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Metric and partial metric topology

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بسيم الله الرحمز الرجيم

(قُلِ اللَّهُمَّ مَالِكَ الْمُلْكِ تُؤْتِي الْمُلْكَ مَن تَشَاءُ وَتَنزِعُ الْمُلْكَ مِمَّن تَشَاءُ وَتُعِزَّ مَن تَشَاءُ وَتُذِلُّ مَن تَشَاءُ^ط بِيَدِكَ الْخَيْرُ^ط إِنَّكَ عَلَىٰ كُلِّ شَيْءٍ قَدِيرٍ)

صدق الله العلي العضيم

سورة آل عمران

الايه 26

Dedication

To whom Allah sent as mercy to the words....

To the prophet Mohammed

To my family.....

To everyone I love

Acknowledgment

It is a will of Allah, the mersciful god, who who surround me with brilliant and supportive people. I would like to express my graduate, my great respect and appreciation to my supervisor Ass.L.Fieras Joad Obead, for his help and support.Thank must go to my dear mother.

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Abstract

Metric spaces are inevitably Hausdorff and so cannot, for example, be used to study non-Hausdorff topologies such as those required in the Tarskian approach to programming language semantics. This paper presents a symmetric generalised metric for such topologies, an approach which sheds new light on how metric tools such as Banach's Theorem can be extended to non-Hausdorff topologies

Introduction

In the study of the denotational semantics of programming languages a topological model is constructed for a programming language defined as a system of logic. More often than not this means a T_0 model for the lambda calculus in the spirit of Scott [12]. However, the necessity in this approach that all suitable models must be T_0 appears to remove any possibility that the theory of metric space (which are of course all T2) can be applied in any way to semantics in Computer Science. Rare exceptions to this rule are the use of quasi-metrics by Smyth in [11] to describe T_0 spaces, or the use of a metric super topology for a T_0 space by Lawson in [8]. If metrics are to be used at all then the more conventional wisdom in Computer Science would dismiss Scott's T_0 approach in favour of a purely metric approach such as that of de Bakker and Zucker [1]. Unfortunately, the latter T_0 approach pays the price of losing the notion of partial ordering inherent in T_0 spaces, a concept of fundamental importance in any Tarskian approach to fixed point semantics [13]. The distinct advantage of using quasi-metrics is that such generalised metrics can be used to define T_0 topologies with partial orderings, and so allow Tarskian semantics. Quasi-metrics are not without their problems though. Being nonsymmetric a quasi-metric is arguably an "unnatural" notion of distance. A more important criticism is that the lack of symmetry sheds little light on how to develop tools for reasoning about programs using quasi-metric ideas. The title Reconciling Domains with Metric Spaces of Smyth's paper [11] indicates a much desired long term goal allegedly argued for by Dana Scott that partial order semantics should one day have a metric foundation. In All Topologies come from Generalised Metrics [5], Kopperman infers that such a foundation might just be possible. This may or may not be of interest to topologists in general as many of the more pleasant T2 properties usually associated with metric spaces may be lost in a process of generalisation.

Chapter one

General topology and Metric space

1.1 Some properties of topological space

1.1.1 *Definition*:

Let X be a set. A topology on X is a collection $T \subseteq P(X)$ of subsets of X satisfying

(i) T contains Ø and X,

(*ii*)*T* is closed under arbitrary unions, i.e. if $U_i \in T$ for $i \in I$ then $U_i \in T$,

(iii) T is closed under finite intersections, i.e. if $U_1, U_2 \in T$ then $U_1 \cap U_2 \in T$

For example :(i) Take any set X and let $curlyT = \{empty, X\}$. Then curlyT is a topology called the trivial topology or indiscrete topology

(ii)Let $X = \{1, 2, 3\}$ and curly $T = \{empty, \{1\}, \{1, 2\}, X\}$. Then curlyT is a topology

1.1.2 Definition :

A topological space (X,T) is said to be T_0 (or to satisfy the T_0 axiom) if for all distinct $x,y \in X$ there exists an open set $U \in \tau$ such that either $x \in U$ and $y \notin U$ or $x \notin U$ and $y \in U$

1.1.3 Definition :

a T_1 space is a topological space in which, for every pair of distinct points, each has a neighborhood not containing the other.

