Republic of Iraq<br>Ministry of Higher Education and<br>Scientific Research<br>University of Al-Qadisiyah<br>College of Education<br>Department of Mathematics

# On S-essential Submodules 

A Graduation Research<br>Submitted to Department of Mathematics college of Education/University of Al-Qadisiyah as partial fulfillment of the Requirements for the Degree of Bachelor of Science in Mathematics.<br>\section*{By}<br>Osama Basim Mohammed Supervised by<br>Asst. Prof. Dr. Akeel Ramadan Mehdi

2018 AC

## Acknowledgments

First and foremost, praises and thank to the God, Almighty, for His blessings throughout my research work to complete the research successfully.

I am most grateful to my supervisor Asst. Prof. Dr. Akeel Ramadan Mehdi for his advice, guidance and helpful suggestions throughout this research .

Also, I with to express my sincere thanks to the staff members of Mathematics Department,College of Education, University of Al-Qadisiyah for all helps during my study.

I am also very grateful to my family for their patience, care and assistance and to all my friends.

## OSAMA

## Supervisor's Certification

I certify that the Graduation Research which
is entitled "On S-essential Submodules" by
Osama Basem Mohammed was made under my Supervision
at the University of AL-Qadisiyah, collage Of Education,
Department of Mathematics as a partial fulfillment of the
requirements of the degree of Bachelor Of Mathematics.

Signature:
Name: Asst. Prof. Dr. Akeel Ramdan Mehdi
Date: / /2018

## Abstract

A submodule $N$ of a left $R$-module $M$ is said to be s-essentail in $M$ if for each small submodule $X$ of $M$ such that $N \cap X=0$, then
$X=0$ [5].
In this work, we give survey of some known properties and results of s-essentail submodules and $\operatorname{Soc}_{\mathrm{s}}(M)$ and rewrite proofs, with more details, for some of them. Also, some new results and examples of s-essentail submodule and $\operatorname{Soc}_{\mathrm{s}}(M)$ are give in this work.

## Contents

| Subjects | Page No. |
| :--- | :---: |
| Introduction | 1 |
| Section 1: Basics Concepts | $\mathbf{3}$ |
| Section 2: On S-essential submodules | $\mathbf{9}$ |
| References | $\underline{20}$ |

## Introduction:

Throughout this work, all ring are associative with identity and all modules are unitary left $R$-modules.

A submodule $N$ of a left $R$-module $M$ is called essential in $M$ if For any submodule $X$ of $M$ such that $N \cap X=0$, then $X=0$ [3]. several authors introduced generalizations of essential submodules.For example: sh. Asgari and A.Haghany introrduced in [2] the concept of t-essential submodule as a proper generalizations of essential submodule. Also, $N$ kh. Abdullah in introduced in [1] the concept of strong essential submodule as a proper generalizations of essential submodules.
D. X. Zohu and $X$. R. Zhang in [5] introduced the concept of s-essential submodule as a proper generalizations of essential submodule. A submodule $N$ of a left $R$-module $M$ is said to be s-essential in $M$ if for each small submodule $X$ of such that $N \cap X=0$, then $X=0$.

This work consists of two sections. In section one, we introduced some basic concept of Module Theory which we will need in the second sections.

In section two, we give survey of some known properties and results of s-essentia submodules and $\operatorname{Soc}_{\mathrm{s}}(M)$ and rewrite proofs, with more details, for some of them. For examples: Proposition 2-13 wich state that (

If $f: M \rightarrow N \quad$ is $\quad$ an $\quad$-homomorphism and $K$ is an s-essential submodule in $N$, then $f^{-1}(K)$ is an s-essential in $M$ ) is appear in [5] but without proof. In this work, we give a proof of this result. Also, in Theorem 2-18 we give a proof with more details for the result $\left(\left(\operatorname{Soc}_{\mathrm{s}}(M)=\cap\left\{N \hookrightarrow M / N \subseteq^{\text {s.e }} M\right\}\right)\right.$ which appears in [5].

Many new results are given in the work. For examples in Proposition 2-2, we prove that every submodule of a semisimple module is s-essential.

Let $\left\{N_{i}\right\}$ be a family of submodules of a left R-module M. We prove in Proposition 2-11 that $\bigcap_{i=1}^{n} N_{i}$ is an S-essential submodule in $M$ if and only if is an $S$-essential Submodule in $M$, for each. In Corollary 2-19, we Prove that if $M$ is a semisimple left R-module, then $\operatorname{Soc}_{\mathrm{s}}(M)=0$.

Finally, in Proposition 2-25, we prove that $\operatorname{Soc}_{\mathrm{s}}\left(\operatorname{Soc}_{\mathrm{s}}(M)\right)=0$, For any left R -module M and give an example of left Z-module $M$ such that $\operatorname{Soc}_{\mathrm{s}}\left(\operatorname{Soc}_{\mathrm{s}}(M)\right) \neq \operatorname{Soc}_{\mathrm{s}}(M)$.

Also, we give an example of left R-module $M$ such that $\operatorname{Soc}_{\mathrm{s}}(N) \neq N \cap \operatorname{Soc}_{\mathrm{s}}(M)$, for some submodule $N$ of $M$.

