Republic of Iraq
Ministry of Higher Education and Scientific Research
University of Al-Qadisiyah / College of Education
Department of Mathematics

On a Certain Subclass of Multivalent Functions

A Research Submitted by

Muntadher Ali Joudah

To the council of the department of mathematics /collage of education, University of Al-Qadisiyah in Partial fulfillment of the requirements for bachelor in mathematics

Supervised by

Dr. Najah Ali Al -Ziadi

2018 A.C. **1439 A.H.**

بسم الله الرحمن الرحيم ....

((اذْكُـرُوا نِعْمَــةَ اللَّهِ عَلَـيْكُمْ إِذْ جَعَـلَ فِـيكُمْ أَنبِيَــاءَ وَجَعَلَكُم مُّلُوكًا وَآتَاكُم مَّا لَمْ يُؤْتِ أَحَدًا مِّنَ الْعَالَمِينَ))

..

صدق الله العلي العظيم
سورة المائدة آية (20)

وعن رسول الله (ص) أنه قال ..

((أن طلبَ العلم فريضة .. ألا أن الله يحبُ بغاة العلم .. وأن الشاخص في طلب العلم كالمجاهد في سبيل الله))

وعن أمير المؤمنين (ع) أنه كان يقول ..

((أيها الناس أعلموا أن كمال الدين طلب العلم و العمل بهِ ألا وأن طلب العلم أوجب عليكم من طلب المال أن المال مقسوم مضمون لكم قد قسمهُ عادلُ بينكم والعلم مخزون عن أهلهِ وقد أمرتم بطلبه من أهلهِ فأطلبوه))

الأهداء

الى النور الذي ينير لي درب النجاح ويامن علماني الصمود مهما
تبدلت الظروف .... أبي وأمي
الى كل من علمني حرفا .... أساتذتي الأعزاء
وبالخصوص أستاذي العزيز الدكتور نجاح علي
الزيادي الذي ساعدني ووقف معي في العمل

Abstract

We presented in this work a certain class MA(ʎ,α,σ,β,m,p) of multivalent analytic functions with linear operator $D_{p,m}^{\sigma,\beta}$ in the open unit disk U. We study coefficient inequality , distortion and growth theorems , radii of starlikeness , convexity and close - to - convexity , weighted mean and arithmetic mean , extreme points .

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Chapter One

Basic Definitions and Standard Results

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Introduction:

In this chapter, we list out all the definitions of the family of functions from analytic, univalent and multivalent (p–valent) and all related terms used during the investigation. We also include in this chapter all the standard theorems and lemmas used in the work.

1.1 Basic Definitions

Definition (1.1.1)[5]: A function$f$ of the complex variable is analytic at a point $z_0$if its derivative exists not only at z_0 but each point z in some neighborhoods of z_0. Itis analytic in region $\mathbb{U}$ if it is analytic at every point in $\mathbb{U}$.

Definition (1.1.2)[5]: A function f is said to be univalent (schilcht) if it does not take the same value twice i.e. $f(z_1) \neq f(z_2)$ for all pairs of distinct points $z_1, z_2 \in U$. In other words, f is one – to – one (or injective) mapping of U onto another domain.

If f assumes the same value more than one, then f is said to be multivalent (p-valent) in$U$. We also deal with the functions which are meromorphic univalent in the punctured unit disk $U^* = \{z \in \mathbb{C}: 0 < |z| < 1\}$. f is said to be meromorphic if it is analytic at every point in Uexcept finite elements in$U$.

As examples, the function $f(z) = z$ is univalent in U while $f(z) = z^2$ is not univalent in U. Also, $f(z) = z + \frac{z^n}{n}$ is univalent in U for each positive integer n.

Example (1.1.1) [5]: The function $f(z) = (1 + z)^2$ is univalent in U.

Let $z_1, z_2 \in U$ and suppose $f(z_1) = f(z_2)$. Then

$$(1 + z_1)^2 = (1 + z_2)^2$$

$$\Rightarrow 1 + 2z_1 + z_1^2 = 1 + 2z_2 + z_2^2$$

$$\Rightarrow z_1^2 - z_2^2 + 2(z_1 - z_2) = 0$$

$$\Rightarrow (z_1 - z_2)(z_1 + z_2 + 2) = 0.$$

Since$|z_1|, |z_2| < 1$, we know that $(z_1 + z_2 + 2) \neq 0$. Hence $z_1 - z_2 = 0$ or $z_1 = z_2$

Definition (1.1.3) [5]: A function $f$is said to be locally univalent at a point$z_0 \in \mathbb{C}$if it is univalent in some neighborhood ofz_0. For analytic function$f$, the condition$f'(z_0) \neq 0$is equivalent to local univalent atz_0.

