل الا ان نخط كلمات صغيرة في حجمها لكنها كبيرة فيما تحمله من معاني الشكر والامتنان الى كل ذي فضل اسهم بجهد او مشورة او دعم في انجاز هذا البحث في البداية اتقدم بالشكر الجزيل الى الدكتورة رجاء جفات (مشرفة البحث) والتي امدتنا بالنصح والمشورة وتفضلت علينا بالاشراف على البحث ليضهربشكله المتكامل.

ونقدم شكرنا وامتناننا لاساتذتنا الافاضل لجهودهم البناءة في ارواء ظمأنا الى معين العلم والمعرفة.

كما نقدم شكرنا الى كل الزملاء في الدراسة والى الاصدقاء الذين امدونا بدعمهم وعونهم ودعائهم.

وختاما نسأل الله العلي القدير ميسر الامور ومفرج الهموم ان نكون قد وفقنا فيما قدمناه في هذا البحث.

Abstract

Let R be a $*$-prime ring with center $Z(A)$, d a non-zero (σ,τ) derivation of R with associated automorphisms σ and τ of R, such that σ and τ and d commute with '$*$' Suppose that U is an ideal of R such that $U^* = U$ and $C_{\sigma,\tau} = \{c\epsilon R: c\sigma(x) = \tau(x)c, for\ all\ x\epsilon R\}$. In the present paper, it is shown that if characteristic of R is different from two and $[d(u), d(v)]_{\sigma,\tau} = \{0\}$, then R is commutative. Commutativity of R has also been established in case if $[d(R), d(R)]_{\sigma,\tau} \subseteq C_{\sigma,\tau}$.

Chapter one

Introduction

Preliminaries

Definition (1.1)(Derivation):-

An additive mapping $d : R \to R$ is said to be a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y\epsilon R$.

Definition (1.2)(inner derivation):-

The mapping $I_a : R \to R$ given by $I_a(x) = [a, x]$ for fixed $a\epsilon R$, is a derivation which is said to be an inner derivation.

Definition (1.3)(prime):- Recall that R is said to be prime if $aRb = \{0\}$ implies $a = 0$ or $b = 0$.

Definition(1.4)(2- torsion):-

A ring R is said to be 2-torsion free, if $2x = 0$ implies $x = 0$.

Definition(1.5)((σ, τ) −derivation):-

For any two endomorphism $\sigma$and τ of R, we call an additive mapping $d : R \to R$ a(σ, τ) -derivation if $d(xy) = d(x)\sigma(y) + \tau(x)d(y)$ for all $x, y\epsilon R$.

Example:-

Let $R = \left\{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \backslash a, b, c \epsilon \mathbb{Z}\right\}$ be the ring of all 2×2 matrices over $\mathbb{Z}$, the ring of integers. Define d, σ, τ: $R \to R$ such that $d\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$, $\sigma\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ and $\tau\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$, It can be easily seen that $\sigma$and τ are auto morphisms of R, and d is a (σ, τ)-derivation which is not a derivation of R. We set $C_{\sigma,\tau} = \{x \epsilon R: x\sigma(y) = \tau(y)x, for\ all\ y \epsilon R\}$ and $[x, y]_{\sigma,\tau} = x\sigma(y) - \tau(y)x$. In particular $C_{1,1} = \mathbb{Z}(R)$, is the center of R, and $[x;\ y]_{1,1} = [x;\ y] = xy - yx$, is the usual Lie product. An additive mapping $x \to x^*$ on a ring R is called an involution if $(x *) * = x$ and $(xy) * = y * x *$ hold for all $x, y\ \epsilon R$.

Definition(1.6)(involution):-

An additive mapping $x \to x^*$ on a ring R is called an involution if $(x *) * = x$ and $(xy) * = y * x *$ hold for all $x, y\ \epsilon R$.

Definition(1.7)($*$-ring):-

A ring equipped with an involution is called a ring with involution or $*$-ring.