## Section One: Basic Concepts

Definition (1-1): [3] Let $R$ be a ring. A left $R$-module is a set $M$ together with:
(1) A binary operation + on $M$ under which $M$ is an abelian group.
(2) A mapping $\cdot: R \times M \rightarrow M$ (is called a module multiplication ) denoted by $r m$, for all $r \in R$ and for all $m \in M$ which satisfies
(a) $(r+s) m=r m+s m$, for all $r, s \in R, m \in M$.
(b) (rs) $m=r(s m)$, for all $r, s \in R$ and $m \in M$.
(C) $r(m+n)=r m+r n$, for all $r \in R$ and $m, n \in M$.

If the ring $R$ has an identity element 1 and
(d) $1 . m=m$, for all $m \in M$, then $M$ is said to be a unitary left $R$-module.

## Examples (1-2):

1) Every Abelian group is $\mathbb{Z}$-module, (in particular, $Q$ and $\mathbb{Z}$ are $\mathbb{Z}$-modules).
2) Every left ideal $(I,+,$.$) of a ring (R,+,$.$) is a left R$-module.
3) Every ring $(R,+,$.$) is a left and right R$-module.
4) Every $F$ - vector space $V$ is an $F$-module, where $F$ is a field.

Definition (1-3): [3] Let $R$ be a ring and let $M$ be a left $R$-module. A left $R$-submodule of $M$ is a subgroup $N$ of $M$ such that $r \bullet n \in N$, for all
$r \in R$, and for all $n \in N$, where $\cdot$ is the module multiplication defined on $M$. We will use $N \hookrightarrow M$ to denote that $N$ is a submodule of $M$.

## Examples (1-4):

1) $<0>$ and $M$ are trivial submodules of $M$.
2) The $R$-submodules of a left $R$-module $R$ are exactly the left ideals of a ring $R$.
3) Let $F$ be a field. Then $<0>$ and $F$ are the only submodules of a left $F$ module $F$.
4) The submodules of a $\mathbb{Z}$-module $\mathbb{Z}$ are $<n>$, for all $n \in \mathbb{Z}$.
5) The submodules of a $\mathbb{Z}-$ module $Z_{4}$ are $\langle 0\rangle,\langle 2\rangle$ and $Z_{4}$.
6) The submodules of a $\mathbb{Z}$-module $\mathrm{Z}_{12}$ are $\left.\langle 0\rangle,\langle 2\rangle,\langle 3\rangle,\langle 4\rangle,<6\right\rangle$ and $Z_{12}$.

Proposition(1-5): Let $\left\{N_{i}\right\}_{i=1,2, \ldots, n}$ be a family of submodules of a left $R$ module $\quad M$. Then $\bigcap_{i=1}^{n-1} N_{1}$ and $\bigcap_{i=1}^{n} N_{1}$ are submodules of $M$. Proof: See[3].

Definition (1-6): [3] A submodule $N$ of a left $R$-module $M$ is said to be a direct summand of $M$ if there is a submodule $K$ of $M$ such that $M=N \bigoplus K$. In other word, there is a submodule $K$ of $M$ such that $M=N+K$ and $N \cap K=0$.

Proposition(1-7): [3] If $X$ is a subset of a left $R$-module $M$, then $<X>$ will denote the intersection of all the submodules of $M$ that contain $X$. This
is called the submodule of $M$ generated by $X$, while the elements of $X$ are called generators of $\langle X\rangle$.

Definition(1-8): [3] A left $R$-module $M$ is said to be simple if $M \neq 0$ and the only left $R$-submodules of $M$ are 0 and $M$.

Definition (1-9): [3] A left $R$-module $M$ is said to be semisimple if every submodule of $M$ is direct summand of $M$.

Definition (1-10): [3] The socle of a left $R$-module $M$ is denoted by $\operatorname{Soc}(M)$ and defined as the sum of the simple submodules of $M$. If $M$ has no simple submodule, then we $\operatorname{set} \operatorname{Soc}(M)=0$.

Proposition (1-11): (see[3]) Let $R$ be a ring, let $M$ be a left $R$-module and let $N$ be a left submodule of $M$. The (additive, Abelian) quotient group $M / N$ can be made into a left $R$-module by defining a module multiplication •:
$R \times(M / N) \rightarrow M / N$ by $r \bullet(x+N)=(r x)+N$, for all $r \in R$ and $x \in M$
Definition (1-12): [3] The left $R$-module $M / N$ is defined in Proposition (1-11) is called quotient (or factor) module.

Definition (1-13) :[3] Let $N$ and $M$ be left $R$-modules

1) A function $f: N \rightarrow M$ is said to be a left $R$-homomorphism if for all $a, b \in N$ and $r \in R$ then $f(a+b)=f(a)+f(b)$ and $f(r a)=r f(a)$.
2) A left $R$-homomorphism is called a monomorphism if it injective and is an epimorphism if it is surjective. A left $R$-homomorphism is called isomorphism if it is both injective and surjective. The modules $N$ and
$M$ are said to be isomorphic, denoted by $N \cong M$, if there is left isomorphism $f N: \rightarrow M$

## Examples (1-14):

1) Let $N \hookrightarrow M$.The mapping $\pi: M \rightarrow M / N$ defined by
$\pi(m)=m+N$ is a left $R$-epimorphism and is called the natural epimorphism.
2) Let $N \hookrightarrow M$. The mapping $i: N \rightarrow M$ defined by $i(m)=m$ for all $m \in N$ is $a$ left $R$-monomorphic.
3) If $M=N \oplus K$, then the epimorphism $\quad \pi_{N}: M \rightarrow N$ defined by $\pi_{N}(n+k)=n, \forall n \in N, k \in K$, it is called the projection epimorphism on $N$.

