Republic of Iraq
Ministry of higher Education and
Scientific Research
Al-Qadisiyah University
College of education
Deprt. Of Mathematics

On Disconnected and connected space

A research

Submitted to the department of education Al- Qadisiyah university as a partial fulfillment requirement for certificate the Bachelor of a science mathematics

By

Muntather Ghaneem Gaber

Supervisor

Ass. L. Firas Joad Obead

2018 A.c

بِسْمِ اللَّهِ الرَّحْمَنِ الرَّحِيمِ

﴿وَوَصَّيْنَا الْإِنسَانَ بِوَالِدَيْهِ إِحْسَانًا حَمَلَتْهُ أُمُّهُ كُرْهًا وَوَضَعَتْهُ كُرْهًا وَحَمْلُهُ وَفِصَالُهُ ثَلَاثُونَ شَهْرًا حَتَّى إِذَا بَلَغَ أَشُدَّهُ وَبَلَغَ أَرْبَعِينَ سَنَةً قَالَ رَبِّ أَوْزِعْنِي أَنْ أَشْكُرَ نِعْمَتَكَ الَّتِي أَنْعَمْتَ عَلَيَّ وَعَلَى وَالِدَيَّ وَأَنْ أَعْمَلَ صَالِحًا تَرْضَاهُ وَأَصْلِحْ لِي فِي ذُرِّيَّتِي إِنِّي تُبْتُ إِلَيْكَ وَإِنِّي مِنَ الْمُسْلِمِينَ﴾

صدق الله العلي العظيم

سورة الاحقاف الاية: ١٥

Dedication

To whom Allah sent as mercy to the worlds….

To the prophet Mohammed ………..

To my family………

To everyone I love.……

ACKNOWLEDGMENT

It is a will of Allah, the merciful god, who surround me with brilliant and supportive people. I would like to express my graduate, my great respect and appreciation to my supervisor Ass. L. Firas Joad Obead Alysaary, for his help and support. Thanks must go to my dear mother.

Abstract

We have discussed the basic properties of connected spaces regarding subspaces, product spaces, preservation under mappings etc. Also we have given several characterizations of these spaces. We begin our research paper of topological properties by making the idea of being connected that is being in one piece. It turns out to be easier to think about the property that is opposite of connectedness, namely the property of being in two or more pieces.

Keywords: Topological, Connected, disconnected.

INTRODUCTION

Two important and interrelated strands in the practice of the exact sciences in the 19th century will be considered in which topological ideas came to be relevant for natural philosophy. In this way, light can be thrown on a part of the causal weave of events that eventually led to the emergence-of topology as a discipline, a part which has largely been neglected up until now in-the historical literature. The first of these two strands was concerned with topological issues that arose in the context of a dynamical theory of physical phenomena, a theory-advocated in particular by British natural philosophers during the last third of the 19th century. These developments will be discussed in the first part of our study. The second strand of events related to speculations about the large-scale topological structure of space will be the focus of the second part of this article.

The emergence of an entirely new discipline within mathematics is a rare event-in the history of science. The creation of topology the science of properties of spaces-and figures that remain unchanged under continuous deformations represents a phenomenon of this kind, but of a distinctly modern variety. Topology bears comparison-with the calculus, probability theory or number theory in that the first ideas about a new field called Analysis Situs or Geometria Situs were communicated among a handful of mathematically minded intellectuals in the late seventeenth and early eighteenth centuries. However, unlike the calculus and number theory, but similar to probability theory, the basic ideas underlying Analysis Situs reveal no ancient roots. Notoriously, ancient-authors treated questions of continuity hardly at all, and if so, then mainly as physical-questions linked to the phenomenon of motion. Moreover, in sharp contrast to these three other fields, during the 18th century no clearly defined domain of mathematical-problems was delineated that should and could be treated by Analysis Situs. Rather, a vague idea about an analysis which dealt not with magnitude, but “position,” left it to-individual mathematicians to decide what should belong to the new field. Only gradually over the course of the 19th century was a consensus reached about the nature of-problems in topology. Nevertheless, after crossing the threshold to a scientific discipline in the full sense of the word in the first decades of this century, topology became one of the core research fields of mathematics, and topological arguments have come to play a role in virtually every other field in mathematics and the mathematical sciences. If one may reasonably speak of genuinely modern mathematical disciplines, then topology-certainly belongs among them. These late beginnings may be one reason why the emergence of topology has only begun to attract historiographical attention comparable to that received by fields like the calculus, number theory, or probability theory. While the invention of the calculus has long since been the object of historical study, and while the emergence of number theory and probability theory have recently been treated from a wide variety of perspectives, the number of historical monographs devoted to the formation of topology remains very small. Apart from these, we have a few survey articles and several research papers-dealing with particular topics within or closely related to topology.

