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# On Ordered Banach Algebra 

A Research<br>Submitted by

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شْكر وتقدبـر
 بـرؤيـتـك.

شْكري وتقتكبـري لlله جل جلاله وإلى من بـلغ الر سالة وأدى 1الأمانـة .. ونـصح الأمة .. إلـى نبـي الرحمة ونـور العالميـز .. رستولنا محمد صاى الاله عليه و1سلم


 الهزبـز

إلى مز كاز د عائها ستر نـجاهـي وصنـانـها بـلـ1م جراهيب.. أمي الهبـبـبة الى من شغد يـده بـيـبي وتحمل اوقاتنـا العصيبـة .. زوجي العزبـز


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In this search we will study ordered Banach algebras. Ordered Banach algebras are necessary objects in analysis. We will study the important concepts and theorems which need in this search like cone and ordered Banach algebra. Also answered the question under which conditions the spectral radius of a positive element $a$ is contained in the spectrum $\sigma(a)$ of that element and showed that the function $f$ is holomorphic on some open neighborhood of $\sigma(a)$, under what conditions of $\sigma(a)$ implies that $f(a) \geq 0$ such that $a \geq 0$.

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In Chapter one, shown three sections:

In section one, will establish some basic concepts and properties to be use in this search.

In section two, shown many properties and theorems about continuous homomorphism in normed algebra.

In section three, shown linear functional $f$ in the symmetric algebra $R$, and the properties about this function.

In Chapter two, will show three sections:

In section one, will shows three sections which contains the preparatory material, define an algebra cone $C$ of a real or complex Banach algebra $A$. It induces on $A$ an ordering that is compatible with the algebraic structure of $A$, and the pair $(A, C)$ is then called an ordered Banach algebra $(O B A)$. We also dene some properties of $C$, of which normality is the most important one. The algebra cone $C$ is said to be normal if there exists a constant $\beta \geq 1$ such that for all $a, b$ in $A$ with $0 \leq a \leq b$, we have that \| $a\|\leq\| b \|$.

In section two, will establish properties of the spectral radius in an $O B A$. The spectral radius $r$ is said to be monotone if $0 \leq a \leq b$ implies that $r(a) \leq r(b)$. proved that, if the algebra cone $C$ is normal, then the spectral radius is monotone. Also answered the question under which conditions the spectral radius of a positive element a is contained in the spectrum $\sigma(a)$ of that element. It turns out that monotonicity of the spectral radius implies this property.

In section three, look at poles of the resolvent function and investigate what role they play in spectral theory in $O B A^{\prime}$ s. First proved several versions of the Krein-Rutman Theorem, which is originally in terms of operators, in the context of $O B A^{\prime}$ s. These theorems describe conditions under which the spectral radius of a positive element will be an eigenvalue of that element, with a positive eigenvector. After that looked at the structure of the spectrum $\sigma(a)$ and what properties this structure forces on a. One of these properties is whether positivity of a implies that $a \geq 1$. More general, for a function $f$ that is holomorphic on some open neighborhood of $\sigma(a)$, under what conditions of $\sigma(a)$ does $a \geq$ 0 implies that $f(a) \geq 0$ ?

# Chapter One 

## Introduction

## Introduction

In this section, we introduce some basic concepts and properties from [6],_[8],[9] to be use in this search

Definition (1.1.1)( linear algebra)[6]:- We shall say that $R$ is a linear algebra if $R$ is a linear space. an operation of multiplication (which in general is not commutative) is in $R$ satisfying the following conditions:-

1) $x(x y)=(x x) y=x(\alpha y)$
2) $x(y z)=(x y z)$
3) $(x+y) z=x z+y z$

For arbitrary $x \in R$ and any number $\alpha$.
In the sequel we shall consider only linear algebras and the term "algebra" will a linear algebra.

Definition (1.1.2)[6]:- In elements $x, y$ in the algebra $R$ are said to commute if $x y=y x$ an algebra is said to be commutative if any two of its elements commute. In the sequel we shall in general that the algebra under consideration are commutative a subset $R_{1} \subseteq R$ is called a sub algebra of the algebra $R$ if the application of the addition scalar multiplication and multiplication to element of $R_{1}$ elements in $R_{1}$

Definition (1.1.3)[6]:- A commutative sub algebra is said to be maximal if it is not contained in any a commutative sub algebra. It follows from the preceding discussion that.

Theorem (1.1.4)[6]:- Every commutative sub algebra is contained in a maximal commutative sub algebra.

Proof:- The set $\sum$ of all all commutative sub algebra of the algebra $R$, which can in a given commutative sub algebra. Is a partially ordered set. Ordered by in which satisfies the condition of zeros lemma: namely. The least upper bound of any linear ordered set of these sub algebra is simply their union on the basis of the Zorn lemma. $\sum$ contains a maximal element which will then be the maximal commutative sub algebra
containing $x$. Since every element $x$ is contained in the commutative sub algebra $R_{a}(x)$, it follows from proposition I that.

Theorem (1.1.5)[6]:- Every element $x$ is contains in a maximal commutative sub algebra.

Example (1.1.6)[6]:- We denote the set of all continuous complex- value function on the topological space $x$ by $C(x)$ in $C(X)$ we define operations of addition- scalar multiplication and multiplication respectively as the addition of function, the multiplication of function by a number and the multiplication of function clearly $C(x)$ will then be an algebra this algebra is commutative.

Example (1.1.7)[6]:- Suppose $x$ is an arbitrary linear, we denote the set of all linear operators in $x$ with domain $x$ by $A(X)$. In $A(X)$ we define operation of addition, scalar multiplication, and multiplication as the corresponding operation on operations (see subsection 6.) then $A(x)$ is an algebra $A(X)$ is commutative only in the case when $X$ is onedimensional.

## Definition (1.1.8) (Algebra with identity)[8]:-

An algebra $R$ iscalled an algebra with identity if $R$ contains an elemente which satisfies the condition: $e x=x e=x$ for all $x \in R$.

The element $e$ itself which satisfies condition (1) is called an identity of the algebra $R$.

Theorem (1.1.9)[8]:- Every algebra $R$ without identity can be considered as a sub algebra of an algebra $R$ with identity.

Theorem (1.1.10)[8]:- A maximal commutative sun algebra $R$, of the algebra $R$ with identity is also an algebra with identity.

Theorem (1.1.11)[8]:- If $x^{-1}$ exists and if $x, y$ commute, then $x^{-1}$ and $y$ also commute.in fact multiplication both members of the equality $x y=$ $y x$ on the left and right by $x^{-1}$, we obtain $y x^{-1}=x^{-1} y$.

Theorem (1.1.12)[9]:- If $x$ is the maximal commutative sub algebra which contains $x$ and $x^{-1}$ exists then $x^{-1} \epsilon X$.

Theorem (1.1.13)[9]:- If every element $x \neq 0$ in the algebra $R$ with identity has a left inverse, then $R$ is a division algebra.

Definition (1.1.14)[8]:- An element $y \in R$ is called a left quasi- inverse of the element $e+x$ in $R, e+y$ is a left inverse of the element $e+x$ in $R$, that mean if $(e+y)(e+x)=e$.

Example (1.1.15)[8]:- The algebra $C(x)$ is an algebra with identity. The identity of this algebra is the function which is identically equal to unity on $x$.

Example (1.1.16)[8]:- The algebra $A(x)$ and $A(x)$ are algebra with identity which is the identity operator.

Definition (1.1.17)[8]:- The center of algebra $R$ is the set of those element $a \epsilon R$ which commutative with all the elements of $R$. The center a commutative sub algebra of the algebra $R$.

Definition (1.1.18)[6]:- A set $I_{1}$ of elements of the algebra $R$ is called a left ideal $R$ if

1) $I_{1} \neq R$.
2) $I_{1}$ is a sub space of the linear space $R$.
3) If $x \in I, A \in R$ then $a x \in I$.

Theorem (1.1.19)[6]:- An element $x$ of an algebra with identity has a left (right) inverse if and only if it is not contained in any left (right)ideal.

Theorem (1.1.20)[6]:- Every left (right) ideal of the algebra $R$ with identity is contained in a maximal left (right) ideal.

Theorem (1.1.21)[6]:- An element $x$ of an algebra with identity has a left (right) inverse if and only if it is not contained in any maximal left(right) ideal.

Theorem (1.1.22)[6]:- Every two- sided ideal of an algebra with identity is contained in a maximal two- sided ideal.

Theorem (1.1.23):- Every regular (right, left, two- sided) ideal can be extended to a maximal (right, left, respectively, two- sided) ideal (which is obviously regular also).

Theorem (1.1.2)[8]:- An element $x$ in the algebra $R$ has a left quasiinverse if and only if for arbitrary maximal regular left ideal $M$, there exists element such that $x+y+y x \in M$.

Theorem (1.1.25)[8]:- An element $x$ in the algebra $R$ dose not have a left a quasi- inverse if and only if $I_{1}=\left\{z+z_{x}\right\}, z \in R$.

Definition (1.1.26)[8]:- An element $x_{o}$ in the algebra $R$ with identity is said to be generalized nilpotent if $\left(e+y x_{o}\right)^{-1}$ exists for an arbitrary element $y \epsilon R$.the set of all generalized nilpotent element in the algebra $R$ is called its (Jacobson) radical.

Theorem (1.1.27)[8]:- The radical of an algebra with identity coincides with intersection of all its maximal left ideal.

Theorem (1.1.28)[8]:- An element $x_{0}$ belong to the radical of an algebra with identity if and only if a two- sided inverse $\left(e+a x_{o}\right)^{-1}$ exists for every element $a$ of the algebra.

Theorem (1.1.29)[8]:- The intersection of all maximal left ideals coincides with the intersection of all maximal right ideals and is the radical of the algebra.

Definition (1.1.30)[8]:- An algebra is said to be semi simple if it is radical consist of only the zero element suppose now that $R$ is an algebra without identity and that $R^{\prime}$ is the algebra obtained form $R$ by adjoining the identity.

Definition (1.1.31)[8]:- An element $x_{o}$ is said to be generalized nilpotent $x x_{o}+z x_{o}$ has a left quasi- inverse for arbitrary $z \epsilon R$ and arbitrary numbers $x$ in this definition $R$ is no large necessarily an algebra with identity.

Theorem (1.1.32)[8]:- In a non- radical algebra, the radical is the intersection of all maximal regular left ideal and also the intersection of all maximal regular right ideal and therefore it is two sides ideal.

Theorem (1.1.33)[8]:- The quotient algebra module the radical is a semi simple algebra.

Theorem (1.1.34)[8]:- Every irreducible algebra $R_{1}$ different form (0), of linear operators in the vector space $x$ is a semi simple algebra.

Definition (1.1.35)[8]:- A mapping $x \rightarrow x^{\prime}$ of the algebra $R$ into an arbitrary algebra $R^{\prime}$ if $x \rightarrow x^{\prime}, y \rightarrow y^{\prime}$ imply that $y x \rightarrow y^{\prime} x^{\prime}, x+y \rightarrow$ $x^{\prime}+y^{\prime}, x y \rightarrow x^{\prime} y^{\prime}$ if $R$ is the image of the algebra $R$, then the homomorphism is called a homomorphism of $R$ onto $R^{\prime}$.

Definition (1.1.36)[8]:- Two algebras $R$ and $R^{\prime}$ are said to be isomorphic if there exists isomorphism of $R$ onto $R^{\prime}$.

Theorem (1.1.37)[8]:- Under a homomorphism of the algebra $R$ into the algebra $R^{\prime}$, the inverse image $I$ of the zero of $R$ is a two sides ideal in $R$.

Theorem (1.1.38)[8]:- Under a homomorphism mapping of the algebra $R$. The inverse image $I$ of the zero element is a two-sided ideal of this algebra and the homomorphic image itself is isomorphic to the quotient algebra $R$ modulo $I$.

Theorem (1.1.39)[6]:- The quotient algebra $R / I$ is simple if and only if $I$ is a maximal two-sided ideal in $R$.

Definition (1.1.40)[6]:- Algebra is the so- called left regular representation of the algebra each element $a \in R$ is assigned the operator $A_{a}$ of left multiplication by a $A_{a} x=a x$.

Theorem (1.1.41)[6]:- Every primitive algebra is isomorphic to an irreducible algebra of linear operators in some vector space.