Definition (1-15) : [4] Let $N$ be a submodule of a left $R$-module $M$. A relative complement of $N$ is denoted by $N^{\mathrm{c}}$ and defined as follows :

1) $N^{c} \hookrightarrow M$.
2) $N \cap N^{c}=0$.
3) If $B \hookrightarrow M$ such that $N \cap B=0$ and $N^{c} \subseteq B$, then $N^{c}=B$.

Definition (1-16) : [3] A left submodule $N$ of a left $R$-module $M$ is said to be maximal submodule of $M$ if $N \neq M$ and for every left submodule $B$ of $M$ with $N \subsetneq B \subseteq M$, then $B=M$.

Definition (1-17): [3] Let $M$ be a left $R$-module the Jacobson radical of $M$ is denoted by $\mathrm{J}(M)$ and defined as the intersection of all maximal left submodules of $M$. If $M$ has no maximal left submodules, then we set $J(M)=M$.

Definition(1-18): [3] A submodule $N$ of a left $R$-module $M$ is said to be small in $M$ and denoted by $N \subseteq{ }^{o} M$ if for all submodule $B$ of $M$ with $N+B=M$, then
$B=M$. A left ideal $I$ of a ring $R$ is said to be small in $R$ if $I$ is a small left submodule in a left $R$-module $R$.

Notation: We will denote to any small and simple submodule $N$ of a module $M$ by $N \subseteq \subseteq^{\text {s.s.s. }} M$.

## Examples (1-19):

1) For every $R$-module $M$ we have 0 is a small submodule in $M$.
2) Let $M$ be any non-zero semisimple left $R$-module. Then 0 is the only small submodule in $M$.
3) The only small submodule in $Z_{6}$ as $\mathbb{Z}$-module is 0 .
4) The only small submodules in $Z_{4}$ as $\mathbb{Z}$-module are 0 and $<2>$.
5) Every finitely generated submodule of a $\mathbb{Z}$-module $\mathbb{Q}$ is small.

Definition (1-20): [3] A submodule $N$ of a left $R$-module $M$ is said to be essential in $M$ and denoted by $N \subseteq^{e} M$ if for all submodules, $B$ of $M$ with $N \cap B=0$, then $B=0$.

## Examples (1-21):

1)For every left $R$-module $M$, we have $M \subseteq^{e} M$.
2) Let $M$ be any non- zero semisimple left $R$-module. Then $M$ is the only essential submodule in $M$.
3) The only essential submodule in $Z_{6}$ as $\mathbb{Z}$-module is $Z_{6}$.
4) The only essential submodules in $Z_{4}$ as $\mathbb{Z}$-module are $<2>$ and $Z_{4}$.

## Section Two : On S-essential Submodules

Definition (2-1): [5] A submodule $N$ of a left $R$-module $M$ is said to be s-essential (and denoted by $N \subseteq \subseteq^{\text {s.e. }} M$ ) if for any small submodule $X$ of $M$ with $N \cap X=0$, then $X=0$.

Preposition (2-2): Every submodule of a semisimple left $R$-module is s-essential.

Proof. Let $M$ be a semisimple left- $R$ module and let $N$ be a submodule in $M$. Let $X$ be a small submodule in $M$, with $N \cap X=0$. Since $M$ is semisimple, 0 is the only small submodule in $M$ (by Example (1-21) (2) ) and hence $X=0$. Thus $N$ is an s-essential submodule in $M$.

Example (2-3): Let $M=\mathrm{Z}_{15}$ as $\mathbb{Z}$-module. Since $M$ is a semisimple Z-module, it follows Proposition (2-2) implies that all submodules of $M$ are s-essential and hence the submodules $0,\langle 3\rangle,<5\rangle$ and $\mathrm{Z}_{15}$ are s-essential in $M=\mathrm{Z}_{15}$ as $\mathbb{Z}$-module.

Example (2-4): Let $M=\mathrm{Z}_{42}$ as $\mathbb{Z}$-module. Since $M$ is a semisimple Z-module, it follows Proposition (2-2) implies that all submodules of $M$ are s-essential. Thus the submodules $0,<2\rangle,\langle 3\rangle,<7\rangle,<6\rangle$ ,$<14>,<21>$ and $Z_{42}$ are s-essential in $\mathrm{Z}_{42}$ as $\mathbb{Z}$-module.

Preposition (2-5): [5] Every essential submodule of a left $R$-module is s-essential.

Proof. Let $M$ be a left $R$-module and let $N$ be any essential submodule in $M$. Let $X$ be any small submodul in $M$ such that $\mathrm{N} \cap X=0$. Since $N$ is essential in $M$, it follows that $X=0$ and hence $N$ is s-essential in $M$.