1.1 Basic of Topological space.

1.1.1Definition :

Let X be a set. A topology on X is a collection $T \subseteq P(X)$ of subsets of X satisfying

1. T contains $\emptyset$ and X;
2. T is closed under arbitrary unions, i.e. if $U_i \in T$ for $i \in I$ then $\bigcup_{i \in I} U_i \in T$;
3. T is closed under finite intersections, i.e. if $U_1, U_2 \in T$ then $U_1 \cap U_2 \in T$.

1.1.2 Definition :

A topological space (X, T) is a set X together with a topology T on it. The elements of T are called open subsets of X. A subset $F \subseteq X$ is called closed if its complement $X \setminus F$ is open. A subset N containing a point $x \in X$ is called a neighborhoods of x if there exists U open with $x \in U \subseteq N$. Thus an open neighbourhood of x is simply an open subset containg x.

Normally we denote the topological space by X instead of (X, T).

1.1.3 Definition :

Let $A \subseteq X$ be a subset of a topological space X. The interior of A is the biggest open subset contained in A. One has $A^\circ = \bigcup_{A \supseteq U}$ open U. Dually the closure of A is the smallest closed subset containing A. One has
$\overline{A} = \bigcap_{A \subseteq F}$ closed F.

1.1.4 Example:

Consider the following set consisting of 3 points; $X = \{a, b, c\}$ and determine if the set $T = \{\emptyset, X, \{a\}, \{b\}\}$ satisfies the requirements for a topology.

This is, in fact, not a topology because the union of the two sets $\{a\}$ and $\{b\}$ is the set $\{a, b\}$, which is not in the set τ

1.1.5 Example:

Find all possible topologies on $X = \{a, b\}$

1. $\emptyset, \{a, b\}$
2. $\emptyset, \{a\}, \{a, b\}$
3. $\emptyset, \{b\}, \{a, b\}$
4. $\emptyset, \{a\},\{b\},\{a,b\}$

1.1.6 Example:

When X is a set and τ is a topology on X, we say that the sets in τ are open. Therefore, if X does have a metric (a notion of distance), then ^{T} ={all open sets as defined with the ball above} is indeed a topology. We call this topology the Euclidean topology. It is also referred to as the usual or ordinary topology.

1.1.7 Example:

If $Y \subseteq X$ and τx is a topology on X, one can define the Induced topology as $\tau y = \{O \cap Y \mid O \in \tau x\}$.

This last example gives one reason why we must only take finitely many intersections when defining a topology.

1.1.8 Remark:

As promised, we can now generalize our definition for a closed set to one in terms of open sets alone which removes the need for limit points and metrics

1.1.9 Definition:

A set C is closed if $X - C$ is open.

Now that we have a new definition of a closed set, we can prove what used to be definition 1.3.3 as a theorem: *A* set *C* is a closed set if and only if it contains all of its limit points.

Proof: Suppose a set *A* is closed. If it has no limit points, there is nothing to check as it trivially contains its limit points. Now suppose *z* is a limit point of *A*. Then if $z \in A$, it contains this limit point. So suppose for the sake of contradiction that *z* is a limit point and *z* is not in *A*. Now we have assumed *A* was closed, so its complement is open. Since *z* is not in *A*, it is in the complement of *A*, which is open; which means there is an open set *U* containing *z* contained in the complement of *A*. This contradicts that z is a limit point because a limit point is, by definition, a point such that every open set about z meets *A*

Conversely: if *A* contains all its limit points, then its complement is open. Suppose *x* is in the complement of *A*. Then it can not be a limit point (by the assumption that *A* contains all of its limit points). So *x* is not a limit point which means we can find some open set around *x* that doesn't meet *A*. This proves the complement is open, i.e. every point in the complement has an open set around it that avoids *A* .

1.1.10 Remark:

Since we know the empty set is open, *X* must be closed.

1.1.11 Remark:

Since we know that *X* is open, the empty set must be closed.

Therefore, both the empty set and X and open and closed.

1.1.12 Example :

When X is a set and τ is a topology on X, we say that the sets in τ are open. Therefore, if X does have a metric (a notion of distance), then τ ={all open sets as defined with the ball above} is indeed a topology. We call this topology the Euclidean topology. It is also referred to as the usual or ordinary topology.

1.1.13 Definition:

A subset S of topological space (x, T) is said clopen if it is both open and closed subset of X.

1.1 .Some properties of Topological space.

1.2.1 Continuity

In topology a continuous function is often called a function. There are 2 different ideas we can use on the idea of continuous functions.

Calculus Style

1.2.2 Definition:

$f : R\ n \to R\ m$ is continuous if for every $\in > 0$ there exists $\delta > 0$ such that when $|x - x0| < \delta$ then $|f(x) - f(x0)| < \in$.

The map is continuos if for any small distance in the pre-image an equally small distance is apart in the image. That is to say the image does not jump

Topology Style. In topology it is necessary to generalize down the definition of continuity, because the notion of distance does not always exist or is different than our intuitive idea of distance.