Theorem (1.1.42)[6]:- Every primitive algebra is semi simple.
Theorem (1.1.43)[6]:- If $I \neq\{0\}$ is a two sided ideal in the primitive algebra $R$ and if a is an arbitrary nonzero element of the algebra $R$, then $I_{a} \neq\{0\}$.

Definition (1.1.44) (topological algebra):-
$R$ is called a topological algebra if :

1) $R$ is an algebra
2) $R$ is a locally convex topological linear space.
3) The product $x y$ is a continuous function of each of the factors $x, y$ provided other factor is fixed.

Definition (1.1.45):- A mapping $x \rightarrow x^{\prime}$ of the topological algebra $R$ into the topological algebra $R^{\prime}$ is call a continuous homomorphism if:

1) $x \rightarrow x^{\prime}$ is a homomorphism of the algebra $R$ into the algebra $R^{\prime}$.
2) $x \rightarrow x^{\prime}$ is a continuous mapping of the topological space $R$ into the topological space $R^{\prime}$.

Definition (1.1.46):- A subset $R_{1} \subseteq R$ is said to be a closed sub algebra of the algebra $R$ if

1) $R_{1}$ is a sub algebra of the algebra $R$.
2) $R_{1}$ is a closed subspace of the topological space $R$.

Theorem (1.1.47):- If $R_{1}$ is a sub algebra of the algebra $R_{1}$ then it's closer $\overline{R_{1}}$ is a closed sub algebra of $R$.

Theorem (1.1.48):- The algebra $R_{1}(s)$ is the closer of the algebra $R_{a}(s)$ : $R_{a}(s)=\overline{R_{a}(s)}$.

Theorem (1.1.49)[6]:- The closer of a commutative sub algebra of a topological algebra is commutative.

Theorem (1.1.50)[6]:- A maximal commutative sub algebra of a topological algebra is closed.

Theorem (1.1.51)[6]:- The set $R_{S}$ of all elements $x$ of a topological algebra $R_{1}$ which commute with all elements of some set $S \subseteq R_{1}$ is a closed sub algebra of the algebra $R$.

Theorem (1.1.52)[6]:- The center $z$ of a topological algebra $R$ is a closed commutative sub algebra in $R$.

Theorem (1.1.53)[6]:- The closer of a left, right, two- sided)ideal in a topological algebra, which does not coincide with the entire algebra, is also (left, right, two sided) ideal in this algebra.

Definition (1.1.54)[6]:- A topological algebra $R$ with identity is called an algebra with continuous inverse if there exists an neighborhood $U_{o}(e)$ posseting the following properties:

1) Every element $x \in U_{o}(e)$ has an inverse $x^{-1}$
2) $x^{-1}$ is a continuous function of $x$ at the point $x=e$.

Definition (1.1.55) (normed algebra)[6]:- $R$ is called normed algebra if

1) $R$ is an algebra
2) $R$ is a normed space
3) for any two elements $x, y \in R \quad|x y|=|x||y|$
4) if $R$ is contains an identity $e$, then $|e|=1$. The norm in a normed algebra $R$ defines a topology in $R$ in a natural manner recall that in this topology, the open balls $\left|x-x_{o}\right|<r$ with center at $x_{o}$ from a neighborhood basis of the element $x_{o} \in R$.

Proposition (1.1.56)[8]:- In the norm topology, the product $x y$ is a continuous function of the variables $x, y$ simultaneous.

In fact, in virtue of (1)

$$
\begin{aligned}
& \left|x y-x_{o} y_{o}\right|=\left|\left(x-x_{o}\right)\left(y-y_{o}\right)+\left(x-x_{o}\right) y_{o}+x_{o}\left(y-y_{o}\right)\right| \\
& \leq\left|x-x_{o}\right|\left|y-y_{o}\right|+\left|x-x_{o}\right|\left|y_{o}\right|+\left|y-y_{o}\right|\left|x_{o}\right| .
\end{aligned}
$$

Now, the assertion follows directly from this since a normed space $R$ is a topological linear space, we conclude from proposition ( )

Proposition (1.1.57)[8]:- In the topology define by the norm, a normed algebra is a topological algebra a normed algebra $R$ is said to be complete if $R$ is a complete normed algebra will also be called a Banach algebra.

Proposition (1.1.58)[8]:- Every non complete normed algebra can be embedded in a complete normed algebra.

Proof:- suppose $R$ is the completion of the normed space $R$. Now define multiplication in $R$ suppose $\bar{x}, \bar{y} \in R$ and $\left\langle x_{n}\right\rangle,\left\langle y_{n}\right\rangle$ be fundamental sequences in $R$. Which define $\bar{x}, \bar{y}$ respectively. It follows from inequality (2) with $x_{n}, x_{m}$ in place of $x, x_{o}$ and $y_{n}, y_{m}$ in place of $y, y_{o}$ that $<$ $x_{n}, y_{n}>$ also is a fundamental sequence. The element in $\bar{R}$ which it define will be considered to be the product $\bar{x} \bar{y}$ of the elements $\bar{x}, \bar{y}$. Again applying inequality (2) it it can also be easily verified that $\bar{x} \bar{y}$, does not depend on the choice of the fundamental sequence.
$\left\langle x_{n}\right\rangle,\left\langle y_{n}\right\rangle$ which define $\bar{x}, \bar{y}$ if in particular $\bar{x}=x \in R, \bar{y} \in R$, then setting $x_{n}=x, y_{n}=y$, we conclude that in this case the product coincides with the product in $R$ passing to the limit in the relations for the
elements in the algebra $R$, it is easily shown that $\bar{R}$ is an algebra and that the inequality $|\bar{x} \bar{y}| \leq|\bar{x}||\bar{y}|$ is satisfy for elements of the ring $\bar{R}$ consequently $\bar{R}$ is a complete normed which contains $R$ a sub algebra.

The algebra $\bar{R}$ is called the completion of the algebra $R$.
Example (1.1.59)[8]:- The algebra $C(T)$ suppose T is topological space. the set $C(T)$ of all bounded continuous function $x(t)$ on $T$ forms a Banach space recall that the norm $|x|$ in $C(T)$ is defined by the formula $|x|=\sup _{t \epsilon T}|x(t)|$.

Multiplication in $C(T)$ can be define as the multiplication of function that mean $(x y)(t)=x(t) \cdot y(t)$

1) this easily seen that the condition $|x y| \leq|x||y|$ will be satisfied so that $C(T)$ becomes a Banach algebra. If $T$ is compact then the boundedness condition on the functions $x(t)$ is redundant in virtue.
2) The algebra $B(x)$. Recall that $B(x)$ denotes the set of all bounded linear operations in the Banach space $x$. We saw above that $B(x)$ is also defined as the multiplication of operator with $|A B| \leq|A||B|$ According to that we proved consequently, $B(x)$ is a Banach algebra
3) The algebra $W$. We denote by $W$ the set of all absolutely convergent series $x(t)=\sum_{n=-\infty}^{\infty} C_{n} e^{i n t}$ with norm $|x|=\sum_{n=-\infty}^{\infty}\left|C_{n}\right|$

We obtain a Banach algebra by defining addition, scalar multiplication as the corresponding operations on
2. Adjunction of the identity. Suppose $R$ a normed algebra without identity and let $R$ be the algebra obtained from $R$ upon adjunction of the identity we may introduce a norm in $R$ by setting

$$
|x e+x|=|x|+|x|
$$

It is easily verified that $R$ then becomes a normed algebra. If $R$ is a complete algebra without identity, then $R^{\prime}$ is also a complete algebra. The proof is simple and so we shall omit it.

## The Radical in a Normed Algebra

Theorem (1.1.60)[8]:- for every $x$ of the normed algebra $R$, $\lim _{n \rightarrow \infty} \sqrt{|x|^{n}}<\infty$ exists.

Theorem (1.1.61)[8]:- If the element $x$ of the normed algebra $R$ belongs to the radical of the algebra $\lim _{n \rightarrow \infty} \sqrt{|x|^{n}}=0$

## Banach Algebra with identity

Theorem (1.1.62)[8]:- Every Banach algebra with identity is an algebra with continuous inverse moreover, every element $x_{1}$ satisfying the inequality $|x-e|<1$ is invertible.

Theorem (1.1.63)[8]:- In a Banach algebra $R$ with identity:

1) The set of all elements $x$ having a (left, right, two- sided) inverse is an open set.
2) The inverse $x^{-1}$ is a continuous function of $x$ at all points for which $x^{-1}$ exists.
3) The closure of a (left, right, two- sided) ideal a (left, right)ideal.
4) Maximal (left, right, two- sided) ideal is closed.
5) The set $R_{x}$ of all regular points of the element $x \in R$ is open and the resolve $X_{\lambda}=(x-\lambda e)^{-1}$ is an analytic function of $\lambda$.
6) The spectrum of every element $x \in R$ is a no avoid set.

Theorem (1.1.64):- (Gelfand and Mazur)[12]:- Every complete normed division algebra is isomorphic to the field of complex number.

Theorem (1.1.65)[12]:- In a Banach algebra $R$ the quotient algebra $R / I$ modulo a closed two- sided ideal $I$ is a Banach algebra.

Theorem (1.1.66)[12]:- If in the Banach algebra $R$ with identity every element $x \neq 0$ has a left inverse. Then $R$ is isomorphic to the field of complex number.

Theorem (1.1.67)[12]:- For $|\lambda|>\lim _{n \rightarrow \infty} \sqrt{|x|^{n}}$ the resolve $X_{\lambda}$ can be expended in absolutely convergent Laurent series.

Theorem (1.1.68)[8]:- For an arbitrary $x \in R, r(x)=\lim _{n \rightarrow \infty} \sqrt[n]{|x|^{n}}$.
Theorem (1.1.69)[8]:- The spectral radius possesses the following properties.

1) $r\left(x^{k}\right)=[r(x)]^{k}$
2) $r(x x)=|x| r(x)$
3) $r(x) \leq|x|$

## Section Two

## Continuous Homomorphlsm of Normed Algehra

In this section we will introduce many properties and theorems about continuous homomorphism in normed algebra

Theorem (1.2.1):- Every continuous homomorphism $x \rightarrow x^{\prime}$ of the normed algebra $R$ into the normed algebra $R^{\prime}$ satisfies in the inequality.

$$
\left|x^{\prime}\right| \leq C|x|
$$

Theorem (1.2.2):- Every continuous homomorphism $x \rightarrow x^{\prime}$ of a normed algebra $R$ into a normed algebra $R^{\prime}$ is uniquely extendible to a continuous homomorphism of the completion $R$ of the algebra $R$ into the completion $R^{\prime}$ of the algebra $R^{\prime}$.
Theorem (1.2.3):- Every continuous isomorphism of a Banach algebra $R$ onto a Banach algebra $R$ is a topological isomorphism.
Theorem (1.2.4):- Under a continuous homomorphism of the Banach algebra $R^{\prime}$. The kernel $I$ of the homomorphism is a closed two- sided ideal in $R$. And the algebra $R^{\prime}$ it self is topologically isomorphism to the quotient algebra $R / I$. Can certainly every closed two- sided ideal $I$ of the Banach algebra $R$ induces a continuous homomorphism (the so- called natural homomorphism) of the algebra $R$ into the algebra $R / I$.
(regular representation of a normed algebra. Recall that the left and right regular representations $a \rightarrow A_{a}$ and $a \rightarrow B_{a}$ of the algebra $R$ are defined by means of the formulas.)
Theorem (1.2.5):- A left (right) regular representation of a normed algebra Ris a continuous. Homomorphism of the algebra $R$ into the algebra $R(a)$ of all bounded linear operations in the space $R$. In fact, the inequalities

$$
\begin{aligned}
& \left|A_{a} X\right| \leq|a||X|,\left|B_{a} X\right| \leq|a||X| \\
& \left|A_{a}\right| \leq|a|,\left|B_{a}\right| \leq|a| .
\end{aligned}
$$

Theorem (1.2.6):- If $R$ is a normed algebra with identity, then a left (right) regular representation of the algebra $R$ is an isometric isomorphism (anti- isomorphism) of the algebra $R$ into the algebra $R(R)$.
In fact, for $x=e$, inequalities (1) go over into equalities, and hence $\left|A_{a}\right|=|a|,\left|B_{a}\right|=|a|$
Theorem (1.2.7):- $R$ is a minimal invariant sub space in $R$ if and only if it is the annihilator of maximal right ideal in $R$.
Theorem (1.2.8):- If $R$ is an algebra with identity, then every closed invariant sub space in $R$, with is distinct from ( 0 ) contains a minimal invariant sub space and consequently, in contains elementary functions.