Remark (2-6) : The convers of Proposition (2-5) is not true in general, for example in $\mathrm{Z}_{15}$ as $\mathbb{Z}$-module, we show that $<3>$ is s-essential in $\mathrm{Z}_{15}$ but it is not essential in $\mathrm{Z}_{15}$.

Example (2.7): Let $M=\mathrm{Z}_{4}$ as $\mathbb{Z}$-module. Since $<2>$ and $\mathrm{Z}_{4}$ are essential submodule in $M$, it follows from Proposition (2-5) that $<2>$, $\mathrm{Z}_{4}$ are s-essential in $\mathrm{Z}_{4}$ as $\mathbb{Z}$-module in $\mathrm{Z}_{4}$. In other hand, $\langle 0\rangle$ is not s-essentail in $Z_{4}$. Let $X=<2>$, it is clear that $X$ is small in $Z_{4}$, and $<0>\cap X=0$, but $X \neq 0$ Hence $<0>$ is not s-essential in $Z_{4}$ as $\mathbb{Z}$-module. The following proposition gives an equivalent statement of an s-essential submodule.

Proposition (2-8): [5] A non-zero submodule $L$ of a left $R$-module $N$ is s-essential if and only if any $0 \neq a \in \mathrm{~N}$, if $R a$ is a small submodule in $N$, then there is $\mathrm{r} \in R$ such that $0 \neq r a \in L$.

Proof. ( $\Rightarrow$ ) Suppose that $L$ is an s-essential submodule of $N$. Let $0 \neq a \in \mathrm{~N}$ with $R a$ is a small submodule in $N$ and $R a \neq 0$. By hypothesis, $L \cap R a \neq 0$ and hence there is $\mathrm{r} \in R$ such that $r a \neq 0$ and $r a \in L$.
$(\Leftarrow)$ Let $X$ be a non-zero small submodule of $N$.
We will prove that $L \cap X \neq 0$. Since $X \neq(0)$, there is $0 \neq \mathrm{a} \in X$ and $R a \subseteq \mathrm{X} \subseteq N$. Since $X$ is a small submodule in $N$, it follows from
[ 3, Lemma 5.1.3 (a), p. 108 ] that $R a$ is a small submodule in $N$. By hypothesis, there is $\mathrm{r} \in R$ such that $0 \neq \mathrm{ra} \in L$. Since $r a \in X$, it follows that $L \cap X \neq 0$ and hence $L$ is an s-essential submodule in $N$.

Proposition (2-9): [5] Let $A, B$ and $N$ be submodules of an $R$-module $M$ such that $A \hookrightarrow B \hookrightarrow N \hookrightarrow M$. If $A$ is an s-essential in $M$, then $B$ is an s-essential in $N$.

Proof. Suppose that $A$ is an s-essential submodule in $M$.
Let $X \hookrightarrow N$ such that $B \cap X=0$. Since $N \subseteq M$, it follows that $N \cap X=0$
Since $A$ is an s-essential in $N, X=0$ and hence $B$ is an s-essential in $N$.
Corollary (2-10): Let $K \subseteq N$ be submodules of a module $M$. If $K$ is an s-essential in $M$, then:

1) $K$ is an s-essential in $N$;
2) $N$ is an s- essential in $M$.

Proof. Suppose that $K \subseteq^{\text {s.e. }} M$

1) By taking $A=B=K$ and applying Proposition (2-9).
2) Consider the following sequence:
$K \subseteq N \subseteq M \subseteq M$.
Since $K \subseteq^{\text {s.e. }} M$, it follows from Proposition (2-9) that $\mathrm{N} \subseteq{ }^{\text {s.e. }} M$.
Proposition (2-11): Let $\left\{N_{i}\right\} i=1,2, \ldots, n$ be a family of submodules of a left $R$-module $M$. Then $\cap_{i=1}^{n} N_{i}$ is an s-essential submodule in $M$ if and only if $N_{i}$ is an s -essential submodule in $M$, for each $i=1,2, \ldots, n$.

Proof. ( $\Rightarrow$ ) Suppose that $\cap_{i=1}^{n} N_{i}$ is an s-essential submodule in $M$. Since $\cap_{i=1}^{n} N_{i} \subseteq N_{j} \subseteq M \quad \forall j=1,2, \ldots, n$, it follows from Corollary (2-10) that $N_{j}$ is an s-essential in $M, \forall \mathrm{j}=1,2, \ldots, n$.
( $\Longleftarrow$ ) Suppose that $N_{i}$ is an s-essential submodule in $M$, for each $i=1,2, \ldots, n$.

We will prove by induction on $n$. For $n=1$ the statement holds by hypothesis. Let $N=\cap_{i=1}^{n} N_{i}$ be an s-essential submodule in $M$. We will prove that $\cap_{i=1}^{n} N_{i}$ is an s-essential in $M$. Let $B$ be a small submodule of $M$ such that $\left(\cap_{i=1}^{n} N_{i}\right) \cap B=0$.
Thus $\left(\cap_{i=1}^{n-1} N_{i}\right) \cap\left(N_{n} \cap B\right)=0$. Since $B \subseteq^{0} M$, it follows from [ 3, lemma 5.1.3 (a), p. 108 ] that $N_{\mathrm{n}} \cap B \subseteq^{\circ} M$.