Example (1.1.2)[5]: Consider the domain

$$D = \left\{z \in \mathbb{C}: 1 < |z| < 2,\ \ 0 < \arg(z) < \frac{3\pi}{2}\right\},$$

and the function$f: D \to \mathbb{C}$given by$f(z) = z^2$. It is clear that$f$is analytic on$D$and local univalent at every point$z_0 \in D$, since$f'(z_0) \neq 2z_0 \neq 0$for all$z_0 \in D$.

However,$f$is not univalent on$D$, since

$$f\left(\frac{3}{2\sqrt{2}} + i\frac{3}{2\sqrt{2}}\right) = f\left(-\frac{3}{2\sqrt{2}} - i\frac{3}{2\sqrt{2}}\right) = \frac{9}{4}i.$$

Definition (1.1.4)[5]: Let $\mathcal{A}$denotes the class of functions f of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad n \in \mathbb{N} \tag{1.1}$$

which are analytic and univalent in the open unit disk U.

Definition (1.1.5)[5]: We say that $f \in \mathcal{A}$ is normalized if f satisfies the conditions $f(0) = 0$ and $f'(0) = 1$.

Definition (1.1.6)[5]: A set $E \subseteq \mathbb{C}$ is said to be starlike with respect to $w_0 \in E$ if the linear segment joining w_0 to every other point $w \in E$ lies entirely in E. In a more picturesque language, the requirement is that every point of $E$is visible

from w_0. The set E is said to be convex if it is starlike with respect to each of its points, that is , if the linear segment joining any two points of E lies entirely in E.

Definition (1.1.7)[5]: A function$f$ is said to be conformal at a point z_0 if it preserves the angle between oriented curves passing through z_0 in magnitude as well as in sense. Geometrically, images of any two oriented curves taken with their corresponding orientations make the same angle of intersection as the curves at z_0 both in magnitude and direction. A function $w = f(z)$ is said to be conformal in the domain D, if it is conformal at each point of the domain.

Definition (1.1.8)[5]: A function $f \in \mathcal{A}$ is said to be starlike function of order α if and only if

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, (\, 0 \leq \alpha < 1;\ z \in U, f(z) \neq 0). \tag{1.2}$$

Denotes the class of all starlike functions of order α in U by $S^*(\alpha)$ and S^* the class of all starlike functions of order 0, $S^*(0) = S^*$. Geometrically, we can say that a starlike function is conformal mapping of the unit disk onto a domain starlike with respect to the origin. For example, the function

$$f(z) = \frac{z}{(1-z)^{2(1-\alpha)}},$$

is starlike function of order α.

Definition (1.1.9)[5]: A function $f \in \mathcal{A}$ is said to be convex function of order α if and only if

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha, (\, 0 \leq \alpha < 1;\ z \in U, f'(z) \neq 0). \tag{1.3}$$

Denotes the class of all convex functions of orderα in U by $C(\alpha)$ and C for the convex function $C(0) = C$.

Definition (1.1.10)[5]: A function $f \in \mathcal{A}$is said to be close – to – convex of order α $(0 \leq \alpha < 1)$ if there is a convex function g such that

$$Re\left\{\frac{f'(z)}{g'(z)}\right\} > \alpha, \qquad (g'(z) \neq 0;\ z \in U). \tag{1.4}$$

We denote by $K(\alpha)$, the class of close – to – convex functions of order α, $f$is normalized by the usual conditions $f(0) = f'(0) - 1 = 0$. By using argument, we can write the condition (1.4) as

$$\left|arg\frac{f'(z)}{g'(z)}\right| < \frac{\alpha\pi}{2}\ , \alpha > 0, z \in U. \tag{1.5}$$

We note that $C(\alpha) \subset S^*(\alpha) \subset K(\alpha)$.

Definition(1.1.11)[6]:A Möbius transformation, or a bilinear transformation, is a rational function $f: \mathbb{C} \to \mathbb{C}$ of the form

$$f(z) = \frac{az + b}{cz + d}\ ,$$

where$a, b, c, d \in \mathbb{C}$are fixed and $ad - bc \neq 0$.