1.1.4 Definition :

Points x and y in a topological space X can be separated by neighbourhoods if there exists a neighbourhood U of x and a neighbourhood V of y such that U and V are disjoint ($U \cap V = \emptyset$). X is a Hausdorff space if all distinct points in X are pairwise neighborhood-separable. This condition is the third separation axiom (after T0 and T1), which is why Hausdorff spaces are also called T2 spaces

for example elmost all spaces encountered in analysis are Hausdorff; most importantly, the real numbers (under the standard metric topology on real numbers) are a Hausdorff space. More generally, all metric spaces are Hausdorff. In fact, many spaces of use in analysis.

The cofinite topology on an infinite set is a simple example of a topology that is T1 but is not Hausdorff (T2). This follows since no two open sets of the cofinite topology are disjoint.

1.2 General definition and metric space

1.2.1 Definition :

A function f from a set X to a set Y is defined by a set of ordered pairs (x, y), such that $x \in X$ and $y \in Y$. This set is subject to the following condition: every element of X is the first component of exactly one ordered pair within this set of pairs. In other words, for every x in X there is exactly one element y, such that the ordered pair (x, y) belongs to the set of pairs defining the function f. Sets of pairs violating this condition do not define functions.

1.2.2 Definition :

a (real) interval is a set of real numbers with the property that any number that lies between two numbers in the set is also included in the set.

For example, the set of all numbers x satisfying $0 \le x \le 1$ is an interval which contains 0 and 1, as well as all numbers between them.

1,2.3 Definition :

An open interval does not include its endpoints, and is indicated with parentheses.

For example, (0,1) *means greater than* 0 *and less than* 1*.*

1.2.4 Definition :

A closed interval is an interval which includes all its limit points, and is denoted with square brackets.

For example, [0,1] means greater than or equal to 0 and less than or equal to 1

1.2.5 Definition :

A lower bound of a subset S of a partially ordered set (P, \leq) is an element a of P such that

(*i*) $a \le x$ for all x in S.

A lower bound a of S is called an infimum (or greatest lower bound, or meet) of S if

(*ii*)for all lower bounds y of S in P, $y \le a$ (a is larger than any other lower bound) for example:)(i).inf{1,2,3,...}=1 (*ii*).inf{ $x \in R/0 < x < 1$ } = 0

1.2.6 Definition:

an upper bound of a subset S of a partially ordered set (P, \leq) is an element b of P such that

(i) $b \ge x$ for all x in S.

An upper bound b of S is called a supremum (or least upper bound, or join) of S if

(*ii*)for all upper bounds z of S in P, $z \ge b$ (b is less than any other upper bound) for example(i).sup{ $(-1)^n - 1/n | n = 1, 2, 3, ... \} = 1$

(*i*).sup{ $x \in R | 0 < x < 1$ } = sup{ $x \in R | 0 \le x \le 1$ } = 1

1.2.7 Definition:

A binary relation is a collection of Sets between two Sets 'M' and 'N' which is the subset of M * N, or we can say that it is a Set of Ordered Pair m, $n \in M*N$. Here set of 'M' and 'N' are known as Domain

For example Suppose there are four objects {ball, car, doll, gun} and four persons {John, Mary, Ian, Venus}. Suppose that John owns the ball, Mary owns the doll, and

Venus owns the car. Nobody owns the gun and Ian owns nothing. Then the binary relation "is owned by" is given as

R = ({ball, car, doll, gun}, {John, Mary, Ian, Venus}, {(ball, John), (doll, Mary), (car, Venus)}).

Thus the first element of *R* is the set of objects, the second is the set of persons, and the last element is a set of ordered pairs of the form (object, owner).