1.2.3 Definition :

A function $f : X \to Y$ is continuous if and only if the pre-image of any open set in Y is open in X. If for whatever reason you prefer closed sets to open sets, you can use the following equivalent definition:

1.2.4 Definition :

A function $f : X \to Y$ is continuous if and only if the pre-image of any closed set in Y is closed in X.

1.2.5 Definition :

Given a point x of X, we call a subset N of X a neighborhood of X if we can find an open set O such that $x \in O \subseteq N$.

1. A function $f : X \to Y$ is continuous if for any neighborhood V of Y there is a neighborhood U of X such that $f(U) \subseteq V$.

2. A composition of 2 continuous functions is continuous

1.2.6 Definition :

A function $f{:}X \to Y$ between two topological spaces is called continuous if every $U \subseteq Y$ open in Y the inverse image $f^{-1}(U)$ is open in X.

1.2.7 Proposition :

The identity function is continuous. A composition of two continuous maps is continuous. Thus topological spaces and continuous maps between them form a category, the category of topological spaces.

1.2.8 Definition :(Homeomorphisms)

A homeomorphism is a function $f : X \to Y$ between two topological spaces X and Y that

1.is a continuous bijection; and
2.has a continuous inverse function f^{-1} .

Another equivalent definition of homeomorphism is as follows.

1.2.9 Definition :

Two topological spaces X and Y are said to be homeomorphic if there are continuous function $f : X \to Y$ and $g : Y \to X$ such that $f \circ g = I_Y$ and $g \circ f = I_X$.

Moreover, the functions f and g are homeomorphisms and are inverses of each other, so we may write f^{-1} in place of g and g^{-1} in place of f.

Here, I_X and I_Y denote the identity maps .

1.2.10 Definition:

Let $\mathfrak{I}$ and $\mathfrak{I}^*$ be two topologies on a given set X. If $\mathfrak{I}^* \supseteq \mathfrak{I}$ then $\mathfrak{I}$ is coarser than $\mathfrak{I}^*$.

1.2.11 Definition :

a topological space *(X,T)* is said to be completely regular space iff every closed subset *F* of X and every point $x \in X\text{-}F$ there exist a continuous function $f:X\to[0,1]$ such that $f(x)=0$, $f(F)=\{1\}$

1.2.12 Definition :(tychonoff)

a tychonoff space or space is completely regular $T_{1\text{-}space}$

1.2.13 Definition :

Say that a family of sets A is linked if for every $A, B \in A$, $A \cap B = \emptyset$.

1.2.14 Definition :(pathwise)

Let X be a topological space, and $x, y \in X$. A continuous function $p : I \to X$ such that $p(0) = x$ and $p(1) = y$ is called a path from x to y. X is called pathwise.

1.2.15 Definition :

A collection U of open subsets of a topological space X is called an (open) cover if its union is the whole of X, *i.e.* $\bigcup_{i \in I} U = \bigcup_{U_0 \in U} U = X$. A subcollection $U_0 \subseteq U$ is called a sub-cover if it is itself a cover.

1.2.16 Definition :

A topological space X is called compact if every open cover admits a finite sub-cover

1.2.16 Definition :(locally compact)

A topological space is locally compact if every point $x \in X$ has a compact neighborhood.

1.2.17 Example 1.2. Any compact space is locally compact

1.2.18 Definition :

Product topology Given two topological spaces *(X, T)* and *(Y, T '),* we define the product topology on X × Y as the collection of all unions $\Upsilon_i\ U_i \times V_i$, where each Ui is open in X and each Vi is open in Y .

1.2.19 Theorem.

Projection maps are continuous Let *(X, T)* and *(Y, T')* be topological spaces. If $X \times Y$ is equipped with the product topology, then the projection map $p_{1=} : X \times Y \to X$ defined by $p_1(x, y) = x$ is continuous. Moreover, the same is true for the projection map $p_2 : X \times Y \to Y$ defined by $p_2(x, y)= y$ □

REVIEW OF LITERATURE

Connectedness plays a very significant role in the study of topological spaces. The first attempt to give a precise definition of these spaces was made by Weierstrass who in fact introduced the notion of arcwise connectedness. However, the notion of connectedness which we use today was introduced by Cantor (1883). Since then a host of leading toplogists notably Jordan (1893), Schoenfliesz (1902), F. Riesz (1906), Lennes (1911), Mazurkiewicz (1920), Vaidyanathaswamy (1947) studies these spaces very extensively and also introduced various generalization too of these spaces.