Example(1.1.3)[5]: Perhaps the most important member of$\mathcal{A}$ is the Koebe function which is given by

$$k(z) = \frac{z}{(1-z)^2} = z + 2z^2 + 3z^3 + \cdots,$$

and maps the unit disk to the complement of the ray $\left(-\infty, -\frac{1}{4}\right]$. This can be verified by writing

$$k(z) = \frac{1}{4}\left(\frac{1+z}{1-z}\right)^2 - \frac{1}{4},$$

and noting that $\frac{1+z}{1-z}$maps the unit disk conformally onto the right half- plane $\{Re\{z\} > 0\}$; see Fig. (1.1.1).

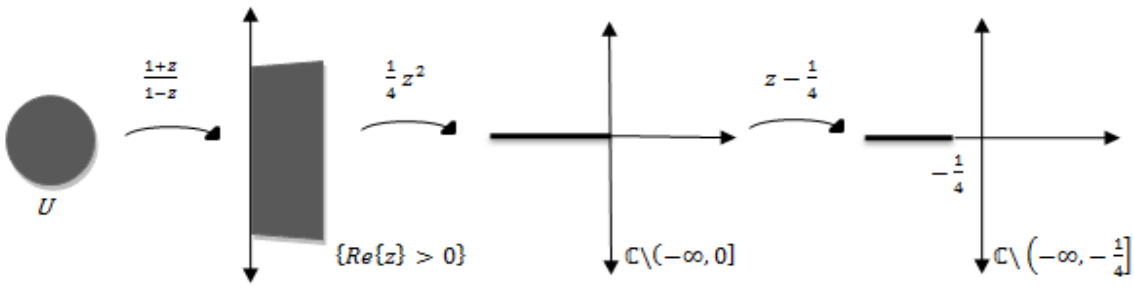

Fig. (1.1.1): The Koebe function maps $\mathbb{U}$conformally onto $\mathbb{C}\backslash\left(-\infty, -\frac{1}{4}\right]$.

We note that $x_1(z) = \frac{1+z}{1-z}$, $\quad x_2(z) = \frac{1}{4}x_1^2(z)$, $\quad x_3(z) = x_2(z) - \frac{1}{4}$.

Now

$$x_3 \circ x_2 \circ x_1(z) = \frac{1}{4}\left(\frac{1+z}{1-z}\right)^2 - \frac{1}{4} = \frac{z}{(1-z)^2}.$$

And x_1Möbius transformation that maps $\mathbb{U}$ onto the right half-plane whose boundary is the imaginary axis. Also, x_2 is the squaring function, while x_3 translates the image one space to the left and then multiplies it by a factor of$\frac{1}{4}$.

Note that the Koebe function is starlike, but not convex.

Definition (1.1.12)[5]: Let $\mathcal{A}(p)$ denote the class of analytic p-valently functions in U of the form:

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n, (z \in U, p \in \mathbb{N} = \{1, 2, \dots\}). \tag{1.6}$$

We say that$f$ is p-valently starlike of orderα, p-valently convex of order α, and p-valently close - to - convex of orderα $(0 \leq \alpha < p)$, respectively if and only if :

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \tag{1.7}$$

$$Re\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\alpha \tag{1.8}$$

$$Re\left\{\frac{f'(z)}{z^{p-1}}\right\}>\alpha. \tag{1.9}$$

Definition (1.1.13)[5]: Let us denote by $\mathcal{A}^*(p)$ the class of meromorphic function f of the form:

$$f(z)=z^{-p}+\sum_{n=p}^{\infty}a_n z^n, \qquad p\in\mathbb{N} \tag{1.10}$$

which are meromorphic and p-valent in the punctured unit disk $U^*=\{z\in\mathbb{C}: 0<|z|<1\}=U-\{0\}$. We say that f is p-valently meromorphic starlike of order α $(0\le\alpha<p)$ if and only if

$$Re\left\{-\frac{zf'(z)}{f(z)}\right\}>\alpha \text{ for } z\in U^*. \tag{1.11}$$

Also,f is p-valently meromorphic convex of order α $(0\le\alpha<p)$ if and only if

$$Re\left\{-\left(1+\frac{zf''(z)}{f'(z)}\right)\right\}>\alpha, \qquad z\in U^*. \tag{1.12}$$

Note that if $p=1$, we have defined univalent meromorphic starlike of order $\alpha(0\le\alpha<1)$, univalent meromorphic convex of order $\alpha(0\le\alpha<1)$ respectively. Denoted by $\mathcal{A}^*(1)$ the class of univalent and meromorphic functions in U^*.