The pair (ball, John), denoted by $_{ball}R_{John}$ means that the ball is owned by John.

Two different relations could have the same graph. For example: the relation

({ball, car, doll, gun}, {John, Mary, Venus}, {(ball, John), (doll, Mary), (car, Venus)})

is different from the previous one as everyone is an owner.

Nevertheless, R is usually identified or even defined as G(R) and "an ordered pair $(x, y) \in G(R)$ " is usually denoted as " $(x, y) \in R$ "

1.2.8 Definition:

A metric space is a set X where we have a notion of distance. That is, if $x, y \in X$, then d(x, y) is the "distance" between x and y. The particular distance function must satisfy the following conditions:

(i). $d(x, y) \ge 0$ for all $x, y \in X$

(ii)d(x, y) = 0 if and only if x = y

(iii)d(x, y)=d(y, x)

 $(iv)d(x, z) \le d(x, y) + d(y, z)$

for example for any space X, let d(x, y) = 0 if x = y and d(x, y) = 1 otherwise. This metric, called the discrete metric, satisfies the conditions one through four.

1.2.9 Definition :

In Euclidean n-space, an (open) n-ball of radius r and center x is the set of all points of distance less than r from x. A closed n-ball of radius r is the set of all points of distance less than or equal to r away from x.

1.2.10 Definition :

A sequence is an enumerated collection of objects in which repetitions are allowed. Like a set, it contains members (also called elements, or terms). The number of elements (possibly infinite) is called the length of the sequence. Unlike a set, the same elements can appear multiple times at different positions in a sequence, and order matters. Formally, a sequence can be defined as a function whose domain is either the set of the natural numbers (for infinite sequences) or the set of the first n natural numbers (for a sequence of finite length n). The position of an element in a sequence is its rank or index; it is the integer from which the element is the image. It depends on the context or of a specific convention, if the first element has index 0 or 1. When a symbol has been chosen for denoting a sequence, the nth element of the sequence is denoted by this symbol with n as subscript; for example, the nth element of the Fibonacci sequence is generally denoted Fn.

For example, (M, A, R, Y) is a sequence of letters with the letter 'M' first and 'Y' last. This sequence differs from (A, R, M, Y). Also, the sequence (1, 1, 2, 3, 5, 8), which contains the number 1 at two different positions

Chapter two

Metric and partial metric topology

2.1 BACKGROUND DEFINITIONS AND RESULTS

2.1.1Definition:

A basis β for a topology is σ -disjoint if there exists B1, B2, ... \subseteq B such that, $\beta = \bigcup \{B_n | n \in \omega\}$ and, $\forall n \in \omega \forall B, B \in B_n$, $B \cap B = \varphi$.

2.1.2 Definition:

A partial ordering is a binary relation $\ll \subseteq U^2$ such that,

- (*i*) $\forall x \in U, x \ll x$
- (*ii*) $\forall x, y \in U, x \ll y \cap y \ll x \Rightarrow x = y$
- (*i*) $\forall x, y, z \in U, x \ll y \cap y \ll z \Rightarrow x \ll z$

Within the field of Computer Science, which originally motivated this work, \ll is used an information ordering in which $x \ll y$ is interpreted as all the information contained in x is also contained in y. We now establish the usual relationship in Computer Science between topology and the information ordering. The topology usually placed upon U will at least be T_0 , and will also be consistent with \ll in the following sense.