Definition(1.8)($*$-prime):-

A ring R equipped with an involution $'*'$ is said to be $*$-prime if $aRb = aRb^* = \{0\}$ (or, equivalently $aRb = a^*Rb = \{0\}$) implies a = 0 or b = 0. If R^o denotes the opposite ring of a prime ring R, then $S = R \times R^o$ equipped with the exchange involution $*_{ex}$ defined by $*_{ex}(x,y) = (y,x)$ is $*_{ex}-$prime, but not a prime ring because of the fact that $(1,0)S(0,1) = 0$. In all that follows, $Sa_*(R)$ will denote the set of symmetric and skew symmetric elements of R, i.e., $Sa_*(R) = \{x \epsilon R \backslash x^* = \mp x\}$: An ideal U of R is said to be a $*-$ideal of R if $U^* = U$. It can also be noted that an ideal of a ring R may not be $*-$ideal of R. As an example, let $R = \mathbb{Z} \times \mathbb{Z}$, and consider the involution $'*'$ on R such that $(a,b)^* = (b,a)$ for all $(a,b) \epsilon R$: The subset $U = \mathbb{Z} \times \{0\}$ of R is an ideal of R but it is not a $*-$ideal of R, because $U^* = \{0\} \times \mathbb{Z} \neq U$. Recently many authors have studied commutativity of prime and semi prime rings with involution admitting suitably constrained derivations. A lot of work have been done by L. Okhtite and co-authors on rings with involution (see for reference [11, 12, 13], where further references can be found). In [10], Lee and Lee proved that if a prime ring of characteristic different from 2 admits a derivation d such that $[d(R), d(R)] \subseteq \mathbb{Z}(R)$, then R is commutative. On the other hand in [7] for $a \epsilon R$. Herstein proved that if $[a, d(R)] = \{0\}$, then $a \epsilon \mathbb{Z}(R)$. Further in the year 1992, Aydin together with Kaya [4] extended the theorems mentioned above by replacing derivation by (σ, τ) $-$derivation and in some of those, R by a non-zero ideal of R. In this note, we investigate the commutativity of $*$ $-$prime ring R equipped with an involution $'*'$ admitting a (σ, τ) $-$derivation d satisfying $[d(U), d(U)]_{\sigma,\tau} = \{0\}$ and $[d(R); d(R)]_{\sigma,\tau} \subseteq C_{\sigma,\tau}$ where U is a nonzero $*$ $-$ideal of R.

Chapter two

the results

In the remaining part of the paper, R will represent a $*-$prime ring which admits a non-zero $(\sigma,\tau\ )-$derivation d with auto morphisms σ and τ such that $'*'$ commutes with d,σ and τ. We shall use the following relations frequently without specific mention.

$[xy,z]_{\sigma,\tau} = x[y,z]_{\sigma,\tau} + [x,\ \tau(z)]y = x[y,\sigma(z)] + [x,z]_{\sigma,\tau}\, y$

$[x,yz]_{\sigma,\tau} = \tau(y)[x,z]_{\sigma,\tau} + [x,y]_{\sigma,\tau}\sigma(z)$. And

$[x,[y,z]]_{\sigma,\tau} + [[x;\ z]_{\sigma,\tau} - [[x,y]_{\sigma,\tau},z]_{\sigma,\tau} = 0.$

Remark(2.1):-

We find that if R is a $*-$prime ring of characteristic different from 2, then R is 2-torsion free. In fact, if $2x = 0$ for all $x \epsilon R$, then $xr(2s) = 0$ for all $r,s \epsilon R$. But since char $R \neq 2$, there exists a non-zero lϵR such that $2l \neq 0$ and hence by the above $xR(2l) = \{0\}$. This also gives that $xR(2l)^{*} = \{0\}$ and $*-$primeness of R yields that $x = 0$, that mean, R is 2-torsion free.

The main result of the present paper states as follows:

Theorem (2.2):-

Let R be a $*-$ prime ring with characteristic different from two and σ,τ be automorphisms of R, and U a $*-$ideal of R. If R admits a non- zero $(\sigma,\tau\ )$-derivation $d : R \to R$ such that $[d(U),d(U)]_{\sigma,\tau} = \{0\}$, then R is commutative. We facilitate our discussion with the following lemmas which are required for developing the proof of our main result. Since every $*-$prime ring is semiprime and every $*-$right ideal is right ideal, hence Lemmas 1.1.4 and 1.1.5 of [5] can be rewritten in case of $*-$prime ring as follows:

Lemma(2.3):-

Suppose that R is a $*-$prime ring and that $a \epsilon R$ is such that $a(ax - xa) = 0$ for all $x \epsilon R$: Then $a \epsilon \mathbb{Z}(R)$.

Lemma(2.4):-

Let R be a $*-$prime ring and U a non-zero $*-$right ideal of R. Then $\mathbb{Z}(U) \subseteq \mathbb{Z}(R)$.

Corollary(2.5):-

Let R be a $*-$prime ring and U a non-zero $*-$right ideal of R. If U is commutative then R is commutative.

Proof:- Since U is commutative, by the Lemma 2.4, we have $U = \mathbb{Z}(U) \subseteq Z(R)$. If for any $x, y \epsilon R$, $a \epsilon U$ we have $ax \epsilon U$ then $ax \epsilon Z(R)$, and hence $(ax)y = y(ax) = ayx$. This further yields $U(xy - yx) = \{0\}$. Since U is a non-zero $*-$right ideal of R, we have $UR(xy - yx) =$

$\{0\} = U^*R(xy - yx)$. Also, since $U \neq \{0\}$ is a right ideal, $*-$primeness of R gives $xy - yx = 0$, for all $x, y\epsilon R$. Hence R is commutative.