2.1 Connected and Disconnected space.

2.1.1 Definition:

Let x be a topological space and Let A and B be any two sets in X. A and B are said to be separated if:

$A \cap \overline{B} = \phi$ *and* $\overline{A} \cap B = \phi$

or $A \cap \overline{B} = \phi$ *and* $\overline{A} \cap B = \phi$

2.1.2 Example:

Let R be the set of real numbers with the usual topology. Consider the sets $A =]1,2[$, $B =]\ 2, 3\ [$ and $C = [2, 3[$ then clearly $A \cap \overline{B} =]1, 2[\cap [2,3] = \phi$ and $\overline{A} \cap B = [1, 2] \cap]2,3[= \phi$ Showing that A and B are separated. However, since $\overline{A} \cap B = [1, 2] \cap 2,3[= \phi$ this shows that A and C are not separated.

2.1.3 Definition:

Let X be a topological space. By a separation of X we mean the existence of a pair of separated subsets of X whose union is X.

2.1.4 Definition:

Let X be a topological space then X is said to be disconnected if there exists two non-empty separated sets A and B such that $X = A \cup B$

2.1.5 Definition:

A topological space X is said to be connected if it is not disconnected .

2.1.6 Example:

We again consider the interval $I = [0, 1]$ and the set $X = [0, 1] \cup [2, 3]$ Here X is disconnected since there exist a separation of X into $[0, 1]$ *and* $[2, 3]$ which are non-empty and disjoint. So X is not connected. While $I = [0, 1]$ has no such separation. So I is not disconnected. Hence $I = [0, 1]$ is connected.

2.1.7 Theorem:

Let X be a topological space then X is disconnected iff X has a non empty proper subset which is both open and closed.

Proof :Let A be a non-empty proper subset of X which is both open and closed. Then $(X - A)$ is also a non-empty proper subset of X which is both open and closed.

So X is the union of two non-empty separated sets, showing that X is disconnected.

Conversely: Let X be a disconnected space then there exists two nonempty separated sets A and B such that $X == AUB$

Since A and B are separated, therefore $A \cap \overline{B} = \phi$ and c

So $\overline{A} \cap B = X,\ A \cap \overline{B} = X$ *and* $A \cap B = \phi$ (i)

Now $A \cup B = X,\ A \cap B = \phi \Rightarrow A = X - B$ (ii)

and $A \cup \overline{B} = X,\ A \cap \overline{B} = \phi \Rightarrow A = X - \overline{B}$ (iii)

also $\overline{A} \cup B = X,\ \overline{A} \cap B = \phi \Rightarrow B = X - \overline{A}$ (iv)

Since A ≠ ϕ, B ≠ ϕ , it follows from (ii) that A is a non-empty proper subset of X and (iii) shows that A is open. (ii) and (iv) both shows that A is closed. Thus X has a non-empty proper subset which is both open and closed.

2.1.8 Theorem:

Let X be a topological space then X is disconnected iff $X == A \cup B$ where A and B are non empty disjoint open sets.

Proof: Let X is disconnected then there exist a nonempty proper subset A of X which is both open and closed then $X - A$ is also a non empty subset of X which is both open and closed. This shows that X is the union of two non-empty disjoint open sets.

Conversely: Let X be the union of two non-empty disjoint open sets A and B, then $X - B = A$. Since B is open this implies A is closed and since B ≠ ϕ this implies A is non empty proper subset of X that is both open and closed. Hence X is disconnected.

2.1.9 Theorem:

Let X be a topological space then X is disconnected iff $X == AUB$ where A and B are non empty disjoint closed sets.

Proof : Let X is disconnected then there exist a non empty proper subset A of X which is both open and closed and $X - A$ is also a non-empty subset of X which is both open and closed this shows that X is union of two non empty disjoint closed set.

Conversely: Let X be the union of two non-empty disjoint closed sets A and B then $X - B = A$. Since B is closed this implies A is open and since $B \neq \phi$ this implies that A is a non empty proper subset of X that is both open and closed. Hence X is disconnected.

2.1.10 Theorem:

Let X be a topological space then X is connected iff the only subsets of X that are both open and closed in X are the empty set and X itself.

Proof: Let X is connected and let A be a non empty proper subset of X which is both open and closed in X. Then the sets A and X – A form a separation of X. Since they are disjoint and nonempty and their union is X. This gives that X is disconnected. Which is a contradiction.

Conversely: Let X be a disconnected space. Let A and B form a separation of X. Then A is non empty and different from X and it is both open and closed in X.

2.1.11 Theorem:

Let X be a topological space. Then X is connected iff one of the following condition hold :

(i) There does not exist a separation of X.

(ii) X can not be decomposed into two disjoint, non empty open sets.

(iii) There does not exist a proper non-empty subset of X which is both open and closed in X.

Proof : Follows from the definition (4), (5) and theorem (10).

2.1.12 Example:

Every indiscrete space is connected. Let X be an indiscrete space. Since empty set and X are the only subset of X which is both open and closed in X. So X is connected.

2.1.13 Example:

Every singleton set is connected .

