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On e-Small Submodules

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بسم الله الرحمن الرحيم

(الله لا إله إلاهو الحي القيوم لاتأخذه سنة ولانوم له مافي السموات وما في الارض من ذا الذي يشفع عنده ألا بأذنه يعلم مابين ايديهم وماخلفهم ولايحيطون بشيء من علمهِ إلا بما شاء وسع كرسيه السماوات والارض ولايؤده حفظهما وهو العلي العظيم)

صدق الله العلي العظيم

Supervisor’s Certification

I certify that the Graduation Research wich is entitled **''On e-small submodules''** by **Ahmed Hadi Salah** was made under my supervision at the University of Al-Qadsisyah, College of Education,Department of Mathematics as a partial fulfillment of th requirements for the degree of Bachelor of Science in Mathematics.

Signature:

Name: Asst. Prof. Dr. Akeel Ramadan Mehdi

Date:

الاهداء

الى من رقدت اجسامهم تحت التراب...

الى الذين ضحوا من اجل الاسلام...

للذين جاهدوا من اجل العراق...

للذين استشهدوا من اجل الحرية

والرد بوجه الظلم والطغيان ...

اهدي ثمرة جهدي هذهِ ...

الى جميع شهداء العراق ...

احمد

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Contents

Abstract

A submodule N of a left R–module M is called e-small if for any essential submodule X of M such that $N+X= M$, then $X=M$ [7].

In this work, we give survey of some known properties and kesults of e-small submodules and $Rad_e(M)$ and rewrite proof, with more details for some of them. Many examples of e-small submodules and $Rad_e(M)$ are given.

Introduction:

A left submodule N of a left R-module M is called small, if for any submodule X of M such that $N+X=M$, then $X=M$ [3,P.106].

Many generalizations of the concept of small submodules were introduced in the literature, for examples: Y. Z hou in [6] introduced δ-small submodules. Also, R. Beyranvand and F. Moradi in [1] introduced the concept of T-small submodules.

In 2011, D. X. Zhou and X. R. Zhang [7] introduced the concept of e-small as a proper generalization of small submodules. A submodule N of a left R-module M is said to be e-small in M if for any essential submodule X of M such that $N+X=M$, then $X=M$.

This work consists of two section. In section one, we state some basic concepts of Module Theory which we will need in the second section.

In section two, we give survey of some known properties and results of e-small submodules and rewrite proofs, with more details, for some of them. For example: we rewrite the proof with details of the result : If $K_1 \hookrightarrow N_1$ and $K_2 \hookrightarrow N_2$ are submodules of a left R-module M such that $M=N_1 \oplus N_2$, then $K_1 \oplus K_2 \subseteq^{e.s.} M$ if and only if $K_1 \subseteq^{e.s.} N_1$ and $K_2 \subseteq^{e.s.} N_2$ (see Proposition 2.17).

Proposition (2.2) states that every submodule of a semisimple left R-module is e-small.

Many example of e-small submodules are given, for example: Example (2.3), Example (2.4), Example (2.7) and Example (2.10).

Proposition (2.8) give a characterization of e-small submodule.

Many properties of e-small submodules are given, for example Proposition (2.13) states that if K and L are two submodules of a left R-module M, then $K+L \subseteq^{e.s.} M$ if and only if $K \subseteq^{e.s.} M$ and $L \subseteq^{e.s.} M$.

In (2018), we recall the definition of $Rad_e(M)$, and Theorem (2.20) gives a characterization of $Rad_e(M)$ by using e-small submodules.

Many properties of $Rad_e(M)$ are stated. For example, Proposition (2.25) shows that if $\alpha: N \longrightarrow M$ be a left R-homomorphism, then $\alpha(Rad_e(N)) \subseteq Rad_e(M)$.