Lemma(2.6):-

Let R be a $*-$prime ring and U a non-zero $*-$right ideal of R. Suppose that $a\epsilon R$ centralizes U. Then $a\epsilon\mathbb{Z}(R)$.

Proof:- Since a centralizes U, for all $u\epsilon U$ and $x\epsilon R$, $aux = uxa$. But $au = ua$, therefore $uax = uxa$, that mean, $u[a, x] = 0$. On replacing u by uy for any $y\epsilon R$, we get $uR[a, x] = \{0\}$ for all $u\epsilon U$, $x\epsilon R$. Also, since U is $*-$right ideal, we get $u^*R[a, x] = \{0\}$. Again since $U \neq \{0\}$, $*-$primeness of R yields that $[a, x] = 0$ for all $x \epsilon R$. Therefore, $a\epsilon Z(R)$.

Lemma(2.7):-

Let R be a $*-$ prime ring with characteristic different from two and suppose that $a\epsilon R$ commutes with all its commutators $ax - xa$ for all $x\epsilon R$. Then $a\epsilon\mathbb{Z}(R)$.

Proof:- Define $d : R \to R$ by $d(x) = ax - xa$ for all $x \epsilon R$: By hypothesis we arrive at

(2.1) $\quad d^2(x) = 0$ for all $x \epsilon R$.

Also, $d^2(xy) = d^2(x)y + 2d(x)d(y) + xd^2(y)$. By (2.1) and using torsion restriction on R, we get $d(x)d(y) = 0$ for all $x, y\epsilon R$: On replacing y by yz for any $z\epsilon R$, we obtain $d(x)Rd(y) = \{0\}$, also $d(x)^*Rd(y) = \{0\}$ for all $x, y\epsilon R$. Using $*-$primeness of R yields that $d(x) = 0$ for all $x\epsilon R$. Recalling that $d(x) = ax - xa$, we obtain $a\epsilon\, Z(R)$.

Lemma(2.8):-

Let R be a $*-$prime ring. Suppose that $ab, a^*b, b\epsilon\, C_{\sigma,\tau}$ for all $a, b\,\epsilon\, R$. Then either $a\epsilon\, \mathbb{Z}(R)$ or $b = 0$.

Proof:- Since $ab\epsilon C_{\sigma,\tau}$ $ab\sigma(x) = \tau(x)ab$ for all $x\epsilon R$. Also since $b\epsilon\, C_{\sigma,\tau}$ that mean $b\sigma(x) = \tau(x)b$ for all $x\epsilon R$, we have $a(b\sigma(x)) = \tau(x)ab$, or $a(\tau\,(x)b) = (\tau\,(x)a)b$, that mean $[a, \tau\,(x)]b = 0$. On replacing x by xy for any $y\,\epsilon\, R$, we get $[a, \tau(x)]Rb = \{0\}$ for all $x\,\epsilon\, R$.

Similarly, since $a^*b\epsilon C_{\sigma,\tau}$,we have

$[a^*, \tau\,(x)]Rb = \{0\}$ for all $x\epsilon R$. On replacing x by x^* in the above relation, we find that

$[a, \tau(x)Rb = \{0\}$ for all $x\,\epsilon\, R$. Therefore, on using $*-$primeness of R, we find that either $[a, \tau(x)] = 0$ or $b = 0$ for all $x\epsilon R$. Hence, we conclude that $a\,\epsilon\mathbb{Z}(R)$ or $b = 0$.

Corollary(2.9):-

Let R be a $*-$prime ring. Suppose that $ab = 0 = a^{*}b, b\epsilon C_{\sigma,\tau}$ for all $\mathfrak{a}, b\epsilon R$. Then either $a = 0$ or $b = 0$.

Proof:- Since $b\epsilon\, C_{\sigma,\tau}$, $b\sigma(x) = \tau(x)b$. Left multiplying by a and a^{*} and on using $ab = 0$ and $a^{*}b = 0$, we obtain $ab\sigma(x) = a\tau(x)b = 0$, for all $x\epsilon R$, that mean $aRb = \{0\}$ and $a^{*}b\sigma(x) = a^{*}\tau(x)b = 0$, for all $x \epsilon R$, that mean, $a^{*}Rb = \{0\}$, respectively. Hence, $*-$primeness of R yields either $a = 0$ or $b = 0$.

Lemma(2.10):-

Let R be a $*-$prime ring and U a $*-$right ideal of R. If $d(U) = \{0\}$, then $d = 0$.