2.1.3Definition:

A weakly order consistent topology is a weaker version of the order consistent topology [2] as used in lattice theory for which in addition suprema of directed sets are their limits. As the work in this paper requires neither directed sets nor lattices we work only with weakly order consistent topologies. An interesting example of a weakly order consistent topology is the topology of all upwardly closed sets, $T(\ll) ::= \{ S \subseteq U | \forall x \in S, x \ll y \Rightarrow y \in S \}.$

Thus, for example, for the usual partial ordering $\leq \subseteq (\omega \cup \{\infty\})^2$ on the nonnegative integers with infinity,

 $T[\leq] = \{\{n, n+1, \dots, \infty\} \mid n \in \omega \cup \{\infty\}\}.$

Each T_0 topology is weakly order consistent if and only if it is a topology of upwardly closed sets. Given any T_0 topology the information ordering can be recovered using

the specialisation ordering defined by $x \ll y \Rightarrow x \in cl(\{y\})$, a topic discussed more fully elsewhere [3]. In Computer Science we are interested in totally ordered sequences $X \in U^w$ of the form $X_0 \ll X_1 \ll X_2 \ll \cdots$, called chains,

of increasing information, the least upper bound lub(X) of which is intended to capture the notion of the amount of information defined by the chain. To ensure that lub(X) cannot contain more information than can be derived from the members of the chain X we insist that our topologies have the following property.

2.1.4 Definition:

A Scott-like topology over a partial ordering $\ll \subseteq U^2$ is a weakly order consistent topology T over U such that for each chain $X \in U^w$, lub(X) exists, and,

 $\forall O \in T$, $lub(X) \in O \Rightarrow \exists k \in \omega \forall n > k$, $X_n \in O$. In other words, the least upper bound of a chain must be a limit of that chain. The term Scott-like topology introduced here is a weaker version of the term Scott topology [2] used in the study of continuous lattices. As the results in this paper do not need the full strength of the Scott topology we work only with the weaker Scott-like topology.

2.1.5 Definition:

A metric is a function $d: U^2 \rightarrow R$ such that,

- (*i*) $\forall x, y \in U, x = y \Leftrightarrow d(x, y) = 0$
- (*ii*) $\forall x, y \in U, d(x, y) = d(y, x)$
- (iii) $\forall x, y, z \in U, d(x, z) \leq d(x, y) + d(y, z)$

2.2 THE PARTIAL METRIC

2.2.1Definition:

A partial metric or pmetric [9] (pronounced "p-metric") is a function $p:U^2 \rightarrow R$ such that,

(i) $\forall x, y \in U, x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y)$

(*ii*) $\forall x, y \in U, p(x, x) \leq p(x, y)$

(*iii*)
$$\forall x, y \in U, p(x, y) = p(y, x)$$

(*iv*) $\forall x, y, z \in U, p(x, z) \le p(x, y) + p(y, z) - p(y, y)$

The pmetric axioms (i) thru (iv) in (2.2.1 difinition) are intended to be a minimal generalisation of the metric axioms(i) thru (iii)) in (2.1.5 difinition) such that each object does not necessarily have to have zero distance from itself. In this generalisation we manage to preserve the symmetry axiom(ii) in (2.1.5 definition) to get(iii) in (2.2.1 difinition) , but have to "massage" the transitivity axiom (iii) in (2.1.5definition) to produce the generalisation (iv) in (2.2.1difinition) (originally suggested to the author in [16]). Consequently a metric is precisely a pmetric

 $p: U^2 \rightarrow R$ such that,

 $\forall x \in U, p(x,x) = 0.$

"Half" of the metric axiom M1 is preserved as,

 $\forall x, y \in U, p(x, y) = 0 \Rightarrow x = y$

However, the converse implication does not generally hold. p(x,x), referred to as the size or weight of x, is a feature used to describe the amount of information contained in x. The smaller p(x,x) the more defined x is, being totally defined if p(x,x) = 0.

2.2.2 Example:

In Computer Science a flat domain is a partial ordering of the form $\ll \subseteq (S \cup \{ \bot \})^2$ consisting of a set S of totally defined objects together with the special undefined object $\bot \notin S$ (pronounced "bottom"), and ordering defined by, $\forall x, y \in S \cup \{\bot\}, x \ll y \Leftrightarrow x = \bot \lor x = y \in S$. Such a domain can be defined by the flat pmetric $p : (S \cup \{\bot\})^2 \rightarrow \{0,1\}$ where \checkmark

 $\forall x, y \in S \ \cup \{ \bot \}, \ p(x, y) = 0 \Leftrightarrow x = y \in S$

Note how the condition p(x,x) = 0 precisely captures the flat domain notion $x \in S$ of total definedness.