Section One

Basic Concepts

Section One: Basic Concepts

Definition (1.1). [3, p.16] Let R be a ring. A left R-module over R is a set M together with

(1) a binary operation + on M under which $(M,+)$ is an abelian group, and

(2) amapping $\bullet: R\times M\to M$ (is called a module multiplication) denoted by rm, for all $r\in R$ and for all $m\in M$ ∈ which satisfies

(a) $(r+s)m = rm + sm$, for all $r, s,\in R, m\in M$

(b) $(rs)m = r(sm)$, for all $r, s\in R, m\in M$ and,

(c) $r(m+n) = rm + rn$, for all $r\in R$, $m, n\in M$.

If the ring R has an identity element 1 and

(d) $1m = m$, for all $m\in M$, then M is said to be a unitary left R-module.

Examples (1.2).

(1) Every abelian group is $\mathbb{Z}$-module, (in particular, $\mathbb{Q}$ and $\mathbb{Z}$ are $\mathbb{Z}$-modules).

(2) Every left ideal $(I,+,.)$ of a ring $(R,+,.)$ is a left R-module.

(3) Every ring $(R,+,\cdot)$ is a left and right R-module.

(4) Every F-vector space V is an F-module, where F is a field.

Definition (1.3). [3, P.17] Let R be a ring and let M be a left R-module. A left R-submodule of M is a subgroup N of M such that $r\circ n\in N$, for all $r\in R$, and for all $n\in N$, where $\circ$ is the module multiplication defined on M. We will use $N\hookrightarrow M$ denote that N is a submodule of M.

Example (1.4).

(1) <0> and M are trivial submodules of M.

(2) The R-submodules of a left R-module R are exactly the left ideals of a ring R.

(3) Let F be a field. Then <0> and F are the only submodules of a left F-module F.

(4) The submodules of a $\mathbb{Z}$-module $\mathbb{Z}$ are <n>, for all $n \in \mathbb{Z}$.

(5) The submodules of a $\mathbb{Z}$–module Z_4 are <0>, <2> and Z_4.

(6) The submodules of a $\mathbb{Z}$-module Z_{12} are <0>, <2>, <3>, <4>, <6> and Z_{12}.

Proposition (1.5). Let $\{Ni\}_{i=1,2,\ldots,n}$ be a family of submodules of a left R-module M. Then $\bigcap_{i=1}^{n} Ni$ and $\sum_{i=1}^{n} Ni$ are submodules of M.

Proof: See [kasch]. □

Definition (1.6). [3, P.31] A submodule N of a left R-module M is said to be a direct summand of M if there is a submodule K of M such that $M=N\oplus K$. In other word, there is a submodule K of M such that $M=N+K$ and $N\cap K=0$.

Definition (1.7). [3, P. 107] A left R-module M is said to be semisimple if every submodule of M is a direct summand of M.

Definition (1.8). [3] Let R be a ring, let M be a left R-module and let N be a left submodule of M. Then (additive, abelian) quotient group M/N can be made into a left R-module by defining a module multiplication by $\bullet: R\times M/N \longrightarrow M/N$ by $r\bullet(x+N) = (rx)+N$, for all $r \in R$, $x+N \in M/N$.

Definition (1.9). [3]

(1) A function $f:N \longrightarrow M$ is said to be a left R-module homomorhism (or just left R-homomorhism) if for all a, $b\in N$ and $r \in R$, then $f(a + b)= f(a) + f(b)$ and $f(ra)= r\,f(a)$.

(2) A left R-homomorphism is a monomorphism if it is injective and is epimorphism if it is surjective .

Example (1.10) .

(1) Let $N \hookrightarrow M$. The mapping $\pi: M \longrightarrow M/N$ defined by $\pi(m) = m + N$ is a left R-epimorphism and is called the natural epimorphism.

(2) Let $N \hookrightarrow M$. The inclusion mapping $i: N \longrightarrow M$ definied by $i(n) = n$, For all $n \in N$ is a left R-monomorphism.