Proof:- For all $u\,\epsilon U$ and $x\epsilon R, 0 = d(ux) = d(u)\sigma(x) + \tau(u)d(x) = \tau(u)d(x)$. On replacing x by xy for any $y\epsilon R$, we get $\tau(u)d(x)\sigma(y) + \tau(u)\tau(x)d(y) = 0$. or, $\tau(u)\tau(x)d(y) = 0$, that mean $\tau(u)Rd(y) = \{0\}$ for all $u\epsilon U$ and $y\,\epsilon R$. Also since U is a $*-$right ideal, we get $\tau(u)^{*}Rd(y) = \{0\}$. Also, $*-$primeness of R yields that $\tau(u) = 0$ for all $u\,\epsilon U$ or $d = 0$. Since $U \neq \{0\}$ we get $d = 0$.

Lemma(2.11):-

Let R be a $*-$prime ring, U a non-zero $*-$ideal of R and $a\epsilon R$. If $ad(U) = \{0\}$ (or, $d(U)a = \{0\}$), then $a = 0$ or $d = 0$.

Proof:- For $u\epsilon U,\ x\epsilon R, 0 = ad(ux) = ad(u)\sigma(x) + a\tau(u)d(x)$. By assumption, we have $a\tau(u)d(x) = 0$, for all $x \epsilon R$: On replacing u by uy for any $y\epsilon R$, we obtain $a\tau(u)Rd(x) = \{0\}$ for all $u \epsilon U, x\epsilon R$: Also, $a(u)Rd(x)^* = \{0\}$. Since R is $*-$prime, we find that either $a\tau(u) = 0$ or $d(x) = 0$. If $a\tau(u) = 0$ for all $u \epsilon U$ or $\tau^{-1}(a)u = 0$, or $\tau^{-1}(a)U = \{0\}$. Now since U is $*-$ideal, we can write $\tau^{-1}(a)U^* = \{0\}$. This implies that $\tau^{-1}(a)RU^* = \{0\} = \tau^{-1}(a)RU^*$. By the $*-$primeness of R, we obtain $\tau^{-1}(a) = 0$, since $U \neq \{0\}$: In conclusion, we get either $a = 0$ or $d = 0$. Similarly, $d(U)a = \{0\}$ implies $a = 0$ or $d = 0$.

Lemma(2.12):-

Let d be a non-zero $(\sigma,\tau)-$derivation of $*-$prime ring R and U a $*-$right ideal of R. If $d(U) \subseteq \mathbb{Z}(R)$, then R is commutative.

Proof:- Since $d(U) \subseteq \mathbb{Z}(R)$, we have $[d(U), R] = \{0\}$. For $u, v\epsilon U$ and $x\epsilon R$,

(2.2) $\qquad [x, d(uv)] = [x, d(u)\sigma(v) + \tau(u)d(v)] = d(u)[x, \sigma(v)] + d(v)[x, \tau(u)] = 0.$

Replacing x by $x\sigma(v), v\epsilon U$ in (2.2), we have

$0 = d(u)[x\sigma(v), \sigma(v)] + d(v)[x\sigma(v), \tau(u)]$

$= d(u)[x, \sigma(v)]\sigma(v) + d(v)(x[\sigma(v), \tau(u)] + [x, \tau(u)]\sigma(v)).$

By using (2.2), we get

(2.3) $\quad d(v)R[\sigma(v), \tau(u)] = \{0\}$, for all $u, v \epsilon U$.

Let $v \epsilon U \cap Sa * (R)$. From (2.3), it follows that

(2.4) $\quad d(v)^* R[\sigma(v), \tau(u)] = \{0\}$, for all $u \epsilon U$.

By (2.3) and (2.4), the $*-$primeness of R yields that $d(v) = 0$ or $[\sigma(v), \tau(u)] = 0$

for any $v \epsilon U \cap Sa * (R)$ and for all $u \epsilon U$. Let $w \epsilon U$, since $w - w^* \epsilon U \cap Sa * (R)$, then $d(w - w^*)= 0$ or $[\sigma(w - w^*)\tau(u)] = 0$. Assume that $d(w - w^*) = 0$. Then $d(w) = d(w^*)$: Replacing v by w^* in (2.3)and since U is $*-$right ideal, we get $d(w^*)R[\sigma(w^*), \tau(u)] = \{0\}$ for all $u \epsilon U$. Consequently,

(2.5) $\quad d(w)R[\sigma(w), \tau(u)]^* = \{0\}$, for all $u, w \epsilon U$.