Definition (1.1.14)[5]: Radius of starlikeness of a function f is the largest $R_1, 0<R_1<1$for which it is starlike in $|z|<R_1$.

Definition (1.1.15)[5]: Radius of convexity of a function f is the largest $R_2, 0<R_2<1$ for which it is convex in $|z|<R_2$.

Definition (1.1.16)[9]: The convolution (or Hadamard product) of the functions$f$and g denoted by $f * g$is defined as following for the functions in $\mathcal{A}(p)$ and $\mathcal{A}^*(p)$respectively:

(i) If

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n, \qquad g(z) = z^p + \sum_{n=p+1}^{\infty} b_n z^n,$$

then

$$(f * g)(z) = z^p + \sum_{n=p+1}^{\infty} a_n b_n z^n.$$

(ii) If

$$f(z) = z^{-p} + \sum_{n=p}^{\infty} a_n z^n, \qquad g(z) = z^{-p} + \sum_{n=p}^{\infty} b_n z^n,$$

then

$$(f * g)(z) = z^{-p} + \sum_{n=p}^{\infty} a_n b_n z^n.$$

and if $p = 1$ in (i), then the convolution (or Hadamard product) for the functions in $\mathcal{A}$. Also, if $p = 1$ in (ii), then the convolution (or Hadamard product) for the functions in $\mathcal{A}^*(1)$.

Definition (1.1.17)[5]: The weighted mean $E_q(z)$ of f and g defined by

$$E_q(z) = \frac{1}{2}[(1-q)f(z) + (1+q)g(z)], \qquad 0 < q < 1.$$

Also,

$$h(z) = \frac{1}{m}\sum_{k=1}^{m} f_k(z),$$

is the arithmetic mean of $f_k(z)(k = 1,2,3,\dots,m)$.

Definition (1.1.18)[8]: Let X be a topological vector space over the field $\mathbb{C}$ and let E be a subset of X. A point $x \in E$ is called an extreme point of$E$ if it has no representation of the form $x = ty + (1-t)z, 0 < t < 1$ as a proper convex combination of two distinct points y and z in E.

1.2 Standard Results

The following lemmas and theorems are essential and has been used in the proofs of the our principal results in the next chapter.

Lemma (1.2.1)[3]: Let $\alpha \geq 0$. Then, $Re(w) > \alpha$ if and only if $|w-(1+\alpha)| < |w+(1-\alpha)|$, where w be any complex number.

Theorem (1.2.1)[5]: (Distortion Theorem)

For each $f \in \mathcal{A}$

$$\frac{1-r}{(1+r)^3} \leq |f'(z)| \leq \frac{1+r}{(1-r)^3}, |z| = r < 1. \tag{1.13}$$

For each $z \in U, z \neq 0$ equality occurs if and only if $f$is a suitable rotation of the Koebe function. We say upper and lower bounds for $|f'(z)|$ as Distortion bounds.

Theorem (1.2.2)[5]: (Growth Theorem)

For each $f \in \mathcal{A}$

$$\frac{r}{(1+r)^2} \leq |f(z)| \leq \frac{r}{(1-r)^2}, |z| = r < 1. \tag{1.14}$$

For each $z \in U, z \neq 0$ equality occurs if and only if $f$is a suitable rotation of the Koebe function.

Theorem (1.2.3)[5]: (Maximum Modulus Theorem)

Suppose that a function f is continuous on boundary of $\mathbb{U}$($\mathbb{U}$ any disk or region). Then, the maximum value of $|f(z)|$, which is always reached , occurs somewhere on the boundary of $\mathbb{U}$and never in the interior.

Chapter Two

On a Certain Subclass of Multivalent Functions

2.1: Introduction

Let A(p) indicate the class of functions of the form:

$$f(Z) = Z^p + \sum_{n=p+1}^{\infty} a_n Z^n \ (Z \in U, p \in N = \{1,2, \dots.\}), (2.1)$$

Which are analytic and multivalent in the open unit disk

$$U = \{Z \in \mathbb{C}: |Z| < 1\}$$

Let M_p denote the subclass of A(p) containing of function of the form:

$$f(Z) = Z^p - \sum_{n=p+1}^{\infty} a_n Z^n \ (a_n \geq 0, p \in N = \{1,2, \dots.\}), (2.2)$$

Which are analytic and multivalent in the open unit disk U.