Let X be a topological space and let $x \in X$. Then $\{x\}$ can not be expressed as the union of two non-empty disjoint sets. So $\{x\}$ has no separation and is therefore connected.

2.1.14 Example:

Every discrete space which contain more than one point is disconnected. Let X be discrete space and let $x \in X$. Then $\{x\}$ is a non empty proper subset of X which is both open and closed in X. Hence X is disconnected.

2.1.15 Example:

The rational Q are not connected.

If Y is a subspace of Q containing two points p and q we can choose an irrational number a lying between p and q such that Y can be written as the union of two disjoint open sets.

2.1.16 Theorem:

Let X be a topological space. Let Y be a subspace of X and let $Y == AUB$ where A and B are non empty and disjoint sets neither of which contain limit point of other, is a separation of Y. Then the space Y is connected if there exists no separation of Y.

Proof: Let $Y = A \cup B$ is a separation of Y. Then A is both open and closed in Y. Let $\overline{A}$ is the closure of A in X. Then closure of A in Y is the set $\overline{A} \cap Y$. Since A is closed in Y then $A = \overline{A} \cap Y$ *or* $\overline{A} \cap B = \phi$. Since $A = \overline{A} \cap Y$, where $D(A)$ is set of all limit points of A. So B contains no limit point of A. In same way we can show that A contains no limit point of B.

Conversely : Let A and B are disjoint non-empty sets whose union is Y and neither of which contain a limit point of other then $\overline{A} \cap B = \phi$ and $\overline{A} \cap B = \phi$. So we have $A = \overline{A} \cap Y$ and $B = \overline{B} \cap Y$. Thus both A and B are closed in Y and since $A = Y - B$ and $B = Y - A$. So they are open in Y

.

2.1.17 Example:

Consider the following subset of the plane R_2 where R_2 is enclosed with the product topology.

$\overline{A} \cap Y =$ A and $\overline{B} \cap Y = B$

Then X is not connected. Since the two sets form a separation of X, because neither contain a limit point of other.

2.1.18 Theorem:

Let X be a topological space and let $X = C \cup D$ where C and D are non empty disjoint open sets in X. Let Y is a connected subset of X then either $Y \subset C$ *or* $Y \subset D$.

Proof : Since C and D are open in X. So the sets $C \cap Y$ and $D \cap Y$ are open in Y and since C and D are disjoint therefore

$C \cap Y$ and $D \cap Y$ are disjoint

and $Y \subset C$ or $Y \subset D$. if $C \cap Y \neq \phi$

and $D \cap Y \neq \phi$ then $C \cap Y$ and $D \cap Y$ form a separation of Y but Y is connected therefore either either $Y \cap C = \phi$ *or* $Y \cap D = \phi$

Hence either $Y \subset C$ *or* $Y \subset D$.

2.1.19 Theorem:

Let X be a topological space and let $A_\alpha \subset C \ \forall \alpha \in I$ collection of connected subsets of X with the property $A_\alpha \subset C \ \forall \alpha \in$ I is connected.

Proof: Let p be any point of $\cap A_\alpha$. We prove that $A_\alpha \subset C \ \forall \alpha \in I$ is connected. Let Y = C UD is a separation of Y and p is in one of the sets C or D. Let p $\in$C . Since the set A_α is connected so either $A \subset C$ *or* $A_\alpha \subset D_\alpha$, it cannot lie in D because it contain the point p of C. Hence $A_\alpha = \subset C \ \forall \alpha \in I$

2.1.20 Theorem :

Let X be a topological space and let A be a connected set in X. If B is any subset of X such that $A \subset B \subset \overline{A}$ then B is also connected subset of X.

Proof: Let A be a connected set in X and let $A \subset B \subset \overline{A}$. Now let $B = C \cup D$ is a separation of B. Since A is connected. So A must lie either in C or in D. Let $A \subset C$ then $\overline{A} \subset \overline{C}$, since $C \cup D = \phi$. So B can not intersect D. contradiction gives that $D = \phi$ and hence B is a connected subset of X. □

2.2. Functions of Connected and Disconnected Space

2.2.1 Definition :

Let X and Y be topological spaces and let $f: X \to Y$ be a function from X into Y. Then f is said to be continuous if any of the following condition is satisfied:

(i)For each open subset V of Y, the set $f : X \to Y$ is an open subset of X.
(ii) For each closed subset B of Y, the set $f : X \to Y$ is closed in X.
(iii) For every subset A of X, we have $f : X \to Y$

2.2.2 Theorem:

Let $f : X \to Y$ be a continuous function from a connected space X into a topological space Y. Then $f : X \to Y$ is connected.