Theorem (1.2.9):- Suppose $R$ is an algebra with identity and let $F \neq 0$ be a function in $R$ there exists an elementary functional which is a weak contact point of functional of the form $f_{a}(x)=f(x a)$. Regular representations of an algebra can also be used in the proof of the following proposition.
Theorem (1.2.10):- Suppose $R$ is a complete topological algebra with identity in which the topology is defined by the norm $|x|$. Then $R$ is topologically isomorphic to a Banach algebra.
Definition (1.2.11)[20]:- $R$ is called a symmetric algebra if:

1) $R$ is an algebra
2) an operation is defined in $R$ which assigns to each element $x$ in $R$ the element $x^{*}$ in $R$ in such a way that the following conditions are satisfies:-
a) $(\lambda x+\lambda y)^{*}=\lambda x^{*}+\mu y^{*}$
b) $x^{* *}=x$
c) $(x y)^{*}=y^{*} x^{*}$

An element $x$ is said to be Hermitian if $x^{*}=x$.
Theorem (1.2.12)[20]:- Every element $x$ of a symmetric algebra can be uniquely represented in the form $x=x_{1}+i x_{2}$, where $x_{1}, x_{2}$ are Hermitian elements.

In fact, if such a representation holds, then $x^{*}=x_{1}-i x_{2}$ consequently
$x_{1}=\frac{x+x^{*}}{2}, x_{2}=\frac{x-x^{*}}{2 i}$
Thus, this representation is unique. Conversely, the elements $x_{1}, x_{2}$ defined by equalities (1) are Hermitian and $x=x_{1}+i x_{2}$.

These elements $x_{1}, x_{2}$ will be called the Hermitian components of the element $x$ an element $x$ is called normal if $x^{*} x=x x^{*}$.

Theorem (1.2.13)[20]:- Every element of the form $x^{*} x$ is Hermitian
In fact, in virtue of c and b$) .\left(x^{*} x\right)^{*}=x^{*} x^{* *}=x^{*} x$
Theorem (1.2.14)[20]:- The identity $e$ is a Hermitian element. In fact $e^{*}=e^{*} e$ is a Hermitian element. Consequently, $e^{*}=e$

If $R$ is asymmetric algebra without identity and $R^{\prime}$ is the algebra obtained from $R$ by adjunction of the identity, then setting $(\lambda e+x)^{*}=\lambda e+x^{*}$ for $x \in R$.

Theorem (1.2.15)[20]:- If $x^{-1}$ exists, then $\left(x^{*}\right)^{-1}$ also exists and $\left(x^{*}\right)^{-1}=\left(x^{-1}\right)^{*}$

Theorem (1.2.16)[6]:- If $R$ is a maximal commutative symmetric sub algebra containing a normal element $x$ and if $x^{-1}$ exists, then $x^{-1} \in R$. In fact since $x$ and $x^{*}$ commute with all elements in $R, x^{-1}$ and $x^{*}=\left(x^{-1}\right)^{*}$

Definition (1.2.17)[6]:- The mapping $x \rightarrow x^{\prime}$ of a symmetric algebra $R$ into the symmetric algebra $R^{\prime}$ is called a symmetric homomorphism if
$\beta$ ) $x \rightarrow x^{\prime}$ is a homomorphism
d) $x \rightarrow x^{\prime}$ implies that $x^{*} \rightarrow x^{\prime *}$.

Theorem (1.2.18)[6]:- The radical of a symmetric two- sided ideal.

## Example (1.2.19)[6]:-

1) The algebra $C(T)$ is a symmetric algebra if we set $x^{*}=\overline{X(t)}$ for $X=$ $X(t)$ (where the vinculum denotes conjugate complex number)
2) Suppose $R$ is a Hilbert space. the algebra $R(R)$ that mean $R(x)$ with $X=R$ is a symmetric algebra if involution is under stood to be passage over to the adjoint operator
3) The algebra $W$ is asymmetric algebra if we set $x^{*}=\sum_{n=-\infty}^{\infty} \bar{C}_{-n} e^{i n t}$ for $x=\sum_{n=-\infty}^{\infty} C_{n} e^{i n t}$

## Somo Propertios of Postituv Finctional

## Definition (1.3.1) (Positive functional)[3]:-

A linear functional $f$ in the symmetric algebra $R$ is said to be real- valued if $f$ assumes real value on all Hemitian elements of the algebra $R$.

Theorem (1.3.2)[27]:- Every linear functional in a symmetric algebra can be represented in the form $f=f_{1}+i f_{2}$ where $f_{1}, f_{2}$ are real valued functional. Namely it suffices to set
$f_{1}(x)=\left[f(x)+\overline{f\left(x^{*}\right)}\right], f_{2}(x)=\frac{1}{2 i}\left[f(x)-\overline{f\left(x^{*}\right)}\right]$.
Then $f_{1}, f_{2}$ are real valued functional and $f(x)=f_{1}(x)+i f_{2}(x)$ these functional $f_{1}, f_{2}$ are called the real components.

Theorem (1.3.3)[27]:- If $f$ is a real- valued functional then $f\left(x^{*}\right)=$ $\overline{f(x)}$ for an arbitrary $x \in R$. In fact setting $x=x_{1}+i x_{2}$ where $x_{1}, x_{2}$ are Hermitian we have $f\left(x^{*}\right)=f\left(x_{1}-i x_{2}\right)=\overline{f\left(x_{1}\right)+\imath f\left(x_{2}\right)=} \overline{f(x)}$ Inasmuch as $f\left(x_{1}\right), f\left(x_{2}\right)$ are real- valued by assumption. A linear functional $f$ is said to be positive if $f\left(x^{*} x\right) \geq 0$ for an arbitrary element $x$ of the algebra $R$.

Theorem (1.3.4)[27]:- For every positive functional $f$ in the symmetric algebra $R$.

1) $f\left(y^{*} x\right)=f\left(\overline{x^{*} y}\right)$
2) $\left|f\left(y^{*} x\right)\right|^{2} \leq f\left(y^{*} y\right) f\left(x^{*} x\right)$
3) $f((\lambda x+u y) \cdot(\lambda x+u y))>0$
4) $|\lambda|^{2} f\left(x^{*} x\right)+\lambda \bar{u} f\left(y^{*} x\right)+\lambda u f\left(x^{*} y\right)+|u|^{2} f\left(y^{*} y\right) \geq 0$

Theorem (1.3.5)[27]:- Every positive functional $f$ in a symmetric
algebra $R$ with identity is real and $\left|f(x)^{2}\right| \leq f(e) f\left(x^{*} x\right)$.
Theorem (1.3.6)[9]:- Suppose $R$ is a symmetric algebra without identity and that $R^{\prime}$ is the symmetric algebra obtained from $R$ by adjunction of the identity. A positive functional $f$ in $R$ can be extended to a positive functional in $R^{\prime}$ if and only if $f$ is real and satisfies inequality
$|f(x)|^{2} \leq c f\left(x^{*} x\right)$ for all $x \in R$ where $c$ is some constant
Theorem (1.3.7)[9]:- If in a symmetric normed algebra $R$
a) $\left|x^{*}\right|=|x|$
b) There exists a set $\left\{e_{x}\right\}$ approximating the identity, then every continuous positive functional in $R$ can be extended to a positive functional in $R^{\prime}$

Definition (1.3.8)[8]:- $R$ is called a normed symmetric algebra if
a) $R$ is a normed algebra
b) $R$ is a symmetric algebra
c) $\left|x^{*}\right|=|x|$

# Chapter Two 

## Main Results

## Section One

Orilered Banach Aldobra

In this section, we will define an algebra cone $C$ of a real or complex Banach algebra $A$ and show that $C$ induces on $A$ an ordering which is compatible with the algebraic structure of $A$. The Banach algebra $A$ is then called an ordered Banach algebra ( $O B A$ ). We also define certain additional properties of $C$.

Definition (2.1.1)[8]:- Let $A$ be a real or complex Banach algebra with unit 1 . We call a nonempty subset $C$ of $A$ a cone if it satisfies the following:

$$
\text { 1. } C+C \subset C \text {, }
$$

2. $\lambda C \subset C$ for all $\lambda \geq 0$.

If in addition $C$ satisfies $C \cap-C=\{0\}$, then $C$ is called a proper cone.
Any cone $C$ on A induces a relation ${ }^{\prime} \leq '$ on $A$, called an ordering, in the following way:
$a \leq b$ if and only if $b, a \in C,(a, b \in A)$.
It can be shown that for every $a, b \in A$ this ordering satisfies

1. $a \leq a$ ( $\leq$ is reflexive),
2. if $a \leq b$ and $b \leq c$, then $a \leq c$ ( $\leq$ is transitive).

The ordering does not have to be antisymmetric.
Proposition (2.1.2)[8]:- The cone $C$ is proper if and only if the ordering is antisymmetric, i.e. $a \leq b$ and $b \leq a$ implies that $a=b$.

Proof:- Let $C$ be a proper cone, $a \leq b$ and $b \leq a$. Then $a-b \in C$ and
$b-a=-(a-b) \epsilon C$, so $a-b \epsilon C \cap-C=\{0\}$ and we have $a=b$. Conversely, let the ordering be antisymmetric and suppose the cone $C$ is not proper. Then there exists an $x \in C$ with $x \neq 0$ such that there is an $a \epsilon C$ with $x=-a$. Now we have $x-a=2 x \epsilon C$ and $a-x=2 a \epsilon C$. So $x \leq a$ and $a \leq x$ and the antisymmetric property gives us $x=a$, which is a contradiction.

So the ordering induced by $C$ is a partial ordering if and only if $C$ is proper. Considering the ordering that $C$ induces, we find that
$C=\{a \epsilon A: a \geq 0\}$, and therefore we call the elements of $C$ positive.
Definition (2.1.3)[8]:- A cone $C$ of a Banach algebra $A$ is called an algebra cone if $C$ satisfies the following conditions:

1. C. $C \subset C$,
2. $1 \epsilon C$.

Denition (2.1.4)[8]:- A real or complex Banach algebra $A$ with unit 1 is called an ordered Banach algebra ( $O B A$ ) if A is ordered by a relation ' $\leq$ ' in such a manner that for every $a, b, c \epsilon A$ and $\lambda \epsilon C$ we have:

1. $a, b \geq 0 \Rightarrow a+b \geq 0$,
2. $a \geq 0, \lambda \geq 0 \Rightarrow \lambda a \geq 0$,
3. $a, b \geq 0 \Rightarrow a b \geq 0$,
$4.1 \geq 0$.
Therefore, if $A$ is ordered by an algebra cone $C$, then $A$, or more specially $(A, C)$ is an $O B A$. Conversely, if $A$ is an $O B A$ the set $C=\{a \in A: a \geq 0\}$ is an algebra cone that induces the ordering on $A$.

Definition (2.1.5)[8]:- An algebra cone $C$ is called normal if there exists a constant $\beta \geq 1$ such that for $a, b \in A$ we have
$0 \leq a \leq b \Rightarrow\|a\| \leq \beta\|b\|$
An alternative definition of normality is
Definition (2.1.6)[8]:- An algebra cone $C$ is called $\alpha$-normal if there exists a constant $\alpha \geq 1$ such that for $a, b, c \in A$ we have
$0 \leq a \leq b \Rightarrow\|b\| \leq \alpha(\max \|a\|,\|c\|)$.
If the normality constant is equal to 1 we say that the $C$ is 1 -normal. It is not hard to prove that the two definitions are equivalent, but the constants and from the definitions need not be the same. If $C$ is normal with constant, $C$ does not have to be $\alpha$-normal.

Proposition (2.1.7)[8]:- If $C$ is a normal algebra cone, then it is a proper algebra cone.