Definition (2.1)[7]: let $\sigma, \beta, m \in N, \sigma \geq 0, \beta \geq 0, m \geq 0,$

$p \in N$ and

$$f(Z) = Z^p + \sum_{k=p+1}^{\infty} a_k Z^k .$$

Then, we wefine the linear operator

$D_{p,m}^{\sigma,\beta} : A(p) \to A(p)$ by

$$D_{p,m}^{\sigma,\beta} f(Z) = Z^p + \sum_{n=p+1}^{\infty} \left(1 + \frac{(n-p)\sigma}{p+\beta}\right)^m a_n Z^n,$$

$$Z \in U . \quad (2.3)$$

With the help of the integral operator we define the class $MA(\lambda, \alpha, \sigma, \beta, m, p)$.

Definition (2.1): A function $f \in M_p$ is said to be in the class $MA(\lambda, \alpha, \sigma, \beta, m, p)$ if and only if

$$\mathrm{Re}\left\{\frac{\left(D_{p,m}^{\sigma,\beta} f(z)\right)' + z\left(D_{p,m}^{\sigma,\beta} f(z)\right)''}{\left(D_{p,m}^{\sigma,\beta} f(z)\right)' + \lambda z\left(D_{p,m}^{\sigma,\beta} f(z)\right)''}\right\} \geq \alpha , \quad (2.4)$$

Where $p \in N$, $o \leq \alpha < \frac{p^2}{\lambda p^2 - \lambda p + p}$, $\sigma \geq 0, \beta \geq o$ and $m \geq 0$

Some of the following properties studied for other classes in [1,2,4] .

2.2: Coefficient bounds

The following theorem gives a necessary and sufficient condition for function to be in the class $MA(\lambda, \alpha, \sigma, \beta, m, p)$.

Theorem (2.1):

Let $f(z) \in M_p$. Then $f(z) \in MA(\lambda, \alpha, \sigma, \beta, m, p)$ if and only if

$$\sum_{n=p+1}^{\infty} \left(1 + \frac{(n-p)\sigma}{p+\beta}\right)^m \left(n^2 - \alpha(\lambda n^2 - \lambda n + n)\right) a_n$$

$$\leq p^2 - \alpha(\lambda p^2 - \lambda p + p), \quad (2.5)$$

Where $p \in N$, $0 \leq \alpha < \frac{p^2}{\lambda p^2 - \lambda p + p}$, $\sigma \geq 0, \beta \geq 0$ and $m \geq 0$.

The result is sharp for the function

$$f(z) = z^p - \frac{p^2 - \alpha(\lambda p^2 - \lambda p + p)}{\left(1 + \frac{(n-p)\sigma}{p+\beta}\right)^m \left(n^2 - \alpha(\lambda n^2 - \lambda n + n)\right)} z^p, (n \geq p+1 \,;\, p \in N) \quad (2.6)$$

Proof: Assume that $f(z) \in MA(\lambda, \alpha, \sigma, \beta, m, p)$, so we have

$$\mathrm{Re}\left\{\frac{\left(D_{p,m}^{\sigma,\beta} f(z)\right)' + z\left(D_{p,m}^{\sigma,\beta} f(z)\right)''}{\left(D_{p,m}^{\sigma,\beta} f(z)\right)' + \lambda z\left(D_{p,m}^{\sigma,\beta} f(z)\right)''}\right\} \geq \alpha$$

$$\mathrm{Re}\left\{\frac{p^2 z^{p-1} - \sum_{n=p+1}^{\infty} n^2 \left(1 + \frac{(n-p)\sigma}{p+\beta}\right)^m a_n z^{n-1}}{(\lambda p^2 - \lambda p + p) z^{p-1} - \sum_{n=p+1}^{\infty} \left(1 + \frac{(n-p)\sigma}{p+\beta}\right)^m (\lambda n^2 - \lambda n + n) z^{n-1}}\right\}$$

$\geq \alpha$

or equivalently

$$\operatorname{Re}\left\{\frac{\left(p^2-\alpha(\lambda p^2-\lambda p+p)\right)z^{p-1}-\sum_{n=p+1}^{\infty}\left(1+\frac{(n-p)\sigma}{p+\beta}\right)^m\left(n^2-\alpha(\lambda n^2-\lambda n+n)\right)a_n z}{(\lambda p^2-\lambda p+p)z^{p-1}-\sum_{n=p+1}^{\infty}\left(1+\frac{(n-p)\sigma}{p+\beta}\right)^m(\lambda n^2-\lambda n+n)z^{n-1}}\right.$$

≥ 0 .