(3) If $M = N \oplus K$, then the epimorphism $\pi_N:M \longrightarrow N$ defined by $\pi_N: (n + k) = n \quad \forall\, n \in N,\ k \in K$. It is called the projection epimorphism on N.

Definition (1.11). [4] Let N be a submodule of a left R-module M. A relative complement of N is denoted by N^c and defined as follows:

(1) $N^c \hookrightarrow M$.

(2) $N \cap N^c = 0$.

(3) If $B \hookrightarrow M$ such that $N \cap B = 0$ and $N^c \subseteq B$ then $N^c = B$.

Definition (1.12). [3] A left submodule N of a left R-module M is said to be a maximal submodule of M if $N \neq M$ and for every left submodule B of M with $N \subsetneq B \hookrightarrow M$, then $B=M$.

Definition (1.13). [3] Let M be a left R-module. The Jacobson radical of M is denoted by $J(M)$ and defined as the intersection of all maximal left submodules of M. If M has no maximal left submodule, then we set $J(M)=M$.

Definition (1.14). [3] A submodule N of a left R-module M is said to be small in M and denoted by $N \subseteq^{\circ} M$ if for all submodule B of M with $N+B=M$, then $B=M$. A left ideal I of a ring R is said to be small in R if I is a small left submodule in a left R-module R.

Examples (1.15).

(1) For every left R-module M we have $0 \subseteq^{\circ} M$.

(2) The only small submodule in Z_6 as $\mathbb{Z}$-module is 0.

(3) The only small submodules in Z_4 as $\mathbb{Z}$-module are 0 and <2>.

Proof: It is clear that 0, <2> and Z_4 are all submodules of Z_4 as $\mathbb{Z}$-module. From(1) above we have that 0 is a small submodule in Z_4 as $\mathbb{Z}$-module. Z_4 is not small submodule in Z_4 as $\mathbb{Z}$-module. Let B be a submodule of Z_4 such that $<2>+B=Z_4$, thus $B=Z_4$ and hence <2> is a small submodule in Z_4 as $\mathbb{Z}$-module.

Definition (1.16). [3] A submodule N of a left R-module M is said to be essential in M and denoted by $N \subseteq^{e} M$, if for all submodule B of M with $N \cap B = 0$, then $B=0$. A left ideal I of a ring R is said to be essential in R, if is an essential submodule in a left R-module R.

Examples (1.17).

(1) For every left R-module M we have $M \subseteq^{e} M$.

(2) The only essential submodule in Z_6 as $\mathbb{Z}$-module is Z_6.

(3) The only essential submodule in Z_4 as $\mathbb{Z}$-module are <2> and Z_4. It is clear that 0, <2> and Z_4 are all submodules of Z_4 as $\mathbb{Z}$-module. From (1) above we have that Z_4 is an essential submodule in Z_4 as $\mathbb{Z}$-module 0 is not essential submodule in Z_4 as $\mathbb{Z}$-module. Let B be a submodule of Z_4 such that $< 2 > \cap B = 0$, thus B=0 and hence <2> is an essential submodule in Z_4 as $\mathbb{Z}$-module.

Section Two

e-Small Submodules

Section Two: e-Small Submodules

Definition (2.1). [7] A submodule N of a left R-module M is said to be e-small and denoted by $N \subseteq^{e.s.} M$, if for any essential submodule X of M with $N+X=M$, then $X=M$.

Proposition (2.2). [5]Every submodule of a semisimple left R-module is e-small.

Proof: Let M be a semisimple left R-module and let N be submodule of M. Let X be any essential submodule of M with $M=N+X$. Since M is semisimple, M is the only essential submodule in M and hence $X=M$. Thus N is an e-small submodule in M. □

Example (2.3). Let $M=Z_{10}$ as $\mathbb{Z}$-module. Then <2>, <5> and Z_{10} are e-small submodules in Z_{10} (by Proposition (2.2)) (since Z_{10} is semisimple $\mathbb{Z}$-module), but all then are not small in Z_{10} as $\mathbb{Z}$-module.