Proof: Let $f : X \to Y$ be a continuous function and let X be connected. We have to prove that $Z = f(X)$ is connected. Since the function obtained from f by restricting its range to the space Z is also continuous. So we consider only the case of continuous subjective function $g : X \to Z$. Let $Z = A \cup B$ is a separation of Z into two disjoint non empty sets open in Z. Then $g : X \to Z$ are disjoint sets such that $X = g : X \to Z$ .They are open in X because g is continuous and non empty as g is subjective, therefore they form a separation of X. This gives a contradiction to the fact that X is connected. Therefore $f(X)$ is connected.

2.2.3 Theorem:

Let X be a connected topological space with topology $\mathfrak{I}$ and $\mathfrak{I}^*$ is coarser than $\mathfrak{I}$. Then the space X with topology $\mathfrak{I}^*$ is also connected.

Proof: Let the space X with topology $\mathfrak{I}^*$ is disconnected then there exist a non empty proper subset A of X which is both open and closed, then A and $X - A$ are both open in $\mathfrak{I}^*$, Since $\mathfrak{I}^* \subseteq \mathfrak{I}$, this implies that A and $X - A$ are both open in $\mathfrak{I}$. This shows that A is a non empty proper subset of X which is both open and closed with respect to $\mathfrak{I}$. So space X with topology $\mathfrak{I}$ is disconnected which gives a contradiction. So space X with topology $\mathfrak{I}^*$ is connected.

2.2.4 Theorem: (Connected sets in the real line).

Let E be a subset of the real line R containing at least two points. Then E is connected iff E is an interval.

Proof: Let E be any subset of real line containing at least two points and let E is not an interval. Let $a, b \in E$ and $p \notin E$ such that $a < p < b$. Let $A = E \cap G$ and $B = E \cap H$ then $a \in G$ and $b \in H$, therefore $E \cap G$ and $E \cap H$ are non empty disjoint sets whose union is E therefore E is disconnected. Now let E is an interval and let E is disconnected. Let G and H form a seperation of E and let $A = E \cap G$ and $B = E \cap H$, then $E = A \cup B$, where A and B are non empty sets. Let $a \in A$ and $b \in B$ such that $a < b$. Let $p \in A = E \cap G$, since [a, b] is a closed set therefore $p \in [a, b]$. Let $p \in A = E \cap G$ then $p < b$ and $p \in G$. Since G is an open set therefore there exists $\delta > 0$ such that $p + \delta \in G$ and $p + \delta < b$. Hence $p + \delta \in E$, then $p + \delta \in A$. This gives a contradiction to the definition of p therefore $p \notin A$. Now let $p \in B = E \cap H$ then $p \in H$ and H is an open set therefore there exist $\delta^* > 0$ such that $[p - \delta^*, p] \subset H$ and

$a < p - \delta^*$ therefore $[p - \delta^*, p] \subset E$ and so $[p - \delta^*, p] \subset B$ Hence $[p - \delta^*, p] \cap A = \phi$ but then $p - \delta^*$ is an upper bound for $A \cap [a, b]$ which is not possible by definition of p therefore $p \notin B$. But this is a contradiction to the fact that $p \in E$. Hence E is connected.

2.2.5 Theorem:

A continuous image of a connected space is connected.

Proof: Let $f : X \to Y$ be a continuous surjection. Suppose Y is disconnected. Then there are disjoint non empty clopen sets Y_1, $Y_2 \subset Y$ such that $Y = Y_1 \cup Y_2$. Put $X_1 = f^{-1}(Y_1)$ and $X_2 = f^{-1}(Y_2)$. Then X_1 and X_2 are disjoint non empty clopen sets in X, and $X = X_1 \cup X_2$. So X is disconnected.

2.2.6 Theorem:

Every Tychonoff space of cardinality more than 1 but less than c is disconnected.

Proof: Suppose X is Tychonoff space such that $1 < |X| < c$. Fix distinct points $p, q \in X$. There is a continuous function $f : X \to R$ such that $f(p) = 0$ and $f(q) = 1$. Since $|X| < c$, $|f(X)| < c$. Since $|[0, 1]| = c$ there is $a \in (0, 1)$ such that $a \notin f(X)$. Then $X_2 = f^{-1}((-\infty, a))$ and $X_2 = f^{-1}((a,\infty))$ are disjoint non empty clopen subsets of X, and $X = X_1 \cup X_2$. So X is disconnected.

2.2.7 Theorem:

Let X be a topological space, and Y and Z subsets of X such that $Y \subset Z \subset \overline{Y}$. If Y is connected then Z is connected.

Proof: Suppose Z is disconnected. Then there are non empty disjoint subsets Z_1=, $Z_2 \subset Z$ such that $Z = Z_1 \cup Z_2$ and Z_1 and Z_2 are clopen in Z. Put $Y_1 = Z_1 \cap Y$ and Y_2 = Z2∩Y . Then Y_1 and Y_2 are disjoint clopen subsets of Y , and $Y = Y_1 \cap Y_2$. Since Y is dense in Z, the sets Y_1 and Y_2 are non empty. So Y is disconnected. A contradiction.