Proof:- Let $C$ be a normal algebra cone. Let $x \in C$ be such that there exists an $a \epsilon C$ with $x=-a$. Then for all scalars $k>0$ we have $a-k a=$ $a+k(-a)=a+k x \epsilon C$, so $k a \leq a$. Because $C$ is normal there exists a constant $\alpha>0$ such that for all $k>0$ we have $k\|a\|=\|k a\| \leq$ $\alpha\|a\|$, so $\|a\|=0$. This means that $a=0$ and therefore
$C \cap-C=\{0\}$.
If $C$ has the property that if $a \epsilon C$ and a is invertible, then $a^{-1} \epsilon C$, then
$C$ is said to be inverse-closed. The following lemma is immediate.
Lemma (2.1.8)[8]:- Let $(A, C)$ be an $O B A$, and let $x, y \epsilon A$ be such that $x y \leq y x$.

1. If $x$ is invertible and $x^{-1} \epsilon C$, then $y x^{-1} \leq x^{-1} y$.
2. If $y$ is invertible and $y^{-1} \epsilon C$, then $y^{-1} x \leq x y^{-1}$.

The following lemma follows with induction.
Lemma (2.1.9)[8]:- Let $(A, C)$ be an $O B A$, and let $x, y, c \in C$. If $y x \leq x y$, then:

$$
(x+y)^{n} \leq \sum_{k=0}^{n}\binom{n}{k} x^{n-k} y^{k}
$$

for every $n \epsilon N \cup\{0\}$.
Proof:- The statement clearly is true for $n=0$. Now let $m>0$ and suppose the statement is true for all $n<m$. We have that $y x \leq$ $x y$ implies $y x^{m-k-1} y^{k} \leq x^{m-k-1} y^{k+1}$ and it follows that

$$
\begin{aligned}
& (x+y)^{m} \leq(x+y) \sum_{k=0}^{m-1}\binom{m-1}{k} x^{m-k-1} y^{k} \\
\leq & \sum_{k=0}^{m-1}\binom{m-1}{k} x^{m-k} y^{k}+\sum_{k=0}^{m-1}\binom{m-1}{k} x^{m-k-1} y^{k+1}
\end{aligned}
$$

$$
=x^{m}+\sum_{k=0}^{m-1}\left(\binom{m-1}{k}\binom{m-1}{k+1}\right) x^{m-k} y^{k}+y^{m}=\sum_{k=0}^{m}\binom{m}{k} x^{m-k} y^{k} .
$$

Let $A$ and $B$ be Banach algebras such that $1 \epsilon B \subset A$, then we have a few easy to prove facts [9].
(i) If $C$ is an algebra cone in $A$, then $C \cap B$ is an algebra cone in $B$ and if $C$ is proper, then $C \cap B$ is proper.
(ii) In the case where B has a finer norm then A, (i.e $\|b\|_{A} \leq\|b\|_{B}$ for all $b \in B$ ) if $C$ is closed in $A$, then $C \cap B$ is closed in $B$.
(iii) If $B$ is a closed subalgebra of $A$ (containing the unit of $A$ ), then the normality of $C$ in A implies normality of $C \cap B$ in $B$
(iv) If $T: A \rightarrow B$ is a homomorphism and if $C$ is an algebra cone of $A$, then $T C=\{T c: c \epsilon C\}$ is an algebra cone in $B$. In particular, if $F$ is aclosed ideal in the $O B A(A, C)$ and if $\pi: A \rightarrow A / F$ is the canonical homomorphism, then $\pi C$ is an algebra cone of $A / F$. We cannot deduce normality or closeness of $\pi C$ from the corresponding properties of $C$.
Now we give some examples of $O B A^{\prime} s$.
Example (2.1.10)[9]:- Let $A=C$ be the Banach algebra with standard norm and $C=R^{+}$. Then $(A, C)$ is an $O B A$ and $C$ is normal.

Proof:- Trivial.
Example (2.1.11)[9]:- Let $\mathbb{C}^{2}$ be equipped with $\|.\|_{\infty}$ and let $A$ be the set of upper triangular $2 \times 2$ complex matrices with the operator norm for bounded operators. Let $C$ the subset of $A$ of matrices with only nonnegative entries. Then $(A, C)$ is an $O B A$ and $C$ is normal.

Proof:- It follows from simple calculations that $(A, C)$ is an $O B A$. From the definition of the operator norm for bounded operators we have for $M \in A$,
$\|M\|=\max \left\{\left\|M_{x}\right\|_{\infty}: x \in \mathbb{C}^{2}\right.$ with $\left.\|x\|_{\infty} \leq 1\right\}$
$=\max \left\{\left|m_{11}\right|+\left|m_{12}\right|,\left|m_{21}\right|+\left|m_{22}\right|\right\}$
Let $\mathrm{M}, \mathrm{N} \in \mathrm{A}$ with $0 \leq M \leq N$, then $m_{i j} \leq n_{i j}$ for all $i, j \epsilon\{1,2\}$. Thus we
see from the definition of the norm that $\|M\| \leq\|N\|$.
Now an example of an infinite-dimensional and semi-simple $O B A, l^{\infty}$, consisting of all bounded sequences of complex numbers.

Example (2.1.12)[9]:- Let $A=l^{\infty}$ with multiplication defined coordinate wise and $C=f\left(c_{1}, c_{2}, \ldots\right) \epsilon l^{\infty}: c_{i} \geq 0$ for all $i \epsilon \mathbb{N}$. Then $(A, C)$ is an $O B A, A$ is semi-simple and $C$ is normal.

Proof:- From the coordinate wise multiplication it follows easily that $A$ is a Banach algebra, with unit $(1,1, \ldots)$. Direct calculation shows that $C$ is an algebra cone. Now we show that $C$ is normal. Suppose that $(0,0, \ldots) \leq\left(x_{1}, x_{2}, \ldots\right) \leq\left(y_{1}, y_{2}, \ldots\right)$ in $A$. By definition of $C$ this means that $0 \leq \mathrm{x}_{k} \leq \mathrm{y}_{k}$ for all $k \in \mathbb{N}$. Hence $\left\|\left(x_{1}, x_{2}, \ldots\right)\right\| \leq \|$ $\left(y_{1}, y_{2}, \ldots\right) \|$, thus $C$ is normal We have $\sigma\left(\left(x_{1}, x_{2}, \ldots\right)\right)=$ $\left\{x_{1}, x_{2}, \ldots\right\}$ for $\left(x_{1}, x_{2}, \ldots\right) \in l^{\infty}$, so $Q N\left(l^{\infty}\right)=\{0\}$.

Now we look at the set consisting of all bounded sequences of upper triangular $2 \times 2$ complex matrices to get an example of an infinite dimensional and not semi-simple $O B A$.

Example (2.1.13)[11]:- Let $A$ be the set of upper triangular $2 \times 2$ matrices, $l^{\infty}(A)$ the set $\left\{x=\left(x_{1}, x_{2}, \ldots\right): x_{i} \in A\right.$ and $\left\|x_{i}\right\|_{A} \leq K_{x}$ for some $K_{x} \in R$, for all $\left.i \in \mathbb{N}\right\}$ and $C$ the set $\left\{\left(c_{1}, c_{2}, \ldots\right) \in l^{\infty}(A) c_{i}\right.$ has only nonnegative entries for all $i \in \mathbb{N}\}$. Then $\left(l^{\infty}(A), C\right)$ is an $O B A, C$ is closed and normal and $l^{\infty}(A)$ is not semi-simple.

Proof:- By defining addition, scalar multiplication and multiplication coordinate wise and the norm to be $\left\|\left(x_{1}, x_{2}, \ldots\right)\right\|=\sup _{j \in \mathbb{N}}\left\|x_{j}\right\|_{A}$ it is not hard to show that $l^{\infty}(A)$ is an Banach algebra with unit $\left(\left(\begin{array}{ll}1 & 0 \\ 0 & 1\end{array}\right),\left(\begin{array}{ll}1 & 0 \\ 0 & 1\end{array}\right), \ldots\right)$. Direct calculation also show that $C$ is an algebra cone of $l^{\infty}(A)$. Now we will prove normality. Suppose $0 \leq x \leq y$, where $0=\left(\left(\begin{array}{ll}0 & 0 \\ 0 & 0\end{array}\right),\left(\begin{array}{ll}0 & 0 \\ 0 & 0\end{array}\right), \ldots\right) . x=\left(\left(\begin{array}{cc}x_{11} & x_{12} \\ 0 & x_{14}\end{array}\right),\left(\begin{array}{cc}x_{21} & x_{22} \\ 0 & x_{24}\end{array}\right), \ldots\right)$. And $y=\left(\left(\begin{array}{cc}y_{11} & y_{12} \\ 0 & y_{14}\end{array}\right),\left(\begin{array}{cc}y_{21} & y_{22} \\ 0 & y_{24}\end{array}\right), \ldots\right)$. From the definition of $C$ we see that $0 \leq x_{j k} \leq y_{j k}$ for all $j \in \mathbb{N}$ and $k=1,2,4$. Therefore $\max \left\{\left|x_{j 1}\right|+\left|x_{j 2}\right| ;\left|x_{j 4}\right|\right\} \leq \max \left\{\left|y_{j 1}\right|+\left|y_{j 2}\right| ;\left|y_{j 4}\right|\right\}$, i.e.
$\left\|\left(\begin{array}{cc}x_{j 1} & x_{j 2} \\ x & x_{j 4}\end{array}\right)\right\| \leq\left\|\left(\begin{array}{cc}y_{j 1} & y_{j 2} \\ y & y_{j 4}\end{array}\right)\right\|$ for all $j \in \mathbb{N}$. It follows that $\sup _{j \in \mathbb{N}}\left\|\left(\begin{array}{cc}x_{j 1} & x_{j 2} \\ x & x_{j 4}\end{array}\right)\right\| \leq \sup _{j \in \mathbb{N}}\left\|\left(\begin{array}{cc}y_{j 1} & y_{j 2} \\ y & y_{j 4}\end{array}\right)\right\|$ i.e. $\|x\| \leq\|y\|$. Thus $C$ is normal. The closedness of $C$ follows easily from the definition of $C$. Since $\left(\left(\begin{array}{ll}0 & 1 \\ 0 & 0\end{array}\right),\left(\begin{array}{ll}0 & 1 \\ 0 & 0\end{array}\right), \ldots\right)$ is an element of the radical, $l^{\infty}(A)$ is not semi-simple.

## Sectlon Two <br> Spectral Properties in OBA,S

In this section we will establish properties of the spectral radius in an $O B A$. We will follow [19].

Definition(2.2.1)[1]:- Let $(A, C)$ be an $O B A$. If $0 \leq a \leq b$ relative to $C$ implies $r(a) \leq r(b)$, then we say that the spectral radius (function) is monotone w.r.t. the algebra cone $C$.

Theorem(2.2.2)[1]:- Let $(A, C)$ be an $O B A$ with a normal algebra cone $C$. Then the spectral radius is monotone w.r.t. $C$.

Proof:- Let $0 \leq a \leq b$, then we see with induction that $0 \leq a^{n} \leq b^{2}$. Let $\alpha$ be the normality constant, then $\left\|a^{n}\right\| \leq \alpha\left\|b^{n}\right\|$ for all $n \in \mathbb{N}$ so $r(a)=\lim _{n \rightarrow \infty}\left\|a^{n}\right\|^{\frac{1}{n}} \leq \lim _{n \rightarrow \infty}\left(\alpha\left\|b^{n}\right\|^{\frac{1}{n}}\right)=\lim _{n \rightarrow \infty} \alpha^{\frac{1}{n}} \cdot \lim _{n \rightarrow \infty}\left\|b^{n}\right\|^{\frac{1}{n}}=$ $r(b)$.

Theorem(2.2.3)[1]:- Let (A;C) be an OBA with algebra cone C such that the spectral radius is monotone. Let $a, b \in A$ be such that $0 \leq a \leq b$ relative to $C$. Then

1. if $b$ is quasinilpotent then $a$ is quasinilpotent,
2. if $b$ is in the radical of $A$ then $a$ is quasinilpotent,
3. if $b$ is in the radical of $A$ and $a$ in the center of $A$ then a is in the radical of $A$.

## Proof:-

1. $r(b)=0$, so from Theorem 2.2 we have $0 \leq r(a) \leq 0$ wich gives $\sigma(a)=0$.
2. From 1. we have $b \in \operatorname{Rad} A \Rightarrow b \in Q N(A) \Rightarrow a$ is quasinilpotent.
3. By 2. $r(a)=0$. Let $x$ be any element of $A$. Then, since $a$ commutes with $x, r(a x) \leq r(a) r(x)=0$, so $a A \subset Q N(A)$. This implies that a is in the radical of $A$.