Example (2.4). Let $M = Z_{30}$ as $\mathbb{Z}$-module. Since M is a semisimple $\mathbb{Z}$-module, all submodules of M are e-small (by Proposition(2.2)), but all non-zero submodules of M are not
small (since M is semisimple).

Proposition (2.5). Every small submodule of a left R-module is e-small.

Proof: Let M be a left R-module and let N be a small submodule of M. Let X be an essential submodule of M with $M=N+X$. Since N is small in M, $X=M$ and hence N is an e-small submodule in M.

Remark (2.6). The converse of Proposition (2.5) is not true in general as we showed that in [examples (2.3)] and (2.4).

Example (2.7). Let $M = Z_4$ as $\mathbb{Z}$-module. Then <0> and <2> are e-small in M (This form Propositin(2.5), because <0> and <2> are small in Z_4 as $\mathbb{Z}$-module). Moreover Z_4 is not e-small in Z_4 as $\mathbb{Z}$-module. Let X = <2>, then X is an essential submodule in Z_4 with $X+Z_4=Z_4$ but $X \neq Z_4$. Hence Z_4 is not e-small in Z_4 as $\mathbb{Z}$-module.

The following proposition gives an equivalent statement of e-small submodules.

Proposition (2.8). A submodule K of a left R-module N is e-small if and only if for each submodule X of N, if $K+X=N$, then X is a direct summand of N.

Proof: ($\Longrightarrow$) Suppose that $K \subseteq^{e.s} N$. Let $X \hookrightarrow N$ such that $K+X=N$. Let X^c be a relative complement of X in N. By [4, 6.1 a,P.215], $X \oplus X^c$ is an essential submodule in N. Since $K+X=N$ (by assumption), it follows that $K+X+X^c=N$. Since $K \subseteq^{e.s.} N$, $X+X^c=N$ and hence $N=X \oplus X^c$. Thus X is a direct summand of N.

($\Longrightarrow$) Suppose that, for any $X \hookrightarrow N$, if $K+X=N$ then X is a direct summand in N. Let $X \hookrightarrow N$ such that $K+X=N$ and X is an essential in N. By hypothesis, X is a direct summand in N. Since N is the only essential direct summand in N, it follows that $X=N$ and hence K is an e-small submodule in N. □

Proposition (2.9). Let K be an e-small submodule in a left R-module M. If X is a submodule of N with $K+X=N$, then M/X is a semisimple left R-module.

Proof: Let $X \hookrightarrow N$ such that $K+X=N$. Since K is e-small in N (by hypothesis), it follows form Proposition (2.8) that X is a direct summand in N. Let Y be a submodule of N such that $X \oplus Y = N$. Let $B \hookrightarrow Y$, thus $K+X+B = N$. By Proposition (2.8), $X+B$ is a direct summand in N. Since $X \cap Y = 0$, it follows that $X \cap B = 0$ and hence $X \oplus B$ is a direct summand in N. Thus there is a submodule C of N such that $X \oplus B \oplus C = N$ and hence B is a direct summand in Y (since $Y \subseteq N$). Thus Y is a semisimple R-module. Since $N/X \cong Y$, it follows that N/X is a semisimple R-module. □

Example (2.10). In the $\mathbb{Z}$-module $\mathbb{Z}$, the zero submodule is the only e-small submodule.