Also by (2.3), we get $d(w)R[\sigma(w), \tau(u)] = \{0\}$, the $*-$primeness of R together with (2.5) assures that $d(w) = 0$ or $[\sigma(w), \tau(u)] = 0$, for all $u \epsilon U$: Now suppose that $[\sigma(v), \tau(u)] = 0$; for all $v \epsilon U \cap Sa * (R)$ and $u \epsilon U$. We have

$[\sigma(w-w^*),\tau(u)] = 0$; for all $u\epsilon U$, or $[\sigma(w),\tau(u)] = [\sigma(w^*),\tau(u)]$. Replacing v by w^* in (2.3), we get $d(w^*)R[\sigma(w^*),\tau(u)] = \{0\}$ for all $u\epsilon U$. Consequently,

(2.6) $\qquad d(w^*)R[\sigma(w),\tau(u)]^* = \{0\}$, for all $u\epsilon U$.

Since $d(w)R[\sigma(w),\tau(u)] = \{0\}$, by (2.3), the $*-$primeness of R together with (2.6) assures that $d(w) = 0$ or $[\sigma(w),\tau(u)] = 0$, for all $u\, \epsilon U$: In conclusion, for all $u\epsilon U$ we have either
$d(w) = 0$ or $[\sigma(w),\tau(u)] = 0$.

Now, define $K = \{w\epsilon U \,/\, d(w) = 0\}$ and $L = \{w\epsilon U/\, [\sigma(w),\tau(u)] = 0$ for all $u\,\epsilon U$. Then $U = K \cup L$. Since $d \neq 0$, we have $d(U) \neq \{0\}$ by Lemma 2.10, therefore, $U \neq K$. By Brauer's trick, we have (2.7) $[\sigma(w),\tau(u)] = 0$ for all $u, w \,\epsilon\, U$. Replacing w by $w\sigma^{-1}\big(\,(v)\big), u\,\epsilon U$, in (2.7) and using (2.7), we get $\sigma(w)([v,u]) = 0$, for all $u, v, w\,\epsilon\, U$. On replacing w by wx for any $x\epsilon R$, we get $\sigma(w)R\tau([v,u]) = \{0\}$, for all $u, v, w\,\epsilon\, U$. Also, since U is $*-$right ideal, we get $\sigma(w)^*R\tau\,([v,u]) = \{0\}$ for all $u, v, w\epsilon\, U$. Since R is $*-$prime, we find that $\sigma(w) = 0$ or $\tau[v,u] = 0$ for all $u, v, w\epsilon U$. Since $U \neq \{0\}$, we have U is commutative. In view of Corollary 2.5, we obtain the commutativity of R.

Using the same technique as in Lemma 4 of [4], we get the following lemma.

Lemma(2.13):-

Let R be a $*-$ prime ring with characteristic different from two, $d_1 : R \to R$ be a $(\sigma, \tau)-$derivation and $d_2 : R \to R$ be a derivation. If $d_1 d_2(R) = \{0\}$, then $d_1 = 0$ or $d_2 = 0$.

Proof:- Let us assume that $d_1 \neq 0$. Then for all $x, y \epsilon R$,

$0 = d_1 d_2(xy) = d_1(d_2(x)y + xd2(y)) = \tau(d_2(x))d_1(y) + d_1(x)\sigma(d_2 2(y)$. That is

(2.8) $\tau(d_2(x))d_1(y) = d_1(x)\sigma(d_2(y))$ for all $x, y \epsilon R$.

If we replace x by $d_2(x)$ in (2.8), we have $\tau(d_2^2(x))d_1(y) = 0$. This further reduces to $\tau(d_2^2(x)) = 0$ for all $x \epsilon R$, in view of Lemma 2.11. Therefore

(2.9) $d_2^2(x) = 0$ for all $x \epsilon R$.

Replacing x by $xd_2(z)$, $z \epsilon R$, in (2.8) and using (2.8) and (2.9), we get

$0 = \tau(d_2(xd_2(z)))d_1(y) + d_1(xd_2(z))\sigma(d_2(y))$

$= \tau(d_2(x))\tau(d_2(z))d_1(y) + d_1(x)\sigma(d_2(z))\sigma(d_2(y))$

$= -\tau(d_2(x))d_1(z)\sigma(d_2(y)) + d_1(x)\sigma(d_2(z))\sigma(d_2(y))$

$= d_1(x)\sigma(d_2(z))\sigma(d_2(y)) + d_1(x)\sigma(d_2(z))\sigma(d_2(y))$

So we obtain,

$2d_1(x)\sigma(d_2(z))\sigma(d_2(y)) \ = \ 0$ for all $x, y, z \epsilon R$.

Since characteristic of R is different from 2. Then by Lemma 2.11, we have

(2.10) $\quad d_2(z)d2(y) = 0$ for all $x, y\epsilon R$.

Again applying Lemma 2.11 to (2.10), we get $d_2 = \ 0$.