2.2.8 Corollary:

The closure of a connected set is connected.

2.2.9 Definition:

Say that a family of sets A is linked if for every A, B ∈ A, $A \cap B = \emptyset$.

2.2.10 Theorem:

Suppose A is a linked family of subsets of a topological space X. If each element of A is connected, then S A is connected.

Proof: Put Y = Y*A*. Suppose Y is disconnected. Then there are non empty disjoint sets Y_1, Y_2 such that $Y = Y_1 \cup Y_2$ and Y_1 and Y_2 are clopen in Y . Pick $y_1 \in Y_1$, $y_2 \in Y_2$. There are A_1, A_2 ∈ A such that $y_1 \in A_1$ and $y_2 \in A_2$. Since A is linked we can pick $z \in A_1 \cap A_2$. Then either $z \in Y_1$ or $z \in Y_2$. In the case $z \in Y_2$ the sets $Y'_1 = Y_1 \cap A_1$ and $Y'_2 = Y_2 \cap A_1$ are non empty disjoint clopen subsets of A_1 such that A1 = $Y'_1 \cup Y'_2$. This contradicts the connectedness of A1. In the case $z \in Y_1$ we arrive to a contradiction with the connectedness of A_2 similarly. So Y is connected.

2.2.11 Theorem:

Suppose A is a family of subsets of a topological space X such that $\bigcap A = \emptyset$. If each element of A is connected, then S A is connected.

2.2.12 Theorem:

A (Tychonoff) product of any family of connected spaces is connected.

Proof: First we prove that the product of two connected spaces, say X and Y is connected. For every x ∈ X and y ∈ Y denote $T_{x,y} = (\{x\} \times Y) \cup (X \times \{y\})$. Then $(\{x\} \times Y)$ and $(X \times \{y\})$ are homeomorphic to Y and X, respectively, and thus are connected. Next, the intersection $(\{x\} \times Y) \cap (X \times \{y\})$ contains point (x, y) and thus is non empty. Thus by Theorem(2.2.11) $T_{x,y}$ is connected. Next, note that:

1. The family $\{T_{x,y} : x \in X, y \in Y\}$ is linked. (Indeed, for every x, x' , y, y' , $(x,y') \in T_{x,y} \cap T_{x',y'}$)
2. $\bigcup \{T_{x,y} : x \in X, y \in Y\} = X \times Y$.

Therefore by Theorem(2.2.10), X × Y is connected.

Second, by induction the statement extends to any finite product of connected spaces.

Last, let X = $\prod_{a \in A} X_a$ be an arbitrary (possibly infinite) product of connected spaces Xa. Fix a point p =(pa : a ∈ A) ∈ X. For a finite subset $F \subset A$, put $P_F = \prod_{a \in A} X_{a,F}$ where

$$X_{a,F} = \begin{cases} X_a & \text{if} \quad a \in F \\ \{p_a\} & \text{otherwise} \end{cases}$$

Remark:

- Each $X_{a,F}$ is homeomorphic it a finite product of connected spaces, and thus, by the previous step, connected.
- The family $\{P_F$: F is a finite subset of $A\}$ has non empty intersection (point p is in each element). Hence by Corollary 9, $\bigcup \{P_F$: F is a finite subset of A$\}$ is connected.
- $\bigcup$ $\{P_F$: F is a finite subset of A$\}$ is a σ-product in X = $\prod_{a \in A} X_a$. As we know from the previous lectures, it follows that it is dense in X.

Therefore by Corollary 6, X is connected since it has a dense connected subspace.

2.2.14 Example :

Suppose R is represented as the union of two non empty disjoint sets: R = $H_1 \cup H_2$. Pick $a \in H_1$ and $b \in H_2$. Without loss of generality we can assume that a < b. Put z = inf([a, b] ∩ H_2). Note that z ∈ $\overline{H1} = \cap\ \overline{H2}$. So (since z must belong to either H_1 or H_2) it is not possible that both sets H_1 and H_2 are closed. Thus R is connected. The same argument shows that any interval of the real line is connected.

2.2.15 Theorem:

Every pathwise connected space is connected.

Proof: Fix x ∈ X. For every y ∈ X fix a path p_y from x to y and denote $A_y = p_y(I)$. Then by Theorem(2.2.5), A_y is a connected subspace of X. The family A = $\{A_y : y \in X\}$ consists of connected subsets of X and has a common point x. So by Corollary 9, $\bigcup A$ is connected. But clearly $\bigcup A$ = X.

2.2.16 Theorem:

A continuous image of pathwise connected space is pathwise connected.

Proof: Let X be a pathwise connected space, f : X → Y a continuous surjection, and x, y ∈ Y . Pick a ∈ f^{-1} (x) and b ∈ f^{-1} (y). Since X is pathwise connected there is a path $p : I \to X$ such that p(0) = a and $p(1) = b$. Then $f \circ p$ is a path between x and y in Y .