This inequality is correct for all $z \in U$. letting $z \to 1^-$ yields

$$\operatorname{Re}\left\{\left(p^2-\alpha(\lambda p^2-\lambda p+p)\right)\right.$$

$$\left.-\sum_{n=p+1}^{\infty}\left(1+\frac{(n-p)\sigma}{p+\beta}\right)^m\left(n^2-\alpha(\lambda n^2-\lambda n+n)\right)a_n\right\}$$

$$\geq 0 .$$

Therefore

$$\sum_{n=p+1}^{\infty}\left(1+\frac{(n-p)\sigma}{p+\beta}\right)^m\left(n^2-\alpha(\lambda n^2-\lambda n+n)\right)a_n$$

$$\leq p^2-\alpha(\lambda p^2-\lambda p+p) .$$

Conversely , let (2.5) hold. We will prove that (2.4) is correct and then $f(z) \in MA(\lambda,\alpha,\sigma,\beta,m,p)$.

By lemma (1.2.1) it is enough , show that

$|w-(p+\alpha)| < |w+(p-\alpha)|$ where

$$W=\frac{\left(D_{p,m}^{\sigma,\beta}f(z)\right)'+z\left(D_{p,m}^{\sigma,\beta}f(z)\right)''}{\left(D_{p,m}^{\sigma,\beta}f(z)\right)'+\lambda z\left(D_{p,m}^{\sigma,\beta}f(z)\right)''}$$

or show that

$$T=\frac{1}{|N(z)|}\left|\left(D_{p,m}^{\sigma,\beta}f(z)\right)'+z\left(D_{p,m}^{\sigma,\beta}f(z)\right)''-(p+\alpha)\left(D_{p,m}^{\sigma,\beta}f(z)\right)'\right.$$

$$\left.-(p+\alpha)\left(D_{p,m}^{\sigma,\beta}f(z)\right)''\right|$$

$$< \frac{1}{|N(z)|}\left|\left(D_{p,m}^{\sigma,\beta} f(z)\right)' + z\left(D_{p,m}^{\sigma,\beta} f(z)\right)'' + (p-\alpha)\left(D_{p,m}^{\sigma,\beta} f(z)\right)' + (p-\alpha)\lambda z\left(D_{p,m}^{\sigma,\beta} f(z)\right)''\right| = Q,$$

where $N(z) = \left(D_{p,m}^{\sigma,\beta} f(z)\right)' + \lambda z\left(D_{p,m}^{\sigma,\beta} f(z)\right)''$ and it is easy to verify that $Q - T > 0$ and so the proof is complete.

Finally, sharpness follows if we take

$$f(z) = z^p - \frac{p^2 - \alpha(\lambda p^2 - \lambda p + p)}{\left(1 + \frac{(n-p)\sigma}{p+\beta}\right)^m \left(n^2 - \alpha(\lambda n^2 - \lambda n + n)\right)} z^n, (n \geq p+1;\ p \in N\ ).$$

Corollary (2.1): Let $f \in MA(\lambda, \alpha, \sigma, \beta, m, p)$. Then

$$a_n \leq \frac{p^2 - \alpha(\lambda p^2 - \lambda p + p)}{\left(1 + \frac{(n-p)\sigma}{p+\beta}\right)^m \left(n^2 - \alpha(\lambda n^2 - \lambda n + n)\right)}, (n \geq p+1; p \in N\ ). \quad (2.7)$$

2.3: Distortion and growth theorems

We introduce here the distortion and growth theorems for the functions in the class $MA(\lambda, \alpha, \sigma, \beta, m, p)$.