The converse of theorem (2.2.2) is in general not true. Also if the algebra cone is not normal, the spectral radius may not be monotone.

Proposition(2.2.4)[1]:- Let $(A, C)$ be an $O B A$ with normal algebra cone $C$ and $a, b \in C$. If $a b \leq b a$ then $r(b a) \leq r(b) r(a), r(a b) \leq r(a) r(b)$ and $r(a+b) \leq r(a)+r(b)$

Proof:- If $\mathfrak{a}, b \in C$ with $a b \leq b a$, then $0 \leq(b a)^{k} \leq b^{k} a^{k}(k \in \mathbb{N})$. The normality of $C$ implies that $\|(b a)\|^{k} \leq$
$\left\|b^{k}\right\|\left\|a^{k}\right\|$. As in the proof of (2.2.2) it follows that $r(b a) \leq$ $r(b) r(a)$.

The second inequality follows in the same way as in the first part, from the observation that $(a b)^{k} \leq(b a)^{k} \leq b^{k} a^{k}$ for every $k \in \mathbb{N}$.

Now we will discuss some results on the connection between the monotonicity of the spectral radius relative to algebra cones of different Banach algebras.

Proposition(2.2.5)[1]:- Let $(A, C)$ be an $O B A$ and $B$ a Banach algebra with $1 \epsilon B \subseteq A$ and such that the spectral radius function in the $O B A$ $(B, C \cap B)$ is monotone. If $a, b \in B$ such that $0 \leq a \leq b$ relative to $C$ and $r(b, B)=r(b, A)$ then $r(a, A) \leq r(b, A)$.

Proof:- Let $a, b \in B$ with $0 \leq a \leq b$ relative to $C$. Since the spectral radius in $(B, C \cap B)$ is monotone, $r(\mathfrak{a}, B) \leq r(b, B)$. Because $B$ is a subalgebra of $A$ we have $\sigma(\mathfrak{a}, A) \subset \sigma(\mathfrak{a}, B)$ and therefore $r(a, A) \leq r(a, B)$. We assumed $r(b, A)=r(b, B)$ and we get $r(a, A) \leq r(a ; B) \leq r(b, B)=$ $r(b, A)$.

If we restrict ourselves to inessential ideals, we can prove a quite similar result in quotient algebras.

Theorem (2.2.6)[1]:- Let $(A, C)$ be an $O B A$ and $B$ a Banach algebra such that $1 \epsilon B \subset A$. Suppose that $I$ is an inessential ideal of both $A$ and $B$ such that $I_{B} \subset I_{A}$, and such that the spectral radius function in the $O B A$ $\left(B / I_{B}, \pi(C \cap B)\right)$ is monotone. If $a, b \in B$ with $0 \leq a \leq b$ relative to $C$ and $\sigma(b, B)=\sigma(b, A)$, then $r\left(\bar{a}, B / I_{B}\right) \leq r\left(\bar{b}, B / I_{B}\right)$ and $r\left(\bar{a}, A / I_{A}\right) \leq$ $r\left(\bar{b}, A / I_{A}\right)$.

Proof:- Let $A, b \in B$ with $0 \leq a \leq b$ relative to $C$. Then $0 \leq a \leq b$ w.r.t. the algebra cone $\pi(C \cap B)$ of $B=I_{B}$. Because the spectral radius $\operatorname{in}\left(B / I_{B}, \pi(C \cap B)\right)$ is monotone $r\left(\bar{a}, B / I_{B}\right) \leq r\left(\bar{b}, B / I_{B}\right)$. Let $\bar{a} \in B / I_{B}$ be invertible, then there exists a $\bar{c} \epsilon B / I_{B}$ such that $\bar{a} . \bar{c}=\overline{1}$. Since $I_{B} \subset$ $I_{A}$ we have $\bar{a} . \bar{c}=\overline{1}$ in $A / I_{A}$ as well, so $\sigma\left(\bar{a}, A / I_{A}\right) \subset \sigma\left(\bar{a}, B / I_{B}\right)$ and therefore $r\left(\bar{a}, A / I_{A}\right) \leq r\left(\bar{b}, B / I_{B}\right)$. and the assumption $\sigma(b, B)=$
$\sigma(b, A)$ imply that $D(b, B, I)=D(b, A, I)$. The ideals $I_{A}$ and $I_{B}$ are inessential now tells us that $\sigma\left(\bar{b}, B / I_{B}\right)^{\wedge}=D(b, B, I)^{\wedge}=D(b, A, I)^{\wedge}=$ $\sigma\left(\bar{b}, A / I_{A}\right)^{\wedge}$. So, $r\left(\bar{b}, B / I_{B}\right)=r\left(\bar{b}, A / I_{A}\right)$. Combining the results, it follows that $r\left(\bar{a}, A / I_{A}\right) \leq r\left(\bar{b}, A / I_{A}\right)$.

Theorem(2.2.7)[8]:- Let $(A, C)$ be an $O B A$ with a closed algebra cone $C$ such that the spectral radius function is monotone. If $a \in C$ then $r(a) \epsilon \sigma(a)$.

Proof:- Let $a \geq 0$ and assume $r(a)=1$. Suppose $1 \neq \sigma(a)$. Choose $0<\alpha<1$ such that $\sigma(a) \subset\{\lambda \in \mathbb{C}: \operatorname{Re} \lambda \leq \alpha\}$. Let t be a positive real number and let $f(z)=e^{t z}$. By the spectral mapping theorem $\sigma\left(e^{t a}\right)=$ $e^{t \sigma(a)} \subset\left\{\lambda \in \mathbb{C}:|\lambda| \leq e^{t}\right\}$ and so $r\left(e^{t a}\right) \leq e^{t}$ for all $t \geq 0$. Since $a \in C$, $t \geq 0$ and $C$ is a closed algebra cone, we have $e^{t a}=1+t a+$ $\left((t a)^{2} /(2!)+\cdots \in C\right.$ so that $0 \leq\left(t^{n}\right) /(n!) a^{n} \leq e^{t a}$, for all $n \in \mathbb{N}$ and $t \geq 0$. By the monotonicity of the spectral radius and $r(a)=1$ we get, $0 \leq r\left(\left(t^{n}\right)=(n!) a^{n}\right)=\left(t^{n}\right)=(n!) \leq e^{t \alpha}$ Substituting $\quad t=n / \alpha$ in this inequality yields a contradiction to Stirling's formula.

Hence $1 \epsilon \sigma(a)$.
This theorem is a stronger version of the following well known theorem:
Theorem(2.2.8)[8]:- Let $(A, C)$ be an $O B A$ with a closed normal algebra cone $C$ and $a \epsilon C$. Then $r(a) \epsilon \sigma(a)$.

Proof:- Because $C$ is normal, the spectral radius is monotone by Theorem (2.2.2) and the result follows from Theorem (2.2.7).

Theorem (2.2.9)[8]:- Let $(\mathrm{A} ; \mathrm{C})$ be an $O B A$ with a closed cone $C$ and let $F$ be a closed ideal of A such that the spectral radius function in $(A / F, C)$ is monotone. If $a \epsilon C$ then $r(\bar{a}, A / F) \epsilon(\bar{a}, A=F)$.

Proof:- Since we cannot deduce from the closedness of C that $\pi C$ is closed, Theorem (2.2.9) does not just follow from Theorem (2.2.7), but the proof is almost the same as that of theorem (2.2.7). There is just one difference to get to the conclusion that $e^{t \bar{a}} \epsilon \pi C$. Let $a \epsilon C$, then we have $\pi\left(e^{t a}\right)=1+t \bar{a}+\left((t \bar{a})^{2}\right)=(2!)+\cdots=e^{t a}$ and $e^{t a} \in C$ because $C$ is closed, so $e^{t \bar{a}} \epsilon \pi C$.

Theorem (2.2.10)[10]:- Let $(A, C)$ be an $O B A$ and B a Banach algebra with $1 \epsilon B \subset A$ such that $C \cap B$ is closed in $B$. Suppose that $I$ is an inessential ideal of both $A$ and $B$ such that $I_{B} \subset I_{A}$ and suppose the spectral radius function in the $O B A\left(B / I_{B}, \pi(C \cap B)\right)$ is monotone. If $a \epsilon C \cap B$ is such that $\sigma(A, A)=\sigma(A, B)$, then $r\left(\bar{a}, B / I_{B}\right) \epsilon \sigma\left(\bar{a}, B / I_{B}\right)$ and $r\left(\bar{a}, A / I_{A}\right) \in \sigma\left(\bar{a}, A / I_{A}\right)$.

Proof:- It follows from Theorem 2.2 .9 that $r\left(\bar{a}, B / I_{B}\right) \epsilon \sigma\left(\bar{a}, B / I_{B}\right)$. Theorem 1.37, Theorem 1.39 and the assumption $\sigma(a ; A)=\sigma(a ; B)$ imply that $D(A, B, I)=D(A, A, I)$. So $D(\mathfrak{a}, B, I)^{\wedge}=D(\mathfrak{a}, A, I)^{\wedge}$ and by Theorem 1.36 .3 we have $r\left(\bar{a}, A / I_{A}\right)^{\wedge}=r\left(\bar{a}, B / I_{B}\right)^{\wedge}$. Hence $r(\bar{a}, A /$ $\left.I_{A}\right)=r\left(\bar{a}, B / I_{B}\right)$. Combining the results it follows that $r(\bar{a}, A /$ $\left.I_{A}\right) \epsilon \sigma\left(\bar{a}, A / I_{A}\right)^{\wedge}$. Consider the polynomial $x+r\left(a, A / I_{A}\right)$, then we have that $\left|2 r\left(a, A / I_{A}\right)\right| \leq\left\|x+r\left(a, A / I_{A}\right)\right\|_{\sigma\left(a, A / I_{A}\right)}$, and we conclude that $r\left(a, A / I_{A}\right) \epsilon \sigma\left(a, A / I_{A}\right)$.

Theorem (2.2.11)[10]:- Let $(A, C)$ be an ordered Banach algebra with $C$ closed, normal and inverse-closed. If $a \epsilon C$, then $\delta(a) \epsilon \sigma(a)$.

Proof:- If $a$ is not invertible then $(a)=0 \epsilon \sigma(a)$. Suppose a is invertible.
Since $a \epsilon C$ and $C$ is inverse-closed we have $a^{-1} \epsilon C$. Also, because $C$ is normal and closed, it follows from Theorem 2.2.8 that $r\left(a^{-1}\right) \epsilon \sigma\left(a^{-1}\right)$. So using the spectral mapping theorem we see that $r\left(a^{-1}\right)=1 / \lambda_{0}$, for some $\lambda_{0} \epsilon \sigma(a)$. We have that $r\left(a^{-1}\right)=1 / \delta(a)$ by Lemma 1.23 , which implies $\delta(a)=\lambda_{0} \epsilon \sigma(a)$.

Lemma (2.2.12)[10]:- Let $(A, C)$ be an $O B A$ with $C$ closed and $I$ a closed inessential ideal of $A$ such that the spectral radius in $\left(\frac{A}{I}, \pi C\right)$ is monotone. Let $a \in C$.

1. If $r(a)$ is a Riesz point of $\sigma(a)$, then $r(a)<r(a)$.
2. If, in addition, the spectral radius in $(A, C)$ is also monotone, then $r(a)$ is a Riesz point of $\sigma(a)$ if and only if $r(\bar{a})<r(a)$.

Proof:- (1) If $r(a)=r(a)$, then, by Theorem 2.2.9, $r(a) \epsilon \sigma(a)$. Therefore by Theorem 1.36.3 $r(a) \epsilon D(a)$, so that $r(a)$ is not a Riesz point of $\sigma(a)$.
(2) Conversely, if $r(\bar{a}) \leq r(a)$ and $r(a)$ is not a Riesz point of $\sigma(a)$, then by Theorem 2.2 .7 we have $r(a) \epsilon D(a)$, so by Theorem 1.36 $r(a) \epsilon \sigma(\bar{a})^{\wedge}$. Therefore $r(a) \leq r(\bar{a})$.