We are now well equipped to prove our main theorem:

Proof of Theorem 2.2. First we will show that if any $a\epsilon Sa_*(R)$ satisfies $[d(U), a]_{\sigma,\tau} = \{0\}$, then $a\epsilon \ Z(R)$.

$0 \ = \ [d(uv), a]_{\sigma,\tau} = [d(u)\sigma(v) + \tau(u)d(v), a]_{\sigma,\tau}$

$= \ d(u)\sigma(v)\sigma(a) + \tau(u)d(v)\sigma(a) - \tau\ (a)d(u)\sigma(v) - \tau(a)\tau(u)d(a)$

By hypothesis, $d(u)\sigma(a) = \tau(a)d(u)$ for all $u\epsilon U$: We have

(2.11) $\quad d(u)\sigma([v, a]) + \tau([u, a])d(v) = 0$ for all $u, v\ \epsilon U$

Replace v by va in (2.11) and use (2.11) to get

$0 = d(u)\sigma([v, a])\sigma(a) + \tau([u, a])(d(v)\sigma(a) + \tau\ (v)d(a))$

$= \{d(u)\sigma([v, a]) + \tau([u, a])d(v)\}\sigma(a) \ + \tau([u, a])\tau(v)d(a)$.

We have $\tau\ ([u, a])\tau(v)d(a) = 0$, for all $u, v\epsilon\ U$. Replacing v by vx for any $x\epsilon R$, we find that $\tau([u, a])\tau(v)Rd(a) = \{0\}$, for all $u, v\epsilon U$. Since

$a\epsilon Sa_*(R)$, the above expression can be rewritten as

$\tau([u,a])\tau(v)Rd(a)^* = \{0\}$ for all $u, v\epsilon U$. On using $*-$primeness of R, we obtain for all $u, v\epsilon U$

(2.12) $\quad \tau([u,a])\tau(v) = 0$ or $d(a) = 0$.

Let us suppose that $d(a) = 0$, then for all $u\epsilon U$, $d([u,a]) = [d(u),a]_{\sigma,\tau} - [d(a),u]_{\sigma,\tau} = 0$. That is

(2.13) $\qquad d([U,a]) = \{0\}$.

On replacing v by $vw, w\epsilon U$, in (2.11), we get

$0 = d(u)\sigma([vw,a]) + \tau([u,a])d(vw)$

$= d(u)\sigma(v)\sigma([w,a]) + d(u)\sigma([v,a])\sigma(w) + \tau([u,a])d(v)\sigma(w)$
$+\tau([u,a])\tau(v)d(w)$

$= d(u)\sigma(v)\sigma([w,a]) + \tau([u,a])\tau(v)d(w) + \{d(u)\sigma([v,a]) +$
$\tau([u,a])d(v)\}\sigma(w)$

By using (2.11), we have

(2.14) $d(u)\sigma(v)\sigma([w,a]) + \tau([u;\ a])\tau(v)d(w) = 0$ for all $u, v, w\epsilon U$.

Replacing w by $[w,a]$ in (2.14) and using (2.13), we get

$d(u)\sigma(v)\sigma([[w,a],a]) = 0$ for all $u, v, w\epsilon U$.

Replacing v by xv for any $x\epsilon R$ in the above relation, we find that

$d(u)R\sigma(v)\sigma\big([[w,a],a]\big)=\{0\}$ for all $u,v,w\epsilon U$: Also since U is $*-$ideal, we may obtain $d(u)^{*}R\sigma(v)\sigma\big([[w,a],a]\big)=\{0\}$ for all $u,v,w\epsilon U$. Using $*-$primeness of R, we get $d(U)=\{0\}$ or $\sigma(v)\sigma([[w,a],a])=0$ for all $u,v,w\epsilon U$. But $d(U)\neq\{0\}$, therefore, $\sigma(v)\sigma([[w,a],a])=0$ for all $u,v,w\epsilon U$: Replacing v by vx, and using U is $*-$ideal, we obtain

$\sigma(U)R\sigma\big([[w,a],a]\big)=\{0\}$ and $\sigma(U)^{*}R\sigma([[w,a],a])=\{0\}$ for all $w\,\epsilon\,U$. Since R is $*-$prime and $\sigma(U)\neq\{0\}$ is $*-$ideal of R, $\sigma\big([[U,a],a]\big)=\{0\}$.