Since $\bigcap_{i=1}^{n-1} N_{i}$ is an s-essential in $M$ (by assumption), $N_{\mathrm{n}} \cap B=0$. Since $N_{\mathrm{n}} \subseteq^{\text {s.e. }} M, B=0$ and hence $\bigcap_{i=1}^{n} N_{i}$ is an s-essential in $M$.

Corollary (2-12): [5, Proposition (2-7) (1-b), p. 1054] Let $N_{1}$ and $N_{2}$ be two submodules of a module $M$. Then $N_{1} \cap N_{2} \subseteq^{\text {s.e. }} M$ if and only if $N_{1} \subseteq^{\text {s.e. }} M$ and $N_{2} \cong^{\text {s.e. }} M$.

Proof. By taking $\mathrm{n}=2$ and applying Proposition (2-11).

The following result appear in [5,p.1054] but without proof.

Proposition (2-13): Let $f: M \rightarrow N$ be an $R$-homomorphism. If $K \subseteq^{\text {s.e. }} N$, then $f^{-1}(K) \subseteq^{\text {s.e. }} M$.

Proof. Suppose that $K \subseteq^{\text {s.e. }} N$.
Let $X \subseteq^{0} M$ such that $f^{-1}(K) \cap X=0$. By [3, Exercise 3(b), p.78], $f\left(f^{-1}(K) \cap X\right)=f\left(f^{-1}(K)\right) \cap f(X) \quad$ and hence $f\left(f^{-1}(K)\right) \cap f(X)=0$.

Since $K \subseteq f\left(f^{-1}(K)\right.$ [3, Lemma 3.1.8, p.44] it follows that $K \cap f(X)=0$.
Since $X \subseteq^{0} M, f(\mathrm{X}) \subseteq^{0} N$ (by [3, Lemma 5.2.3 (c), p.10].
Since $\mathrm{X} \subseteq^{\text {s.e. }} M$ (by hypothesis ), $f(X)=0$ and hence
$X \subseteq \operatorname{ker}(f) \subseteq f^{-1}(K)$.
Thus $X=f^{-1}(K) \cap X=0$ and hence $f^{-1}(K) \subseteq{ }^{\text {s.e. }} M$.
The following proposition was stated in [5, p. 1054] but without proof.
Proposition (2-14): Let $K_{1} \subseteq^{\text {s.e. }} N_{1}$ and $K_{2} \subseteq^{\text {s.e. }} N_{2}$ be submodules of a module $M$.

Then $K_{1} \oplus K_{2} \subseteq^{\text {s.e. }} N_{1} \oplus N_{2}$ if and only if $K_{1} \subseteq{ }^{\text {s.e. }} N_{1}$ and $K_{2} \subseteq^{\text {s.e }} N_{2}$.
Proof. ( $\Leftarrow)$ Suppose that $\quad K_{1} \subseteq^{\text {s.e. }} N_{1} \quad$ and $\quad K_{2} \simeq^{\text {s.e. }} N_{2}$.
Let $\quad \pi_{N_{1}}: N_{1} \oplus N_{2} \rightarrow N_{1}$ and $\pi_{N_{2}}: N_{1} \oplus N_{2} \rightarrow N_{2}$ be the projection epimorphism. By Proposition (2-13), $\quad \pi_{N_{1}}^{-1}\left(K_{1}\right) \subseteq^{\text {s.e. }} N_{1} \oplus N_{2} \quad$ and $\pi_{N_{2}}^{-1}\left(K_{2}\right) \subseteq^{\text {s.e. }} N_{1} \oplus N_{2}$. Since $\pi_{N_{1}}^{-1}\left(K_{l}\right)=K_{l} \oplus N_{2}$ and $\pi_{N_{2}}^{-1}\left(K_{2}\right) N_{l} \oplus K_{2}$, it follows that $K_{1} \oplus N_{2} \subseteq^{\text {s.e. }} N_{1} \oplus N_{2}$ and $\quad N_{1} \oplus K_{2} \subseteq^{\text {s.e. }} N_{1} \oplus N_{2}$.

It is clear that $K_{1} \oplus K_{2}=\left(K_{1} \oplus N_{2}\right) \cap\left(N_{1} \oplus K_{2}\right)$.

Since $\quad\left(K_{l} \oplus N_{2}\right) \cap\left(N_{l} \oplus K_{2}\right) \subseteq^{\text {s.e. }} N_{l} \oplus N_{2} \quad($ by Corollary (2-12)), it follows that $K_{l} \oplus K_{2} \subseteq^{\text {s.e. }} N_{l} \oplus N_{2}$.
$(\Rightarrow)$ Let $i_{N_{1}}: N_{l} \rightarrow N_{l} \oplus N_{2}$ and $i_{N_{2}}: N_{2} \rightarrow N_{l} \oplus N_{2}$ be the injection $R$-monomorphisms. Since $K_{l} \oplus K_{2} \cong^{\text {s.e. }} N_{l} \oplus N_{2}$ ( by hypothesis ), it follows from Proposition (2-13). That $i_{N_{1}}^{-1}\left(K_{1} \oplus K_{2}\right) \subseteq^{\text {s.e. }} N_{1} \quad$ and $i_{N_{2}}^{-1}\left(K_{1} \oplus K_{2}\right) \subseteq^{\text {s.e. }} N_{2}$. Since $i_{N_{1}}^{-1}\left(K_{1} \oplus K_{2}\right)=K_{1}$ and $i_{N_{2}}^{-1}\left(K_{1} \oplus K_{2}\right)=K_{2}$, then $K_{1} \subseteq^{\text {s.e. }} N_{1}$ and $K_{2} \simeq^{\text {s.e. }} N_{2}$.