Lemma (2.2.13)[4]:- Let I be a two-sided closed inessential ideal in the Banach algebra $A$. Then for every $a \in A$ the set $\sigma(a, A) \backslash \sigma\left(\bar{a}, A / I_{A}\right)$ is the union of the Riesz points of $\sigma(a)$ relative to $I$ and some of the holes of $\sigma\left(a, A / I_{A}\right)$.

Proof:- See Theorem 6.1 in [4].
Theorem (2.2.14)[4]:- Let $(A, C)$ be an $O B A$ with $C$ closed and $I$ a closed inessential ideal of $A$ such that the spectral radius in $(A / I, \pi C)$ is monotone. If $a \in C$ is such that $r(a)$ is a Riesz point of $\sigma(a)$, then $\operatorname{psp}(a)$ consists of Riesz points of $\sigma(a)$.

Proof:- Let $\lambda \in \operatorname{spp}(a)$. If $\lambda \in r(\bar{a})$, then $r(a)=|\lambda| \leq r(\bar{a})$, so that $r(a)=r(\bar{a})$. But by Lemma 2.2 .12 this is a contradiction with the fact that $r(a)$ is a Riesz point of $\sigma(a)$. Therefore $p \operatorname{sp}(a) \subset \sigma(a) \backslash \sigma(\bar{a})$ and Lemma 2.2.13,now tells us that $p \operatorname{sp}(a)$ consists of Riesz points of $\sigma(a)$

## Section Three

## Poles of The Resolvent in OBA's

In this section we investigate the role of poles of the resolvent in spectral theory. First we state versions of the Krein-Rutman Theorem in an $O B A$,then we take a closer look at the structure of the spectrum.

Lemma (2.3.1)[8]:- Let $A$ be a Banach algebra and $a \in A$. If $\lambda_{0}$ is an isolated point of $\sigma(a)$ then

$$
(z-a)^{-1}=\sum_{n=-\infty}^{\infty}\left(z-\lambda_{0}\right)^{n} a_{n}
$$

For $0<\left|z-\lambda_{0}\right|<r_{0}=d\left(\lambda_{0}, \sigma(a) \backslash\left\{\lambda_{0}\right\}\right)$, where

$$
a_{n}=\frac{1}{2 \pi i} \int_{\gamma}\left(z-\lambda_{0}\right)^{-n-1}(z-a)^{-1} d z
$$

for any positively oriented circle centered at $\lambda_{0}$ with radius $<r_{0}$. The isolated point $\lambda_{0}$ is a pole of order $k \geq 1$ if and only if $a_{-k} \neq 0$ and $a_{-n}=0$ for all $n>k$.

Proof:- The first part is Lemma 6.11 in [3] and the series follows from the usual Laurent series development that can be found in Theorem 1.11 in [3]. The statement for the pole of order $k$.

Proposition (2.3.2)[8]:- Let $A$ be a Banach algebra and $a \epsilon A$. If $\lambda_{0}$ is an isolated point of $\sigma(a)$ and $n \geq 1$, then $\lambda_{0}$ is a pole of the resolvent function $R(z, a)=(z-a)^{-1}$ of order $n$ if and only if $\left(\lambda_{0}-\right.$ $a)^{n} p\left(a, \lambda_{0}\right)=0$ and $\left(\lambda_{0}-a\right)^{n-1} p\left(a, \lambda_{0}\right) \neq 0$.

Proof:- Let $(z-a)^{-1}=\sum_{n=-\infty}^{\infty}\left(z-\lambda_{0}\right)^{n} a_{n}$ as in Lemma 2.3.1. Now $\lambda_{0}$ is a pole of order n if and only if $a^{n} \neq 0$ and $a_{-k}=0$ for $k>n$. Let $\Gamma$ be a positively oriented system of curves such that $\sigma(a)\left\{\lambda_{0}\right\} \subseteq$ ins $\Gamma$ and $\lambda_{0} \in$ out $\Gamma$. Let $\gamma$ be a circle centered at $\lambda_{0}$ and contained in out $\Gamma$. Let $e(z) \equiv 1$ in a neighborhood of $\gamma \cup$ ins $\gamma$ and $e(z) \equiv 0$ in a neighborhood of $\Gamma \cup$ ins $\Gamma$. So e $e \operatorname{Hol}(a)$ and $e(a)=p\left(a, \lambda_{0}\right)$. If $k \geq$ 1 ,

$$
\begin{gathered}
a_{-k}=\frac{1}{2 \pi i} \int_{\gamma}\left(z-\lambda_{0}\right)^{k-1}(z-a)^{-1} d z \\
=\frac{1}{2 \pi i} \int_{\gamma+\Gamma} e(z)\left(z-\lambda_{0}\right)^{k-1}(z-a)^{-1} d z \\
=p\left(a, \lambda_{0}\right)\left(a-\lambda_{0}\right)^{k-1}
\end{gathered}
$$

The last stap follows from the functional calculus, since $\sigma(a) \subseteq$ ins $(\gamma+\Gamma)$.The proposition follows.

Since $p\left(a, \lambda_{0}\right)$ is an idempotent it directly follows that:
Corollary (2.3.3)[8]:- Let $A$ be a Banach algebra and $a \in A$. If $\lambda_{0}$ is an isolated point of $\sigma(a)$ and $n \geq 1$, then $\lambda_{0}$ is a pole of order n of the resolvent if and only if $\left(\lambda_{0}-a\right) p\left(a, \lambda_{0}\right)$ is a nilpotent element of $A$ of order $n$.

## (2.3.4) Krein-Rutman Theorems[9]

We will now state $O B A$ versions of the Krein-Rutman Theorem, which is concerned with operators,. The Krein-Rutman Theorem describes conditions under which the spectral radius of a positive operator is an eigenvalue of that operator, with a positive eigenvector. For more information on this theorem we refer to [7]. First we state a version in which the condition that ensures that if $a$ is positive, $r(a)$ is an eigenvalue of a with positive eigenvector, is in terms of $r(a)$

Theorem (2.3.5)[9]:- Let A be $O B A$ with a closed algebra cone $C$ and let $0 \neq a \epsilon C$ be such that $r(a)>0$. If $r(a)$ is a pole of the resolvent of a, then there exists $0 \neq u \epsilon C$ such that $u a=a u=r(a) u$ and $a u a=$ $r(a)^{2} u$.

Proof:- Suppose that $r(a)$ is a pole of order $k$ of the resolvent of $a$. Then we have according to Lemma 2.3.1 the following Laurent series development of the resolvent:
$R(z, a)=\sum_{n=-k}^{\infty}(z-r(a))^{n} a_{n}, 0<|z-r(a)|<\operatorname{dist}(r(a), \sigma(a) \backslash$ $\{r(a)\})$.

From the Laurent expression it follows that $a_{-k}=\lim _{z \downarrow r(a)}(z-$ $r(a))^{k} R(z, a)$. We show that $a \mathrm{k}$ is a possible choice for u . It is clear that a commutes with $a_{-k}$. From the Neumann series $R(z, a)=$ $\sum_{j=0}^{\infty} \frac{a_{j}}{z^{j+1}}(z>r(a))$ for $R(z, a)$
and the fact that C is a closed algebra cone it follows that $\mathrm{R}(z ; a)$, and hence $a_{-k}$, is an element of $C$. From the proof of Proposition 2.3.2 it follows that $0=a_{-}(k+1)=(r(a)-a) a_{-k}=a_{-k}(r(a)-a)$, which yields the first part of the theorem, with $u=a_{-k}$. Since $a u=u a=$ $r(a) u$, it follows that $a u a=r(a)^{2} u$.

From the proof we see that if the pole $r(a)$ of the resolvent function is of order $k$, a possible choice for $u$ is the coefficient $a_{-k}$ from the Laurent series expression of the resolvent. To distinguish $a_{-k}$ from possible other eigenvectors, we will call $a_{-k}$ the (positive) Laurent eigenvector of the eigenvalue $r(a)$ of $a$. From the proof we see that we have more generally:

Theorem (2.3.6)[9]:- Let $A$ be a Banach algebra and $a \in A$. If is a pole of the resolvent of a of order $k$, so that

$$
(z-a)^{-1}=\sum_{n=-\infty}^{\infty}\left(z-\lambda_{0}\right)^{n} a_{n}
$$

and $0 \neq u=a_{-k}$, then $a u=u a=u$.
Now we state another version of the Krein-Rutman theory in an $O B A$ context

Theorem (2.3.7)[9]:- Let $A$ be a semisimple $O B A$ with a closed normal algebra cone $C$ and let $a \in C$ be such that $r(a)>0$. If there exists a closed inessential ideal I in $A$ such that a is Riesz w.r.t $I$, then there exists $0 \neq u \epsilon C$ such that $u a=a u=r(a) u$ and $a u a=r(a)^{2} \epsilon u$.

Before we can give the proof we need a few other theorems and lemmas.
Theorem (2.3.8)[9]:- Let $A$ be a semi simple Banach algebra and $I$ an inessential ideal of $A$. Then $I \subset k h(\operatorname{soc}(A))$.

From this theorem we get the following corollary
Corollary (2.3.9)[9]:- Let A be a semi simple Banach algebra, $a \in A$ and $I$ a closed inessential ideal of $A$. If a is Riesz relative to $I$ then a is Riesz relative to $\operatorname{soc}(A)$.

Proof:- Suppose a is Riesz relative to I. According to Theorem $1.40 \sigma(a)$ is finite or a sequence converging to zero, and for every $0 \neq \epsilon \sigma(a)$ the spectral projection $p(a, \alpha)$ lies in $I$. By Theorem 2.3.8 we have $I \subset$ $k h(\operatorname{soc}(A))$, so that all these spectral projections are in $k h(\operatorname{soc}(A))$. tells us that $\operatorname{soc}(A)$ and $k h(\operatorname{soc}(A))$ have the same projections, so it follows that all these spectral projections are in $\operatorname{soc}(A)$. Thus a is Riesz relative to $\operatorname{soc}(A)$.

Lemma (2.3.10)[9]:- Let $A$ be a semisimple Banach algebra and $a \in A$. If a is in $\operatorname{soc}(A)$ and a is quasinilpotent, then $a$ is nilpotent.

Theorem (2.3.11)[9]:- Let $A$ be a semisimple Banach algebra, $a \in A$ and $I$ a closed inessential ideal of $A$ such that a is Riesz relative to I. If $0 \neq$ $\alpha \epsilon \sigma(a)$ then is a pole of the resolvent of $a$.

Proof:- If $a$ is Riesz relative to $I$, then by Corollary 2.3 .9 we have that a is Riesz relative to $\operatorname{soc}(A)$. If $0 \neq \alpha \epsilon \sigma(a)$, then that is an isolated point of $\sigma(a)$ and $p(a, \sigma)$ is in $\operatorname{soc}(A)$. Since $\operatorname{soc}(A)$ is an ideal, we have $(a-$ $\alpha) p(a, \alpha) \epsilon \operatorname{soc}(A)$. that $(a-\alpha) p(a, \alpha)$ is quasinilpotent, so from Lemma 2.3.10 we see that $(a-\alpha) p(a, \alpha)$ is nilpotent. It follows from Corollary 2.3.3 that is a pole of the resolvent of $a$.

In a similar way as for the previous theorem, we can prove the following related theorem. We do not use it to prove Theorem 2.3.7, but we will use it later on.

Theorem (2.3.12)[8]:- Let $A$ be a semisimple Banach algebra, $I$ an inessential ideal of $A$, and $a \in A$. Then a point in $\sigma(a)$ is a Riesz point of $\sigma(a)$ relative to $I$ if and only if is a pole of the resolvent of $a$ and $p(a, \alpha) \epsilon I$.

Proof:- One implication is trivial. For the other implication let be a Riesz point of $\sigma(a)$ relative to $I$. Then by definition is an isolated point of $\sigma(a)$ and $p(a, \alpha) \epsilon I$. From Theorem 2.3.8 and the fact that $k h(\operatorname{soc}(A))$ and $\operatorname{soc}(A)$ have the same spectral projections we see that $p(a, \alpha) \in \operatorname{soc}(A)$. Since $\operatorname{soc}(A)$ is an ideal, we have $(a-\alpha) p(a, \alpha)) \epsilon \operatorname{soc}(A)$. Then $(a-$ $\alpha)$ is quasinilpotent, so from Lemma 2.3 .10 we see that $(a-\alpha) p(a, \alpha)$ is nilpotent. It follows from Corollary 2.3.3 that is a pole of the resolvent of $a$.