Proof: Since 0 is small in $\mathbb{Z}$, 0 is e-small in $\mathbb{Z}$. Assume that there is a nonzero submodule, say K in $\mathbb{Z}_\mathbb{Z}$. Since $\mathbb{Z}$ is a Principal ideal domsin, there *is* $n \in \mathbb{Z}^+$ such that $K = \langle n \rangle$. Let $X = \langle n+1 \rangle$, thus $0 \neq X \neq \mathbb{Z}$ and $K+X=\langle n\rangle+\langle n+1\rangle = \mathbb{Z}$. Since $K \subseteq^{e.s.} \mathbb{Z}_\mathbb{Z}$ (by assumption), it follows form Proposition (2.8) that X is a direct summand in $\mathbb{Z}$ and this is contradication (since 0 and $\mathbb{Z}$ are the only direct summand in $\mathbb{Z}$). Thus 0 is the only e-small submodule in $\mathbb{Z}_\mathbb{Z}$. □

Proposition (2.11). [7] Let K, L and N be submodules of a module M with $K \subseteq N$. If N is $\subseteq^{e.s.}$ in M, then $K \subseteq^{e.s.} M$ and $N/K \subseteq^{e.s.} M/K$.

Proof: let $X \subseteq^e M$ such that $K+X = M$. Then $N + X = M$. Since $N \subseteq^{e.s} M$, $X=M$. Hence $K \subseteq^{e.s} M$. Let $X/K \subseteq^e M/K$ such that $N/K + X/K = M/K$. Thus $N+X/K = M/K$ and hence $N+X = M$. Let $\pi: M \longrightarrow M/K$ be the natural epimorphism. Since $X/K \subseteq^e M/K$, it follows from [3, Lemma, 5.1.5 (c), p.110] that $\pi^{-1}(X/K) \subseteq^e M$. Then $X \subseteq^e M$. Since $N \subseteq^{e.s} M$, $X=M$ and hence $X/K=M/K$. Thus $N/K \subseteq^{e.s.} M/K$. □

Remark (2.12). The converse of Proposition (2.11) is not true in general, for example: let $M=Z_{12}$ as $\mathbb{Z}$-module, let N =<3> and K= <6>. Let $X \hookrightarrow M \ni K+X=M \Longrightarrow$ <6> $+X = Z_{12} \Longrightarrow X=Z_{12}$ and hence $K \hookrightarrow^{\circ} M$. Also, since $M/K = Z_{12}/$<6> is a semisimple $\mathbb{Z}$-module, it follows from Proposition (2.2), that N/K is e-small in M/K. Since $N +$ <2> $=M$ and <2>$\neq M$, thus N is not e-small in M . Then $K \subseteq^{\circ} M$ and $N/K \subseteq^{e.s.} M/K$ but N is not e-small in M. □

Proposition (2.13). [7] Let K, L be submodules of a left R-module M. Then $K+L \subseteq^{e.s.} M$ if and only if $K \subseteq^{e.s.} M$ and $L \subseteq^{e.s.} M$.

Proof: ($\Longrightarrow$) Suppose that $K+L \subseteq^{e.s.} M$. Since K and L are subsets of $K+L$, it follows Proposition(2.11) implies that $K \subseteq^{e.s.} M$ and $L \subseteq^{e.s.} M$. ($\Longleftarrow$) Let $X \subseteq^e M$ such that $(K+L) + X = M$. Since $X \subseteq^e M$, it follows from [3, Lemma 5.1.5 (a), p.109] that $L+X \subseteq^e M$. Since $K \subseteq^{e.s.} M$, $L+X=M$. Since $L \subseteq^{e.s.} M$, $X=M$ and hence $K+L \subseteq^{e.s.}$ M. □

Corollary (2.14). $N_i \subseteq^{e.s.} M$, i=1,2,…,n if and only if $\sum_{i=1}^{n} N_i \subseteq^{e.s.} M$.

Proof: ($\Longrightarrow$) Suppose that $N_i \subseteq^{e.s.} M$, $i = 1,2,\ldots,n$. By Proposition (2.13), $N_1+N_2 \subseteq^{e.s.} M$. By applying Proposition (2.13) n-times we get, $\sum_{i=1}^{n} N_i \subseteq^{e.s.} M$.