Corollary (2-15): Let $K_{i} \hookrightarrow N_{i}$ be submodules of a module $M$, $\forall i=1,2,3, \ldots, n$. Then $\oplus_{i=1}^{n} K_{i} \subseteq^{\text {s.e. }} \oplus_{i=1}^{n} N_{i}$ if and only if $K_{i} \subseteq^{\text {s.e. }} N_{i}$, $\forall_{i}=1,2, \ldots, n$.

Proof. ( $\Rightarrow$ ) Suppose that $\oplus_{i=1}^{n} K_{i} \subseteq^{\text {s.e. }} \oplus_{i=1}^{n} N_{i}$. By Proposition (2-14), $K_{l} \subseteq^{\text {s.e. }} N_{l}$ and $\oplus_{i=2}^{n} K_{i} \subseteq^{\text {s.e. }} \oplus_{i=2}^{n} N i$. By applying Proposition (2-14) again ( n -times ), we get that $K_{i} \subseteq^{\text {s.e. }} N_{i}, \forall i=1,2, \ldots, n$.
$(\Longleftarrow)$ Suppose that $\quad K_{i} \subseteq^{\text {s.e. }} N_{i}, \forall i=1,2, \ldots, n$. By Proposition (2-14), $K_{l} \oplus K_{2} \subseteq^{\text {s.e. }} N_{l} \oplus N_{2}$.

By applying Proposition (2-14) (n- times), we get $\oplus_{i=1}^{n} K_{i} \subseteq^{\text {s.e. }} \oplus_{i=1}^{n} N_{i}$.
Corollary (2-16): Let $M=\oplus_{i=1}^{n} M_{i}$ and let $K \hookrightarrow M$. Then the following statements are equivalent:

1) $K \cap M_{i} \subseteq^{\text {s.e. }} M_{i} \forall i=1,2, \ldots, n$.
2) $\oplus_{i=1}^{n}\left(K \cap M_{i}\right) \subseteq^{\text {s.e. }} M$.
3) $K \varrho^{\text {s.e. }} M$.

## Proof.

( $1 \Rightarrow 2$ ). By Corollary (2-15).
$(2 \Rightarrow 3)$. Since $K \cap M_{i} \subseteq^{\text {s.e. }} K$, it follows that $\oplus_{i=1}^{n}\left(K \cap M_{i}\right)$.
Since $\oplus_{i=1}^{n}(K \cap M i) \subseteq^{\text {s.e. }} M$ (by hypothesis ),
it follows from Corollary (2-10) that $K \subseteq \subseteq^{\text {s.e. }} M$.
$(3 \Rightarrow 1)$. Let $0 \neq m_{i} \in M_{i}, \forall i=1,2, \ldots, n$ with $R m_{i} \subseteq^{o} M_{i}$.
Thus $R m_{i} \subseteq^{o} M$ ( by [3, 5.1.3, p. 108 ]).
Since $B \subseteq^{\text {s.e }} M$ (by hypothesis), there is $\mathrm{r} \in R \ni 0 \neq r m \in B$ (by Proposition (2-8)). Since $r m_{i} \in M_{i}$, thus $r m_{i} \in \mathrm{~B} \cap M_{i}, \forall i=1,2, \ldots, n$. By Proposition (2-8), $B \cap M_{i} \subseteq^{\text {s.e. }} M_{i}, \forall i=1,2, \ldots, n$.

Definition (2-17): (See [5]) Let $M$ be a left $R$-module.
Define $\operatorname{Soc}_{\mathrm{s}}(M)=\Sigma\{N \hookrightarrow M \mid N$ is small and simple submodule of $M$, if $M$ has a small simple submodule.

If $M$ has no small simple submodule, then define $\operatorname{Soc}_{\mathrm{s}}(M)=0$.
Theorem (2-18) : ( See [5]) Let $M$ be left $R$-module.
Then $\operatorname{Soc}_{\mathrm{s}}(M)=\cap\left\{N \hookrightarrow M \mid N \subseteq^{\text {s.e. }} M\right\}$.
Proof. Let $S$ be any small simple submodule of $M$ and let $K$ be any s-essential submodule in $M$ ( $i$.e., $K \subseteq \subseteq^{\text {s.e. }} M$ ). Since $S$ is simple, $S \neq 0$. Since $K \subseteq^{\text {s.e. }} M$, it follows that $K \cap S \neq 0$ ( since $S \subseteq^{0} M$ ).

Since $S$ is simple and $K \cap S \subseteq S$, thus $K \cap S=S$.
Since $K \cap S \subseteq K$, it follows $S \subseteq K$.
Hence $S \subseteq \cap\left\{K \mid K \subseteq^{\text {s.e. }} M\right\}$, for any small simple submodule $S$ of $M$.