2.2.17 Theorem:

Any product of pathwise connected spaces is pathwise connected.

Proof: Let X = $\prod_{a \in A} Xa$ be a product such that each Xa is pathwise connected. Let x, y ∈ X. For each a ∈ A, there is a path $pa : I \to Xa$ such that $pa(0) = x(a)$ and $pa(1) = y(a)$. Then the diagonal product $p = \Delta\{pa : a \in A\}$ is a pass between x and y.

2.2.18 Example:

The Topological Sin Curve is connected but not locally connected. (And of course any more than one point discrete space is locally connected but not connected.)

2.2.19 Theorem:

The following conditions are equivalent:

i. X is locally connected;

ii. Components of open subspaces of X are open.

Proof: (i)⇒(ii) Let U be an open set in X. Then X is locally connected. Let C be a component of U and let x ∈ C. Since U is locally connected there is a connected open set V such that x ∈ V ⊂ U.

By the definition of component, $V \subset C$. So C contains every point together with some neighborhood, i.e. C is open.

(ii) ⇒ (i) Let $x \in U \subset X$ where U is open, and let C be a component of x in U. Being a component, C is connected; by (2), C is open in U and thus in X.

2.2.20 Theorem:

In a locally connected space, components are equal to quasicomponents.

Proof: Let X be locally connected $x \in X$; let C be the component of x, and Q the quasicomponent of x. By Proposition 29, C is open. But also for every $y \in X \setminus C$, the component of y in X is open and is a subset of $X \setminus C$. Thus $X \setminus C$ is open. So C is closed and thus clopen. Thus $Q \setminus C$, and so by Proposition 16, $Q = C$.

2.2.21 Remark:

A continuous image of a locally connected space does not have to be locally connected.

2.2.22 Example:

Take X_0 from Example(2.2.18) and put $X'_1 = \{(-1, y) : y \in [-1, 1]\}$ and $X' = X_0 \cup X'_1 \subset R^2$. Then X' is locally connected and X from Example (2.2.18) is a continuous image of X' (under a condensation); we know that X is not locally connected. □

REFERENCES

[1]. BING, R.H., A connected countable Hausdorff space. Proc. Amer. Math. Soc. 4 (1953) 474

[2]. Brown, M. A countable connected T-space. Bull. Amer. Math. Soc. 59 (1953), 367

[3]. Charatonik, J.J. Local connectedness and connected open functions. Portugaliae mathematica, Vol. 53 Fasc 4–1996

[4]. Charatonik, J.J. And Janusz, R. Prajs. On Local connectedness of Absolute Retracts Pacific Journal of Mathematics Vo. 201, No. 1, 2001

[5]. DUGUNDJI J. Topology, Allyn and Bacon, 1965

[6]. De Groot, J. and Mc Dowell R.H. Locally connected space and their compactification Illions J. Math 11 (1967) 353–364.

[7]. Gustin, W. countable connected spaces, Bull Amer Math Soc. 53 (1946), 101–106

[8]. Jankovic, D.S. A note on mapping of Extremelly disconnected spaces Acta. Math. Hungar 46 (1–2) (1985) (83–92).

[9]. Jankovic, D.S. A note on almost locally connected spaces. Math. Japonica, 30 (1985), 393–397.

[10]. Reilly, I.L. and M.K. Vamahamurthy On the topology of semilocal connectedness Math. Nachr 129 (1986) 109–113.

[11]. RON LIVNE Monograph on Topology Unpublished.

[12]. Vermer, J. The smallest basically disconnected Preimage of a space. Topology and its Application 171 (1984) 217–232.

[13]. Alexander, J.W. and BRIGGS, G. B., On types of knotted curves, Annals of Mathematics 28 (1927), 562–586.

REFERENCES

[14]. Andersson,K. G., Poincar´e's discovery of homoclinic points, Archive for History of Exact Sciences 48 (1994), 133–147.

[15]. Archibald, T., Connectivity and smoke rings: Green's second identity in its first fifty years, Mathematics Magazine 62 (1989), 219–232.

[16]. Bertrand, J. L. F., Theoreme relatif au movement le plus general d'un fluide, Comptes Rendus 66 (1868), 1227–1230.

[17]. Biggs, N. L., Lloyd, E. K., and Wilson, R. J., Graph Theory, 1736–1936, Oxford: Clarendon Press, 1976.

[18]. Bollinger, M., Geschichtliche Entwicklung des Homologiebegriffs, Archive for History of ExactSciences 9 (1972), 94–170.

[19]. Dehn, M. and Heegaard, P., Art. "Analysis situs," in: Encyklop¨adie der mathematischen Wissenschaften, III AB, Leipzig: Teubner, 1907–1910, pp. 153–220; completed January 1907.