In other words, if we define $I_a(x)=[x,a]$ an inner derivation determined by a then we have $I_a^2(U)=\{0\}$. By Lemma 2.13, $I_a=\{0\}$, that mean $[a,U]=\{0\}$, and so by Lemma 2.6, $a\epsilon\mathbb{Z}(R)$. In view of (2.12) let us now suppose that $\tau([u,a])\tau(v)=0$ for all $u,v\epsilon U$: On replacing v by xv for any $x\epsilon R$, the above equation reduces to $\tau([u,a])R\,(v)=\{0\}$, for all $u,v\epsilon\,U$. Also, U being a $*-$ideal, we get $\tau([u,a])R\,(v)=\{0\}$. Using the $*-$primeness of R yields either $\tau([U,a])=\{0\}$ or $\tau(U)=\{0\}$. Since $\tau(U)=\{0\}$ is not possible, it reduces to $\tau([U,a])=\{0\}$ and so $[U,a]=\{0\}$. In view of Lemma 2.6, we find that $a\epsilon Z(R)$. Hence by our hypothesis we obtain that $d(U)\subseteq\mathbb{Z}(R)$: So by Lemma 2.12, R is commutative.

Theorem(2.14):-

Let R be a $*-$ prime ring with characteristic different from two and σ, τ be automorphisms of R. If R admits a non-zero $(\sigma, \tau\ )-$derivation $d: R \rightarrow R$ such that $[d(R), d(R)]_{\sigma,\tau} \subseteq C_{\sigma,\tau}$, then R is commutative.

Proof:- First we will show that for any $a \epsilon Sa_*(R)$ satisfying $[d(R), a]_{\sigma,\tau} \subseteq C_{\sigma,\tau}$, we have $a\ \epsilon \mathbb{Z}(R)$: Suppose on contrary that $a \notin \mathbb{Z}(R)$. Using the hypothesis we have $[d(a^2), a]_{\sigma,\tau} \epsilon C_{\sigma,\tau}$,

$$[d(a^2), a]_{\sigma,\tau} = [d(a)\sigma(a) + \tau(a)d(a), a]_{\sigma,\tau}$$

$$= d(a)\sigma(a)\sigma(a) - \tau(a)\tau(a)d(a)$$

$$= [d(a), a^2]_{\sigma,\tau} = \tau(a)[d(a), a]_{\sigma,\tau} + [d(a), a]_{\sigma,\tau}\sigma(a)$$

$$= 2\ \tau(a)[d(a);\ a]_{\sigma,\tau}$$

Since char $R \neq 2$, we have$\tau(a)[d(a), a]_{\sigma,\tau} \epsilon C_{\sigma,\tau}$. Since $a \epsilon Sa_*(R)$, we also have $\tau(a)\sigma[d(a), a]_{\sigma,\tau} \epsilon C_{\sigma,\tau}$. In view of the hypothesis and Lemma 2.8, we get either $\tau(a)\ \epsilon\ \mathbb{Z}(R)$ or $[d(a), a]_{\sigma,\tau} = 0$. Since by our assumption a/2 Z(R), we have

(2.15) $[d(a), a]_{\sigma,\tau} = 0$. On the other hand, since $[d(R), a]_{\sigma,\tau} \subseteq C_{\sigma,\tau}$, for any $x \epsilon R$, $[d([a, x]), a]_{\sigma,\tau} \epsilon C_{\sigma,\tau}$. Therefore

$[d([a, x]), a]_{\sigma,\tau} = [[d(a), x]_{\sigma,\tau}, a]_{\sigma,\tau} - [[d(x), a]_{\sigma,\tau}, a]_{\sigma,\tau}$. We obtain

(2.16) $[[d(a), x]_{\sigma,\tau}, a]_{\sigma,\tau}, \epsilon C_{\sigma,\tau}$ for all $x \epsilon R$.

Replacing x by ax in (2.16)

$$[[d(a), ax]_{\sigma,\tau}, a]_{\sigma,\tau} = [\tau(a)[d(a), x]_{\sigma,\tau} + [d(a), a]_{\sigma,\tau}\sigma(x), a]_{\sigma,\tau}$$

$$= [\tau(a)[d(a), x]_{\sigma,\tau}, a]_{\sigma,\tau}.$$

$$= \tau(a)[[d(a), x]_{\sigma,\tau}, a]_{\sigma,\tau} + [\tau(a), \tau(a)][d(a), x]_{\sigma,\tau}.$$

We get $\tau(a)[[d(a), x]_{\sigma,\tau}, a]_{\sigma,\tau} \epsilon C_{\sigma,\tau}$ for all $x \epsilon R$: Since $a \epsilon Sa_*(R)$, we have $\tau(a)^*[[d(a), x]_{\sigma,\tau}, a]_{\sigma,\tau} \epsilon C_{\sigma,\tau}$ for all $x \epsilon R$. In view of (2.16), together with above two relations and Lemma 2.8, we obtain $\tau(a) \epsilon \, \mathbb{Z}(R)$ or $[[d(a), x]_{\sigma,\tau}, a]_{\sigma,\tau} = 0$. Since $a \notin \mathbb{Z}(R)$, we have

(2.17) $\quad [[d(a), x]_{\sigma,\tau}, a]_{\sigma,\tau} = 0$ for all $x \epsilon R$.