By Definition (2-17) , $\operatorname{Soc}_{\mathrm{s}}(M) \subseteq \cap\left\{K \mid K \subseteq^{\text {s.e. }} M\right\}$.
Conversely, since $\cap\left\{N \hookrightarrow M \mid N \subseteq^{\text {s.e. }} M\right\} \subseteq \cap\left\{N \mid N \subseteq^{\mathrm{e}} M\right\}$ and since $\operatorname{Soc}(M)=\cap\left\{N \hookrightarrow M \mid N \subseteq^{\mathrm{e}} M\right\} \quad$ (by [3, Theorem 9.1.1 (b), p . 213 ], thus $\cap\left\{N \hookrightarrow M \mid N \subseteq^{\text {s.e. }} M\right\} \subseteq \operatorname{Soc}(M) \quad$ and $\quad$ hence $\cap\left\{M \hookrightarrow M \mid N \subseteq^{\text {s.e. }} M\right\} \quad$ and $\quad \operatorname{Soc}(M) \quad$ are $\quad$ semisimple modules. Assume that $\cap\left\{N \subseteq^{\text {s.e. }} M\right\} \neq \operatorname{Soc}_{\mathrm{s}}(M)$. Thus
$\cap\left\{N \subseteq^{\text {s.e. }} M\right\} \nsubseteq \operatorname{Soc}_{\mathrm{s}}(M)$ and hence there is a simple submodule $B$ such that $B \subseteq \cap\left\{N \subseteq^{\text {s.e. }} M\right)$ and $B$ is not small in $M$.

Let $C$ be a proper submodule of $M$ such that $B+C=M$.
i) If $B \cap C \neq 0$, then $B \cap C=B$ ( since $B$ is simple module ) and hence $B \subseteq C$. Thus $C=M$ and this is a contradiction.
ii) If $B \cap C=0$, then $M=B \oplus C$.

We will prove that $C \subseteq^{\text {s.e. }} M$. Let $H \subseteq^{\circ} M$ such that $C \cap H=0$, then $H+C \neq M($ since $C \neq M)$. Since $H \cap C=0$, thus $\frac{H \oplus C}{C} \cong H$.

Since $\frac{H \oplus C}{C} \underset{\neq}{\neq}$, it follows that $H_{\neq}^{\hookrightarrow} \frac{M}{C}$. Since $\frac{M}{C}=\frac{H \oplus C}{C} \cong B$ and $B$ is simple, thus $\frac{M}{C}$ is simple $R$-module and hence $H=0$.

Thus $C \subseteq^{\text {s.e. }} M$ and hence $B \subseteq \cap\left\{N \subseteq^{\text {s.e. }} M\right\} \subseteq C$. Since $B+C=M$, then $C=M$ and this is a contradiction from i) and ii), we have that $B \cap C=0$ and $B \cap C \neq 0$ and this is a contradiction.

Thus $\cap\left\{N \subseteq \subseteq^{\text {s.e. }} M\right\}=\operatorname{Soc}_{\mathrm{s}}(M)$.

Corollary (2-19): If $M$ is a semisimple $R$-module, then $\operatorname{Soc}_{\mathrm{s}}(M)=0$.
Proof. Suppose that $M$ is a semisimple $R$-module. By Propsition (2-2), 0 is s-essential submodule of $M$. Since $\operatorname{Soc}_{\mathrm{s}}(M)=\bigcap\left\{N \subseteq^{\text {s.e. }} M\right\}$ (by Theorem (2-18) ), it follows that $\operatorname{Soc}_{\mathrm{s}}(M)=0$.

Remark (2-20) : For any left $R$-module $M$ we have $\operatorname{Soc}_{\mathrm{s}}(M) \subseteq \operatorname{Soc}(M)$ (by Theorem (2-18)). The other inclusion is not true in general, for example:
if $M=Z_{6}$ as $\mathbb{Z}$-module, then from Corollary (2-19) we have that $\operatorname{Soc}_{\mathrm{s}}(M)=0 \subsetneq \operatorname{Soc}(M)=Z_{6}$.

## Examples (2-21) :

1) $\operatorname{Soc}_{\mathrm{s}}\left(\mathbb{Z}_{z}\right)=0$ and this from Remark (2-20) (because Soc $\left(\mathbb{Z}_{z}\right)=0$.
2) $\operatorname{Soc}_{\mathrm{s}}\left(Z_{4}\right)=<2>$ (by theorem (2-18) and Example (2-7) ).

Lemma (2-22): Let $\propto: M \rightarrow N$ be a left $R$-homomorphism.
If $S$ is a simple submodule of $M$, then either $\propto(S)=0$ or $\propto(S)$ is a simple submodule of $N$.

Proof. Suppose that $\propto(S) \neq 0$. We will prove that $\propto(S)$ is a simple submodule of $N$. Assume that $\propto(S)$ is not simple submodule of N , thus there is a submodule B of $N$ such that $0 \underset{\neq}{\hookrightarrow} B \underset{\neq}{\hookrightarrow} \propto(S)$.

Define $\quad f: S \rightarrow \propto(S) \quad$ by $\mathrm{f}(x)=\propto(x), \quad \forall x \ni S$. It is clear that $f$ is an epimorphism (i.e. $f(S)=\propto(S)$ ).