Now we can give the proof of Theorem 2.3.7.
Proof:- By Theorem 2.2.2 and Theorem 2.3.7, $r(a) \epsilon \sigma(a)$. By assumption $r(a) \neq 0$, so by Theorem 2.3.11 $r(a)$ is a pole of the resolvent of a. The theorem now follows from Theorem 2.3.5.

## (2.3.12) More spectral theory

In this section we are going to investigate the influence that the structure of the spectrum $\sigma(a)$ has on some properties of $a$. First we discuss the case in which the spectrum consists of one element. Then we also consider spectra consisting of multiple elements. The property of a we focus on is whether positivity of a implies that $a^{-1}$ is positive, i.e. $a \geq 1$. Later on we discuss the more general case, if $f \in \operatorname{Hol}(a)$ and $f(a)$ defined by the functional calculus, whether $a \epsilon C$ implies $f(a) \epsilon C$.

Theorem (2.3.13)[8]:- Let $(A, C)$ be an $O B A$ with $C$ closed and let $a \in C$. If $\lambda>r(a)$, then $(\lambda-a)^{-1} \geq 0$.

Proof:- For $|\lambda|>r(a)$, the resolvent of $a$ has a Neumann series representation $(\lambda-a)^{-1}=\sum_{n=0}^{\infty}\left(a^{n} / \lambda^{n+1}\right)$. Since $\lambda>0$, all the terms in the series are positive, so because $C$ is closed, we have $(\lambda-a)^{-1} \geq 0$.

Theorem (2.3.14)[8]:- Let $A$ be a Banach algebra and $a \in A$ such that $\sigma(a)=\left\{\lambda_{0}\right\}$. If $\lambda \neq \lambda_{0}$, then $(\lambda-a)^{-1}=\sum_{n=1}^{\infty} b_{-n}\left(\lambda-\lambda_{0}\right)^{n-1}$

Where $b_{-n}=\left(a-\lambda_{0}\right)^{n-1}$
Proof:- If $\lambda \neq \lambda_{0}$ then $\left|\lambda-\lambda_{0}\right|>=0=r\left(\lambda-\lambda_{0}\right)$, so that

$$
\begin{aligned}
(\lambda-a)^{-1}= & \left(\left(\lambda-\lambda_{0}\right)-\left(a-\lambda_{0}\right)\right)^{-1}=\sum_{n=0}^{\infty} \frac{\left(a-\lambda_{0}\right)^{n}}{\left(\lambda-\lambda_{0}\right)^{n+1}} \\
& =\sum_{n=1}^{\infty} \frac{\left(a-\lambda_{0}\right)^{n-1}}{\left(\lambda-\lambda_{0}\right)^{n}}
\end{aligned}
$$

Hence the result follows.
Theorem (2.3.15)[8]:- Let $A$ be a Banach algebra and $a \epsilon A$ such that $\sigma(a)=\left\{\lambda_{0}\right\}$. $\lambda_{0}$ is a pole of order $k$ of the resolvent of $a$, then

$$
\lim _{\lambda \rightarrow \lambda_{0}}\left(\lambda-\lambda_{0}\right)^{k}(\lambda-a)^{-1}=\left(a-\lambda_{0}\right)^{k-1} .
$$

Now we can state some conditions which imply that if $a \in C$ and $\sigma(a)$ $=\{r(a)\}$ with $r(a) \geq 1$, then $a^{-1} \in C$.

Theorem (2.3.16)[8]:- Let $A$ be a Banach algebra and $a \in A$ such that $\sigma(a)=\{r(a)\}$.

1. If $r(a)$ is a pole of order $k$ of the resolvent of $a$, then $(a-r(a))^{k}=$ 0 .
2. If $r(a)$ is a simple pole of the resolvent of $a$, then $a=r(a)$. It follows that, if $C$ is an algebra cone of $A$, then $r(a) \geq 1 \Rightarrow a-1 \epsilon C$ Suppose that $C$ is a closed algebra cone of $A$, and $a \in C$.
3. If $r(a)$ is a pole of order $k$ of the resolvent of $a$, then $(a-$ $r(a))^{k-1} \epsilon C$.
4. If $r(a)$ is a pole of order 2 of the resolvent of $a$, then $a \geq r(a)$. Proof:-
5. Follows directly from Theorem 2.3.15.
6. Follows from 1.
7. From Theorem 2.3 .15 we have $(a-r(a))^{k-1}=\lim _{\lambda \rightarrow r(a)}(\lambda-$ $r(a))^{k}(\lambda-a)^{-1}$ so we certainly have $(a-r(a))^{k-1}=\lim _{\lambda \rightarrow r(a)^{+}}(\lambda-$ $r(a))^{k}(\lambda-a)^{-1}$. Since $C$ is closed, it follows from Theorem 2.3.13 that $(a-r(a))^{k-1} \epsilon C$
8. Follows from 3.

Now we state some results about the following question: if $a \in C$, for which functions $f \epsilon \operatorname{Hol}(a)$ does it follow that $f(a) \epsilon C$ ?

Theorem (2.3.17)[8]:-Let $(A, C)$ be an $O B A$ and $a \epsilon C$.

1. If $p(\lambda)=\alpha_{n} \lambda^{n}+\cdots+\alpha_{1} \lambda_{1}+\alpha_{0}$ with $\alpha_{n}, \ldots, \alpha_{0}$ real and positive, then $p(a) \epsilon C$.
2. Suppose, in addition, that $C$ is closed. If $f(\lambda)=e^{\lambda}$, then $f(a) \epsilon C$

Proof:- Follows from the functional calculus.
Theorem (2.3.18)[8]:- Let $A$ be a Banach algebra and $a \in A$ such that $r(a)$ is a pole of order $k$ of the resolvent of $a$. Let $f$ be a complex valued function that is analytic in the open $\operatorname{disk} D(r(a), R)$ for some $R>0$. Let $g(\lambda)=f(\lambda)(\lambda-a)^{-1}$ and let $\sum_{n=-\infty}^{\infty}(\lambda-r(a))^{n} a_{n}$ be the Laurent series of $g$ around $r(a)$.

1. If $f(r(a))=0$ and the order of $f$ at $r(a)$ is $k$, then $a_{-1}=0$.

Suppose, in addition, that $(A, C)$ is an $O B A$ with $C$ closed, $a \in C$ and $f(\lambda)>0$ for all $\lambda$ in the open real interval $(r(a), r(a)+R)$.
2. If the order of f in $r(a)$ is equal to $j \geq 0$, then $a_{-k+j} \epsilon C$ and $a_{j}=0$ for $l \leq-k+j$.

## Proof:-

1. If $f(r(a))=0$ and the order of $f$ at $r(a)$ is $k$, then the order of $g$ at $r(a)$ is zero, so its residue is zero. Hence $a_{-1}=0$.
2. If the order of $f$ in $r(a)$ is equal to $j \geq 0$, then the order of $g$ at $r(a)$ is $k-j$, so $a_{-k+j}=\lim _{\lambda \rightarrow r(a)}(\lambda-r(a))^{k-j} g(\lambda)$. Restricting $\lambda$ to the interval $(r(a), r(a)+R)$, we get $a_{-k+j}=\lim _{\lambda \rightarrow r(a)}(\lambda-r(a))^{k-j} f(\lambda)(\lambda-$ $a)^{-1}$. For $\lambda$ in $(r(a), r(a)+R)$ we have that $\mathrm{f}(\lambda)>0$ by assumption and $(\lambda-a)^{-1} \epsilon C$ by Theorem 2.3.13, so $(\lambda-r(a))^{k-j} f(\lambda)(\lambda-$ $a)^{-1} \epsilon C$. Since $C$ is closed, $a_{-k+j} \epsilon C$. It is clear that $a_{l}=0$ for $l<$ $-k+j$.

If we take $f=1$ we know that $a_{-1}$ is equal to the spectral projection $p(a, r(a))$, so that the above Theorem gives us.

Corollary 2.3.19[8]:- Let $(A, C)$ be an $O B A$ with $C$ closed, and $a \in C$ such that $r(a)$ is a simple pole of the resolvent of $a$, then $p(a, r(a)) 2 C$.

Theorem (2.3.20)[8]:-Let $A$ be a Banach algebra and $a \in A$ such that $\sigma(a)=\left\{\lambda_{1}, \ldots, \lambda_{m}\right\}(m \geq 1)$ where $\lambda_{1}=r(a)$ and $\lambda_{j}$ is a pole of order $k_{j}$ of the resolvent of $a(j=1, \ldots, m)$. Let $f \epsilon \operatorname{Hol}(a)$, such f has a zero of order $k_{j}$ at $\lambda_{j}$ for $j=2, \ldots, m$.

1. If $f(r(a))=0$ and the order of f at $r(a)$ is $k_{1}$, then $f(a)=0$. Suppose, in addition, that $(A, C)$ is an $O B A$ with $C$ closed, $a \epsilon C$ and $f(\lambda)>0$ in the real interval $(r(a), r(a)+b)$, for some $b>0$.
2. If order of $f$ at $r(a)$ is $k_{1}-1$, then $f(a) \epsilon C$

Proof:- Let $\Gamma$ be the union of circles with centers $\lambda_{1}, \ldots, \lambda_{m} \mathrm{~m}$ and resp. radii $r_{1}, \ldots, r_{m}$ such that they are disjoint. Then the functional calculus gives us $f(a)=\frac{1}{2 \pi i} \int_{\Gamma} g(\lambda) d \lambda=\sum_{j=1}^{m} \frac{1}{2 \pi i} \int_{\mathrm{C}\left(\lambda_{j}, r_{j}\right)} g(\lambda) d \lambda$ with $g(\lambda)=$ $f(\lambda)(\lambda-a)^{-1}$. Since the order of $g$ at $\lambda_{j}$ is zero, it follows that $\int_{\mathrm{C}\left(\lambda_{j}, r_{j}\right)}$ $g(\lambda) d \lambda=0$ for $j=2, \ldots, m$, so that $f(a)=\frac{1}{2 \pi i} \int_{\Gamma} g(\lambda) d \lambda$. We can choose the radius $r_{1}$ such that $r_{1}$ is analytic in a deleted neighbourhood of $r(a)$ containing $C\left(r(a), r_{1}\right)$. Therefore $\frac{1}{2 \pi i} \int_{\Gamma} g(\lambda) d \lambda$ is the residue of $g$ at $r(a)$. So $f(a)=a_{-1}$, with $a_{-1}$ the coefficient of $(\lambda-r(a))^{-1}$ in the Laurent series of $g$ around $r(a)$. The results now follow from Theorem 2.3.18.

We now give some corollaries of Theorem 2.3.20
Corollary (2.3.21)[8]:- Let $A$ be a Banach algebra and $a \in A$ such that $r(a)=k \pi \epsilon \sigma(a)$ with $k \in \mathbb{N}$ an even number, and
$\sigma(a) \backslash r(a) \subset\{n \pi: n \in\{0, \mp 1, \ldots, \mp k\}\}$.

1. If each value in $\sigma(a)$ is a simple pole of the resolvent of $a$, then sina $=0$. Suppose, in addition, that $(A, C)$ is an $O B A$ with $C$ closed, and $a \in C$.
2. If each element of $\sigma(a) \backslash r(a)$ is a simple pole and $r(a)$ is a pole of order 2 of the resolvent of a, then $\sin a \in C$

Proof:- Let $f(\lambda)=\sin \lambda$. Then $f$ has simple zeros at all the values of the spectrum of a and $f(\lambda)>0$ for all $\lambda$ in the real interval $(r(a), r(a)+\pi)$. Since $f(a)=$ sina,

1. Follows from Theorem 2.3.20(1).
2. Follows from Theorem 2.3.20(3).

Corollary (2.3.22)[8]:- Let $(A, C)$ be an $O B A$ with $C$ closed, and $a \epsilon C$ such that $r(a)=\left(k+\frac{1}{2}\right) \pi \epsilon \sigma(a)$ with $k \in \mathbb{N}$ an even number, and $\sigma(a) \backslash r(a) \subset\{n \pi: n \in\{0, \bar{\mp} 1, \ldots, \bar{\mp} k\}\}$.