Now, applying the relation

$$[x, [y, z]]_{\sigma,\tau} + [[x, z]_{\sigma,\tau}, y]_{\sigma,\tau} - [[x, y]_{\sigma,\tau}, z]_{\sigma,\tau} = 0$$

to (2.17) and using (2.15), we obtain

(2.18) $\quad [d(a), [a, x]]_{\sigma,\tau} = 0$ for all $x \epsilon R$.

In other words, if we define $I_a(x) = [a, x]$ an inner derivation determined by a and $I_{d(a)}(x) = [d(a), x]_{\sigma,\tau}$ a (σ, τ)-derivation determined by $d(a)$, in view of (2.18), we find that $I_{d(a)} I_a(x) = 0$, for all $x \epsilon R$: By Lemma 2.13, either $I_{d(a)} = 0$ or $I_a = 0$. That is, $d(a) \epsilon C_{\sigma,\tau}$, or $a \epsilon \mathbb{Z}(R)$. Since $a \, \mathbb{Q} \notin Z(R)$, this gives us $d(a) \epsilon C_{\sigma,\tau}$

On the other hand, since $[d(R), a]_{\sigma,\tau} \subseteq C_{\sigma,\tau}$. For $x\epsilon R$, $[d(ax), a]_{\sigma,\tau}\epsilon C_{\sigma,\tau}$

Then$[d(ax), a]_{\sigma,\tau} = [d(a)\sigma(x) + \tau(a)d(x), a]_{\sigma,\tau}$

$= d(a)\sigma(x)\sigma(a) + \tau(a)d(x)\sigma(a) - \tau(a)d(a)\sigma(x) - \tau(a)\tau(a)d(x).$

Now since we have $d(a)\epsilon C_{\sigma,\tau}$, the above equation reduces to

$[d(ax), a]_{\sigma,\tau} = d(a)\sigma(ax) + \tau(a)d(x)\sigma(a) - d(a)\sigma(ax) -$

$\tau(a)\tau(a)d(x),$

or,

(2.19) $\quad d(a)\sigma([x, a]) + \tau(a)[d(x), a]_{\sigma,\tau}\epsilon C_{\sigma,\tau}$ for all $x\epsilon R$.

Commuting (2.19) with a and using $d(a), [d(x), a]_{\sigma,\tau}\epsilon C_{\sigma,\tau}$. we get

$$0 = [d(a)\sigma([x, a]) + \tau(a)[d(x), a]_{\sigma,\tau}, a]_{\sigma,\tau}$$

$$= d(a)\sigma([x, a])\sigma(a) + \tau(a)[d(x), a]_{\sigma,\tau}, \sigma(a) - \tau(a)d(a)\sigma([x, a])$$

$$-\tau(a)\tau(a)[d(x), a]_{\sigma,\tau}$$

$= d(a)\sigma([x, a]a) + \tau(a)[d(x), a]_{\sigma,\tau}\sigma(a) - d(a)\sigma(a[x, a])$

$-\tau(a)[d(x), a]_{\sigma,\tau}\sigma(a) = d(a)\sigma([[x, a], a]).$

Also since $a \,\epsilon Sa_*(R)$, we have $d(a)\sigma([[x, a], a])^* = 0$: Therefore, by Corollary 2.9, $d(a) = 0$ or $[a, [a, x]] = 0$ for all $x\epsilon R$. If $[a, [a, x]] = 0$, for all $x\epsilon\, R$, we have by Lemma 2.7, $a\epsilon\mathbb{Z}(R)$; a contradiction. Therefore, $d(a) = 0$. Now (2.19) can be rewritten as $\tau(a)[d(x), a]_{\sigma,\tau}\epsilon C_{\sigma,\tau}$,for

$all\ x\ \epsilon R$. Also $\tau(a)\sigma[d(x), a]_{\sigma,\tau}\epsilon C_{\sigma,\tau}$, for all $x\epsilon R$. But $[d(x), a]_{\sigma,\tau}\epsilon C_{\sigma,\tau}$, yields by Lemma 2.8 either $\tau(a)\ \epsilon\mathbb{Z}(R)$ or $[d(x), a]_{\sigma,\tau} = 0$, for all $x\epsilon R$. Now in application of Theorem 2.2, we obtain $a\epsilon\mathbb{Z}(R)$. This contradicts our assumption. Hence, $a\epsilon\mathbb{Z}(R)$. By our hypothesis we have $d(R) \subseteq Z(R)$, and hence R is commutative by Lemma 2.12.

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