Thus $f\left(f^{-1}(B)\right)=B$. Since $S$ is simple, either $f^{-1}(B)=0$ or $f^{-1}(B)=S$.

1) If $f^{-1}(B)=0$, then $B=f\left(f^{-1}(B)\right)=f(0)=0$ and this is contradiction.
2) If $f^{-1}(B)=S$, then $B=f\left(f^{-1}(B)\right)=f(S)=\propto(S)$ and this is contradiction.

Proposition (2-23): [5] Let $\propto: M \rightarrow N$ be a left $R$-homomorphism. Then $\propto\left(\operatorname{Soc}_{\mathrm{s}}(M)\right) \subseteq \operatorname{Soc}_{\mathrm{s}}(N)$.

Proof. $\propto\left(\operatorname{Soc}_{\mathrm{s}}(M)\right)=\propto\left(\sum_{\subseteq}{ }_{\subseteq}\right.$ s.s. $\left.A\right) \quad($ by $\operatorname{Definition~(2-17)})$ $\propto\left(\operatorname{Soc}_{\mathrm{s}}(M)\right)=\left(\sum_{\mathrm{A} \subseteq^{\text {s.s. }}} \propto(\mathrm{A})\right) . \quad$ Since $\quad$ (if $A \subseteq^{\text {s.s. }} M$, then from Lemma (2-22) and [3, Lemma 5.1.3(c)] we have either $\propto(A)=0$ or $\left.\propto(A) \subseteq^{\text {s.S. }} N\right)$, it follows that
$\sum_{A \subseteq \text { s.s. } M} \propto(A) \subseteq \propto \alpha\left(\sum_{A \subseteq^{\text {s.s. }} M} B\right)=\operatorname{Soc}_{\mathrm{s}}(N)$.

Therefore, $\propto\left(\operatorname{Soc}_{\mathrm{s}}(M)\right) \subseteq \operatorname{Soc}_{\mathrm{s}}(N)$.

Corollary (2-24) : If $N \hookrightarrow M$, then $\operatorname{Soc}_{\mathrm{s}}(N) \subseteq \operatorname{Soc}_{\mathrm{s}}(M)$.

Proof. Let $i: N \hookrightarrow M$, be the inclusion $R$-homomorphism.
By Proposition (2-23), $i\left(\operatorname{Soc}_{\mathrm{s}}(N) \subseteq \operatorname{Soc}_{\mathrm{s}}(M)\right.$.

Thus $\operatorname{Soc}_{\mathrm{s}}(N) \subseteq \operatorname{Soc}_{\mathrm{s}}(M)$.

Proposition (2-25): $\quad \operatorname{Soc}_{\mathrm{s}}\left(\operatorname{Soc}_{\mathrm{s}}(\boldsymbol{M})\right)=0, \quad$ for any left $R$-module $M$. Proof. Since $\operatorname{Soc}_{\mathrm{s}}(\mathrm{M}) \subseteq \operatorname{Soc}(\mathrm{M}) \quad$ (by Remark (2-20)) and $\operatorname{Soc}(\mathrm{M})$ is semisimple module, it follows that $\operatorname{Soc}_{s}(\mathrm{M})$ is semisimple module. By Corollary (2-19), $\operatorname{Soc}_{s}\left(\operatorname{Soc}_{s}(M)\right)=0$.

Remark (2-26): In general, it is not true that $\operatorname{Soc}_{\mathrm{s}}\left(\operatorname{Soc}_{\mathrm{s}}(M)=\operatorname{Soc}_{\mathrm{s}}(M)\right.$, for example, if $M=Z_{4}$ as $\mathbb{Z}$-module, then
$\operatorname{Soc}_{s}\left(\operatorname{Soc}_{\mathrm{s}}(M)\right)=0 \neq \operatorname{Soc}_{\mathrm{s}}(M)=<2>$.
Remark (2.27): If $N \hookrightarrow M, \quad$ then it is not necessary that $\operatorname{Soc}_{\mathrm{s}}(N)=N \cap \operatorname{Soc}_{\mathrm{s}}(M)$.

For example:
Let $M=Z_{4}$ as $\mathbb{Z}$-module and $\left.N=<2\right\rangle$, then $\operatorname{Soc}_{s}(N)=0($ since $N$ is semisimple $\mathbb{Z}$-module ), but $N \cap \operatorname{Soc}_{\mathrm{s}}(M)=\langle 2\rangle \cap\langle 2\rangle=\langle 2\rangle$.

## References:

[1] N. kh. Abdullah, strong essential submodules and strong uniform modules, Tikrit J of pure science 21 (1) (2016), pp.112-117.
[2] Sh. Asgari and A. Haghany, t-Extending modules and t-Baer
Modules, Comm. Algebra, 39 (2011), pp. 1605-1623.
[3] F. Kasch, Modules and rings, Academic Press, London, New York, 1982.
[4] T. Y. Lam, Lectures on Modules and Rings, Springer-Verlag, New York, 1999.
[5] D. X. Zhou and X. R. Zhang, small- essential submodules and Morita duality, Southeast Asian Bull. Math., 35 (2011), pp. 1051-1062.