($\Longleftarrow$) By Proposition(2.13) $N_1 \subseteq^{e.s.} M$, and $\sum_{i=2}^{n} N_i \subseteq^{e.s.} M$, and by applying (2.13) n-times, we get $N_i \subseteq^{e.s.} M$, $\forall i = 1,2,\ldots,n$. □

Proposition (2.15). [7] Let $f : N \longrightarrow M$ be an R-homomorphism. If $B \subseteq^{e.s.} N$, then $f(B) \subseteq^{e.s.} M$.

Proof : Let $X \subseteq^{e} M$ such that $f(B)+X=M$. Thus $N = B+f^{-1}(X)$. By [3, lemma 5.1.5 (c), p.110], $f^{-1}(X) \subseteq^{e} N$. Since $B \subseteq^{e.s.} N$ (by hypothes), $f^{-1}(X) = N$ and hance $B \subseteq f^{-1}(X)$. Thus $f(B) \subseteq f(f^{-1}(X)) \subseteq X$ (by [3, Lemma 3.1.8(3), p.44]). Hence $f(B)+X=X$ and so $X=M$. Therefore, $f(B) \subseteq^{e.s.} M$. □

Corollary(2.16). Let $A \hookrightarrow B$ be submodules of a module M. If $A \subseteq^{e.s.} B$, then $A \subseteq^{e.s} M$.

Proof: Let $i : B \longrightarrow M$ be the inclusion homomorphism. Since $A \subseteq^{e.s} B$ (by hypothesis), $i(A) \subseteq^{e.s.} M$ (by Proposition (2.15)). Hence $A \subseteq^{e.s} M$.

□

The following result stated in [7] but with very short proof.

In following, we will give the proof in more details.

Proposition (2.17). [7] Let $K_1 \hookrightarrow N_1$ and $K_2 \hookrightarrow N_2$ be submodules of M such that $M = N_1 \oplus N_2$. Then $K_1 \oplus K_2 \subseteq^{e.s.} M$ if and only if $K_1 \subseteq^{e.s.} N_1$ and $K_2 \subseteq^{e.s.} N_2$.

Proof: ($\Longrightarrow$) Suppose that $K_1 \oplus K_2 \subseteq^{e.s.} M$. Let $\pi_{N_1} : M \longrightarrow N_1$ and $\pi_{N_2} : M \longrightarrow N_2$ be the projection epimorphisms. Since $K_1 \oplus K_2 \subseteq^{e.s.} M$ (by hypothesis), it following from Proposition (2.15) that

π_{N_1}: $(K_1 \oplus K_2) \subseteq^{e.s} N_1$ and $\pi_{N_2}(K_1 \oplus K_2) \subseteq^{e.s.} N_2$. Since $\pi_{N_1}(K_1 \oplus K_2) = K_1$ and $\pi_{N_2}(K_1 \oplus K_2) = K_2$ it follows that $K_1 \subseteq^{e.s.} N_1$ and $K_2 \subseteq^{e.s.} N_2$.

($\Longleftarrow$) Suppose that $K_1 \subseteq^{e.s.} N_1$ and $K_2 \subseteq^{e.s.} N_2$. By Corollary (2.16), $K_1 \subseteq^{e.s} M$ and $K_2 \subseteq^{e.s.} M$. By Proposition (2.13), $K_1 \oplus K_2 \subseteq^{e.s.} M$. □

Definition (2.18). [7] If a left R-module M has an essential maximal submodule, then we define

$Rad_e(M) = \cap\{N \hookrightarrow M / N$ is an essential maximal in $M\}$. If M has no essential maximal submodule, then we define $Rad_e(M) = M$.

Lemma (2.19): Let M be a module and let $x \in M$. Then Rx is not e-small in M if and only if $x \notin K$, for some essential maximal submodule K of M.

Proof. See[2, Proposition 3.1, p.216]. □

Theorem(2.20). [7, Theorem (2.10), p. 1055] If M is a module, then $Rad_e(M) = \sum\{N \hookrightarrow M | N \subseteq^{e.s.} M\}$.