2.2.3 Definition:

An open ball for a pmetric $p : U^2 \to R$ is a set of the form, $B_{\epsilon}^P(x) ::= \{y \in U | p(x,y) < \epsilon\}$

for each $\epsilon > 0$ and $x \in U$.

Note that, unlike their metric counterparts, some pmetric open balls may be empty.

For example, if p(x,x) > 0 then $B^{p}_{p(x,x)}(x) = \varphi$.

2.2.4Theorem:

The set of all open balls of a pmetric $p: U^2 \rightarrow R$ is the basis of a topology T[p] over U.

Proof: As, $U = \bigcup_{x \in U} B_{p(x,x)+1}^{P}(x)$ and, for any balls $B_{\epsilon}^{P}(x)$ and $B_{\delta}^{P}(y)$, $B_{\epsilon}^{P}(x) \cap B_{\delta}^{P}(y) = \bigcup \{B_{\eta}^{P}(z)/z \in B_{\epsilon}^{P}(x) \cap B_{\delta}^{P}(y)\}$ where, $\eta ::= p(z,z) + min\{\epsilon - p(x,z), \delta - p(y,z)\}.$

2.2.5 Theorem:

For each pmetric p, open ball $B_{\epsilon}^{P}(a)$, and $x \in B_{\epsilon}^{P}(a)$, there exists $\delta > 0$ such that $x \in B_{\delta}^{P}(x) \subseteq B_{\epsilon}^{P}(a)$.

Proof: Suppose $x \in B_{\epsilon}^{P}(a)$. Then $p(x,a) < \epsilon$. Let $\delta ::= \epsilon - p(x,a) + p(x,x)$. Then $\delta > 0$ as $\epsilon > p(x,a)$. Also, $p(x,x) < \delta$ as $\epsilon > p(x,a)$.

Thus
$$x \in B_{\delta}^{P}(x)$$
.
Suppose now that $y \in B_{\delta}^{P}(x)$.
 $\therefore p(y,x) < \delta$.
 $\therefore p(y,x) < \epsilon - p(x,a) + p(x,x)$.
 $\therefore p(y,x) + p(x,a) - p(x,x) < \epsilon$.
 $\therefore p(y,a) < \epsilon$ (by P4).
 $\therefore y \in B_{\epsilon}^{P}(a)$.
Thus $B_{\delta}^{P}(x) \subseteq B_{\epsilon}^{P}(a)$.

Using the last result it can be shown that each sequence $X \in U^w$ converges to an object $a \in U$ if and only if,

 $\lim_{n\to\infty} p(X_n, a) = p(a, a).$

2.2.6Theorem:

Each pmetric topology is T_0 .

Proof: Suppose $p: U^2 \to R$ is a pmetric, and suppose $x \neq y \in U$, then, from P1 & P2 (wlog) p(x,x) < p(x,y), and so,

 $x \in B_{\epsilon}^{P}(x) \land y \notin B_{\epsilon}^{P}(x),$

where, $\epsilon ::= (p(x,x) + p(x,y))/2$

So far we have shown that a partial metric p can quantify the amount of information in an object x using the numerical measure p(x,x), and also that p has an open ball topology. This would not be of much use in Computer Science without a partial ordering.

2.2.7Definition:

For each pmetric $p: U^2 \rightarrow R$, $\ll_p \subseteq U^2$ is the binary relation such that,

 $\forall x, y \in U, x \ll_p y \Leftrightarrow p(x, x) = p(x, y).$

2.2.8 Theorem:

For each pmetric p, \ll_p is a partial ordering.

Proof: We prove (i) thru (iii) in (2.2.1 difinition).