If each value in $\sigma(a)$ is a simple pole of the resolvent of $a$, then $\sin a \epsilon C$.

Proof:- Let $f(\lambda)=\sin \lambda$. Then $f$ has simple zeros at all the values of $\sigma(a) \backslash r(a)$. Furthermore, $f(r(a))=1>0$ and $f(\lambda)>0$ for all $\lambda$ in the real interval $\left(r(a), r(a)+\frac{\pi}{2}\right)$. Since $f(a)=\sin a$, the result follows from Theorem 2.3.20(2).

Corollary (2.3.23)[8]:- Let $A$ be a Banach algebra and $a \in A$ such that $\sigma(a)=\{r(a)\}$ with $r(a)>0$.

1. If $r(a)=1$ is a simple pole of the resolvent of $a$, then $\log a=0$.

Suppose, in addition, that $(A, C)$ is an $O B A$ with $C$ closed, and $a \in C$
2. If $r(a)$ is a simple pole of the resolvent of $a$ and $r(a)>1$, then $\log a \in C$.
3. If $r(a)=1$ is a pole of order 2 of the resolvent of $a$, then $\log a \in C$.

Proof:- Let $f(\lambda)=\log \lambda(\log \lambda$ is the principal branch of the complex logarithm), then $f$ is analytic on the right half plane, so because $r(a)>$ $0, f \epsilon \operatorname{Hol}(a)$. Also, $f$ has a simple zero at 1 , and $f(\lambda)>0$ for all real $\lambda>1$. Hence the results follow from Theorem 2.3.20.

Corollary (2.3.23)[8]:- Let $(A, C)$ be an $O B A$ with $C$ closed and $a \epsilon C$ such that $\sigma(a)=\{1, r(a)\}$, with $r(a)>1$. If both 1 and $r(a)$ are simple poles of the resolvent of $a$, then $\log a \in C$.

Proof:- Let $f=\log \lambda$, then as in the proof of the previous corollary we have $f \epsilon \operatorname{Hol}(a)$ and $f(\lambda)>0$ for all real $\lambda>1$. Furthermore, 1 and $r(a)$ are both simple poles, hence the result follows from Theorem 2.3.20(2).

Corollary (2.3.24)[8]:- Let $(A, C)$ be an $O B A$ with $C$ closed and $a \in C$ such that $\sigma(a)=\{1, r(a)\}$, with $r(a)>1$. If both 1 and $r(a)$ are simple poles of the resolvent of $a$, then $\log a \in C$.

Proof:- Let $f=\log \lambda$, then as in the proof of the previous corollary we have $f \epsilon \operatorname{Hol}(a)$ and $f(\lambda)>0$ for all real $\lambda>1$. Furthermore, 1 and $r(a)$ are both simple poles, hence the result follows from Theorem 2.3.20(2).

Now we discuss the case of $C$ being inverse-closed. First a theorem that complements Theorem 2.3.17 and 2.3.20.

Theorem (2.3.25)[8]:- Let $(A, C)$ be an $O B A$ with $C$ inverse-closed, and $a \in C$. Let $p(\lambda)=\alpha_{n} \lambda^{n}+\cdots+\alpha_{1} \lambda+\alpha_{0}$ and $q(\lambda)=\beta_{m} \lambda^{m}+\cdots+$ $\beta_{1} \lambda+\beta_{0}$ with $\alpha_{n}, \ldots, \alpha_{0}, \beta_{m}, \ldots, \beta_{0}$ real a positive. Suppose that $q(\lambda)$ has no zeroes in $\sigma(a)$ and let $t(\lambda)=p(\lambda) / q(\lambda)$. Then $t(a) \epsilon C$.

Proof:- From Theorem 2.3.17(1) it follows that $p(a) \epsilon C$ and $q(a) \epsilon C$. According to the Spectral Mapping Theorem $\sigma(q(a))=q(\sigma(a))$, and $q(\lambda)$ has no zeroes in $\sigma(a)$, so $q(a)$ is invertible and $q^{-1} \epsilon \operatorname{Hol}(a)$. Since $C$ is inverse-closed, $(q(a))^{-1} \in C$. From the functional calculus we have $t(a)=p(a)(q(a))^{-1}$, so it follows that $t(a) \epsilon C$.

Now we give some conditions under which it is true that $a \epsilon C$ and $\sigma(a)=\{1\}$ imply that $a-1 \epsilon C$, under the assumption that $C$ is inverseclosed.

We begin with an obvious lemma
Lemma (2.3.26)[8]:- Let $(A, C)$ be an $O B A$ with a and b invertible elements of $A$ such that $a \leq b$ and $a^{-1}, b^{-1} \geq 0$. Then $b^{-1} \leq a^{-1}$.

Theorem (2.3.26)[8]:- Let $(A, C)$ be an $O B A$ with C closed and inverseclosed. If $a \in C$ and a is invertible, then

1. $a \geq \alpha$ for all $\alpha \geq 0$ with $\alpha<\delta(a)$.
2. $a \leq \beta$ for all $\beta>r(a)$.

## Proof:-

1. For $\alpha=0$ it is obviously true. Let $0<\alpha<(a)$, then $(1 / \delta(a))<$ $(1 / \alpha)$, so that $(1 / \alpha)>r\left(a^{-1}\right)$. It follows from Theorem 2.3.13 that $\left.(1 / \alpha)-a^{-1}\right)^{-1} \geq 0$ Because $C$ is inverse-closed $(1 / \alpha)-a^{-1} \geq 0$, so we have $a^{-1} \leq(1 / \alpha)$ The result now follows from Lemma 2.3.26.
2. If $\beta>r(a)$, then according to Theorem 2.3.13, $(\beta-a)^{-1} \geq 0$. Since $C$ is inverse-closed, it follows that $\beta-a \geq 0$, and hence $a \leq \beta$.

Theorem (2.3.27)[8]:- Let $(A, C)$ be an $O B A$ with $C$ closed and inverseclosed, and let $a \epsilon C$. Then we have

1. $\delta(a) \leq a \leq r(a)$.

Suppose, in addition, C is proper. Then,
2. $\sigma(a) \subset\{z \in \mathbb{C}:|z|=1\} \Rightarrow a=1$.
3. $\sigma(a)=\{1\} \Rightarrow a=1$.

## Proof:-

1. Let $\left(\alpha_{n}\right)$ be a sequence of real numbers such that $0 \leq$ $\alpha_{n}<\delta(a)$ and $\alpha_{n} \rightarrow \delta(a)$ as $n \rightarrow \infty$. By Theorem 2.3.27(1), $a \geq \alpha_{n}$, i.e. $\left(a-\alpha_{n}\right) \epsilon C$ for all $n$. Therefore $\lim _{n \rightarrow \infty}\left(a-\alpha_{n}\right)=a-\delta(a) \epsilon C$, because $C$ is closed. Let $\left(\beta_{n}\right)$ be a sequence of real numbers such that $r(a)<\beta_{n}$ and $\beta_{n} \rightarrow r(a)$ as $n \rightarrow \infty$. Then $a \leq \beta_{n}$, by Theorem 2.3.27(2), so as before we have that $a \leq r(a)$.
2. If $\sigma(a) \subset\{z \in \mathbb{C}:|z|=1\}$, then $\delta(a)=1=r(a)$, so by 1 . we have that
$1 \leq a \leq 1$. Therefore, because $C$ is proper, it follows that $a=1$.
3. Follows from 2.

Lemma (2.3.28)[8]:- Let $A$ be a Banach algebra and $a \in A$. If there exist $k \in \mathbb{N}$ and $\lambda_{0} \epsilon C$ such that $\operatorname{psp}\left(a^{k}\right)=\left\{\lambda_{0}\right\}$, then $\# p s p(a) \leq k$.

Proof:- If $\lambda \in \operatorname{psp}(a)$, then (by the Spectral Mapping Theorem) $\lambda^{k} \operatorname{\epsilon psp}\left(a^{k}\right)$, so $\lambda^{k}=\lambda_{0}$. Hence every $\lambda \epsilon p s p(a)$ is a $k-t h$ root of $\lambda_{0}$ and thus $\# p s p(a) \leq k$.

Theorem (2.3.29)[8]:- Let $(A, C)$ be an $O B A$ with $C$ closed and the spectral radius function monotone. If $a \in A$ and there exist $k \in N$ and $\alpha>0$ such that $a^{k} \geq \alpha$, then

1. $\operatorname{psp}\left(a^{k}\right)=\left\{r(a)^{k}\right\}$.
2. $\# p s p(a) \leq k$.

## Proof:-

1. Since $p \operatorname{sp}(a)=\beta p s p(a)$ for all $\beta 0$, we may assume without loss of generality that $r(a)=1$. Let $b=a^{k}-\alpha$. Then $b \geq 0$. Since $a^{k}=$ $b+\alpha$, it follows that $1=r\left(a^{k}\right)=r(b+\alpha)$, so that $1=\sup \{\mid \lambda+$ $\alpha \mid: \lambda \epsilon \sigma(b) g$. Since $r(b) \epsilon \sigma(b)$, by Theorem 2.2.7, this supremum is exactly $r(b)+\alpha$. Hence $r(b)=1-\alpha$, so that $\sigma\left(a^{k}\right) \subset\{\lambda+\alpha$ :
$|\lambda| \leq 1-\alpha\}$.Now suppose $z \operatorname{\epsilon psp}\left(a^{k}\right)$. Then $z=\lambda+\alpha$ with $|\lambda| \leq$ $1-\alpha$, so that $|z| \leq 1-\alpha$, and $|z|=1$. Consequently $z \in \bar{D}(\alpha, 1-$ $\alpha) \cap\{z \in \mathbb{C}:|z|=1\}$. Let $z=c+d i$. Then $(c-\alpha)^{2}+d^{2} \leq(1-$ $\alpha)^{2}$ and $c^{2}+d^{2}=1$, so that $2 \alpha c \geq 2 \alpha$, and hence $c \geq 1$, since $\alpha>0$. Since $c^{2}+d^{2}=1$, it follows that $c=1$ and $d=0$, so that $z=1$. Hence the result follows.
2. Follows from 1. and Lemma 2.3.29.

Now with Theorem 2.3.27(1) and 2.3.30(1) we come to
Theorem (2.3.30)[8]:- Let $(A, C)$ be an $O B A$ with $C$ closed, inverseclosed and the spectral radius function monotone. If $a \epsilon C$ is an invertible element, then $p s p(a)=\{r(a)\}$.

## (2.3.31)Representation theorems for OBA's

Let $A$ be a Banach space. With $A^{*}$ we denote the dual space of $A$ and with $w k^{*}$ the weak-star topology of this space.

We state a corollary of the Hahn-Banach Theorem .
Corollary (2.3.32)[8]:- If $A$ is a normed space and $x \in A$, then $\|x\|=$ $\sup \left\{|f(x)|: f \in A^{*}\right.$ and $\left.\|f\| \leq 1\right\}$

Moreover, this supremum is attained.

If $X$ is a normed space, denote by $\operatorname{ball}(X)$ the closed unit ball in $X$. So $\operatorname{ball}(X)=\{x \in X:\|x\| \leq 1\}$.

Theorem (2.3.33) (Alaoglu's Theorem)[8]:- If $X$ is a normed space, then $\operatorname{ball}\left(X^{*}\right)$
is $w k^{*}$ compact.

## Preference

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في هذا البحث سنقوم بدراسة جبر بناخ المرتب. بناخ الجبرا المرتب من المواضيع المهمة في التحليل. نبين المفاهيم و المبر هنات التي نحتاجها بالبحث مثل المخروطو بناخ الجبرا المرتب كذلك الاجابة على السؤال عن شروط نصف القطر الطيفي للعنصر الموجب a والمحتوى في
o(a). وبينا انه الدالة f هي هومومورفك على بعض الجوارات ل f (a)

$$
\text { } \sigma(a) \text { التي تحقق } f(a) \geq 0 \text { حيث } 0 \text { ح }
$$

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قسم الرياضيات

## حول بناخ (لجبرا المرتب

بحث مقدم الى مجلس قسم الرياضيات / كلية التربية كجزء من متطلبات نيل درجة بكالوريوس علوم في الرياضيات

|عد/د الطالبة<br>فاتن قاسم ترتيب