Proof: Let $x \in Rad_e(M)$, thus $x \in N$, for any essential maximal submodule N of M. Assame that Rx is not e-small in M. By Lemma (2.19), there is an essential maximal submodule K of M such that $x \notin K$ and this is a contradiction. Thus $Rx \subseteq^{e.s.} M$ and hance a$\in \sum\{N \hookrightarrow M | N \subseteq^{e.s} M\}$. There for, $\text{Rad}_e(M) \subseteq \sum\{N \hookrightarrow M | N \subseteq^{e.s.} M\}$.

Conversely, let $B \subseteq^{e.s} M$ and let K be any essential maximal submodule in M. Assume that $B \not\subseteq K$. Since K is a maximal in M, $B+K = M$. Since $B \subseteq^{e.s.} M$, $K=M$ and this is a contradiction. Thus $B \subseteq K$ and hence $B \subseteq Rad_e(M)$. Therefore $\Sigma\{N \hookrightarrow M | N \subseteq^{e.s.} M\} \subseteq Rad_e(M)$. Hence $Rad_e(M) = \Sigma\{N \hookrightarrow M / N \subseteq^{e.s.} M\}$. □

Corollary(2.21). Let M be a left R-module and let $x \in M$. Then $x \in Rad_e(M) \longleftrightarrow Rx \subseteq^{e.s.} M$.

Proof: ($\Longrightarrow$) By lemma (1.19).

($\Longleftarrow$)By Theorem (2.20). □

Corollary(2.22). If M is a semisimple left R-module, then $Rad_e(M)=M$.

Proof: Suppose that M is a semisimple left R-module. By Proposition (2.2), $M \subseteq^{e.s.} M$. Since $Rad_e(M) = \sum\{N \subseteq^{e.s.} M\}$ (by Theorem (2.20)), it follows that $Rad_e(M)=M$. □

Remark(2.23). The converse of Corollary (2.22) is not true in general, for example: let $M=\mathbb{Q}$ as $\mathbb{Z}$-module. Since $J(M) = M$, it follows that $Rad_e(M)=M$, but M is not semisimple $\mathbb{Z}$-module.

Example(2.24).

(1) $Rad_e(Z_6)=Z_6$ as $\mathbb{Z}$-module, this from Corollary(2.22) (since Z_6 is a semisimple $\mathbb{Z}$-module).

(2) $Rad_e(Z_4) = <2>$ as $\mathbb{Z}$-module, since $<2>$ is the only essential maximal submodule in Z_4 as $\mathbb{Z}$-module.

(3) $Rad_e(\mathbb{Q}_Z) = \mathbb{Q}_Z$.

(4) $Rad_e(\mathbb{Z}_\mathbb{Z}) = 0$, since 0 is the only e-small submodule in $\mathbb{Z}_\mathbb{Z}$.

Proposition(2.25). [ ] Let $\alpha:N \longrightarrow M$ be a left R-homomrphism. Then $\alpha(Rad_e(N)) \subseteq Rad_e(M)$.

Proof: $\alpha(Rad_e(N)) = \alpha(\sum_{B \subseteq^{e.s.} N} B = \sum_{B \subseteq^{e.s.} N} \alpha(B)$. Since $B \subseteq^{e.s.} N$ it follows from Proposition (2.15) that $\alpha(B) \subseteq^{e.s.} M$. By Theorem (2.20), $\sum_{B \subseteq^{e.s.} N} \alpha(B) \subseteq Rad_e(M)$ and hence $\alpha(Rad_e(N)) \subseteq Rad_e(M)$. □

Corollary (2.26). If $N \hookrightarrow M$, then $Rad_e(N) \subseteq Rad_e(M)$.

Proof: Let $i : N \longrightarrow M$ be the inclusion R-homomorphsim. By Proposition (2.25), $i(Rad_e(N)) \subseteq Rad_e(M)$ and hence $Rad_e(N) \subseteq Rad_e(M)$. □

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