(i) $\forall x \in U, x \ll_p x \text{ as } p(x,x) = p(x,x).$ (ii) $\forall x, y \in U, x \ll_p y \land y \ll_p x$ $\Rightarrow p(x,x) = p(x,y) = p(y,y) (by P3)$ $\Rightarrow x = y (by P1).$ (iii) $\forall x,y,z \in U, x \ll_p y \land y \ll_p z$ $\Rightarrow p(x,x) = p(x,y) \land p(y,y) = p(y,z).$ But by P4, $p(x,z) \leq p(x,y) + p(y,z) - p(y,y)$ $\therefore p(x,z) \leq p(x,x)$ $\therefore p(x,z) = p(x,x) (by P2)$ $\therefore x \ll_p z.$

2.2.9 Example:

The concept of a vague real number might be constructed as a nonempty closed interval on the real line. The function $p : \{[a,b] | a \leq b\}^2 \rightarrow R$ over all such intervals where,

 $\forall [a,b], [c,d], p([a,b], [c,d]) ::= max\{b,d\} - min\{a,c\}$

is a pmetric such that $[a,b] \ll_p [c,d] \Leftrightarrow [c,d] \subseteq [a,b]$, read as [c,d] is a more precise version of [a,b]. Also we can use p([a,b],[c,d]) to measure the degree of vagueness of a vague number [a,b].

For each pmetric $p, T[p] \subseteq T[\ll_p]$, that is, T[p] is a weakly order consistent topology over \ll_p .

Proof: It is sufficient to show that, $\forall x \in U \forall \epsilon > 0$

 $B_{\epsilon}^{p}(x) = U\{\{z|y \ll_{p} z\}|y \in B_{\epsilon}^{p}(x)\}$. Suppose $x, y, z \in U$ and $\epsilon > 0$ are such that $y \in B_{\epsilon}^{p}(x)$ and $y \ll_{p} z$. Then,

 $p(x,z) \le p(x,y) + p(y,z) - p(y,y)$ (by P4)

 $= p(x,y) as y \ll_p z$

$$<\epsilon$$
 as $y \in B^p_{\epsilon}(x)$.

Thus,
$$z \in B^p_{\epsilon}(x)$$
.

Thus T[p] is a Scott-like topology over \ll_p if each chain X has a least upper bound and if,

 $\lim_{n\to\infty} p(Xn, Xm) = p(lub(X), \, lub(X))$

2.2.11 Theorem:

For each pmetric $p: U^2 \rightarrow R$, $T[p] = T[\ll_p]$, if and only if,

 $\forall x \in U, \exists \epsilon > 0 \ B_{\epsilon}^{p}(x) = \{y | x \ll_{P} y\}$

Proof: Suppose first that, $\forall x \in U \exists \epsilon > 0 \cdot B_{\epsilon}^{p}(x) = \{y|x \ll_{p} y\}$. Then, $\forall O \in T[\ll_{p}]$,

$$O = \bigcup_{x \in O} \{ y | x \ll_P y \} = \bigcup_{x \in O} B_{\epsilon}^p(x) \in T[p]$$

$$\therefore T[\ll_p] \subseteq T[p]$$

 $:: T[p] = T[\ll_p] (by Theorem 3.5)$

Suppose now that, $T[p] = T[\ll_p]$. Then, $\forall x \in U, \{y|x \ll_p y\} \in T[p]$. Thus by (2.2.5Theorem) $\forall x \in U \exists \epsilon > 0, x \in B_{\epsilon}^p(x) \subseteq \{y|x \ll_p\}$. But, if $x \in B_{\epsilon}^p(x)$ then $\{y|x \ll_p y\} \subseteq B_{\epsilon}^p(x)$. Thus, $\forall x \in U \exists \epsilon > 0, B_{\epsilon}^p(x) = \{y|x \ll_p y\}$.

Having now established the relationship of the open ball topology T[p] to both the upward closure $T[\ll_p]$ and the weakly order consistent topology.

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