Theorem (2.2): Let the function $f(z)$ defined by (2.2) be in the class $MA(\lambda, \alpha, \sigma, \beta, m, p)$. Then, for $|z| = r(0 < r < 1)$

$$r^p - \frac{\left(p^2 - \alpha(\lambda p^2 - \lambda p + p)\right) r^{p+1}}{\left(1 + \frac{\sigma}{p+\beta}\right)^m \left((p+1)^2 - \alpha(\lambda(p+1)^2 - \lambda(p+1) + (p+1))\right)} \leq |f(z)|$$

$$\leq r^p + \frac{\left(p^2 - \alpha(\lambda p^2 - \lambda p + p)\right) r^{p+1}}{\left(1 + \frac{\sigma}{p+\beta}\right)^m \left((p+1)^2 - \alpha\left(\lambda(p+1)^2 - \lambda(p+1) + (p+1)\right)\right)}, (2.8)$$

for $z \in U$. The result (2.8) is sharp .

Proof: Since $f(z) \in MA(\lambda, \alpha, \sigma, \beta, m, p)$, in view of theorem (2.1), we have

$$\left(1+\frac{\sigma}{p+\beta}\right)^{m}\left((p+1)^{2}-\alpha\left(\lambda(p+1)^{2}-\lambda(p+1)+(p+1)\right)\right)$$

$$\sum_{n=p+1}^{\infty} a_n$$

$$\leq \sum_{n=p+1}^{\infty}\left(1+\frac{(n-p)\sigma}{p+\beta}\right)^{m}\left(n^{2}-\alpha(\lambda n^{2}-\lambda n+n)\right)a_n$$

$$\leq p^{2}-\alpha(\lambda p^{2}-\lambda p+p)\ ,$$

which immediately yields

$$\sum_{n=p+1}^{\infty} a_n$$

$$\leq \frac{p^{2}-\alpha(\lambda p^{2}-\lambda p+p)}{\left(1+\frac{\sigma}{p+\beta}\right)^{m}\left((p+1)^{2}-\alpha\left(\lambda(p+1)^{2}-\lambda(p+1)+(p+1)\right)\right)}$$

Consequently, for $|z| = r\ (0 < r < 1)$, we obtain

$$|f(z)| \leq r^{p}+r^{p+1}\sum_{n=p+1}^{\infty} a_n$$

$$\leq r^{p}+\frac{\left(p^{2}-\alpha(\lambda p^{2}-\lambda p+p)\right)r^{p+1}}{\left(1+\frac{\sigma}{p+\beta}\right)^{m}\left((p+1)^{2}-\alpha\left(\lambda(p+1)^{2}-\lambda(p+1)+(p+1)\right)\right)}$$

and

$$|f(z)| \geq r^{p}-r^{p+1}\sum_{n=p+1}^{\infty} a_n$$

$$\geq r^{p}-\frac{\left(p^{2}-\alpha(\lambda p^{2}-\lambda p+p)\right)r^{p+1}}{\left(1+\frac{\sigma}{p+\beta}\right)^{m}\left((p+1)^{2}-\alpha\left(\lambda(p+1)^{2}-\lambda(p+1)+(p+1)\right)\right)}$$

This completes the proof of theorem (2.2). Finally, by taking the function.

$$f(z) = z^p - \frac{p^2-\alpha(\lambda p^2-\lambda p+p)}{\left(1+\frac{\sigma}{p+\beta}\right)^m\left((p+1)^2-\alpha\left(\lambda(p+1)^2-\lambda(p+1)+(p+1)\right)\right)} z^{p+1} \ , \ (2.9)$$

we can show that the result of theorem (2.2) is sharp.

Theorem (2.3): If $f(z) \in MA(\lambda, \alpha, \sigma, \beta, m, p)$, then

$$pr^{p-1} - \frac{(p+1)\left(p^2-\alpha(\lambda p^2-\lambda p+p)\right)r^p}{\left(1+\frac{\sigma}{p+\beta}\right)^m\left((p+1)^2-\alpha\left(\lambda(p+1)^2-\lambda(p+1)+(p+1)\right)\right)} \leq \left|f^{/}(z)\right|$$

$$\leq pr^{p-1} + \frac{(p+1)\left(p^2-\alpha(\lambda p^2-\lambda p+p)\right)r^p}{\left(1+\frac{\sigma}{p+\beta}\right)^m\left((p+1)^2-\alpha\left(\lambda(p+1)^2-\lambda(p+1)+(p+1)\right)\right)} .$$

The result is sharp for the function f is given by (2.9)

Proof: The proof is similar to that of theorem (2.2).

2.4: Radii of starlikeness, convexity and close – to - convexity

Using the inequalities (1.7) , (1.8) , (1.9) and theorem (2.1), we can compute the radii of starlikeness, convexity and close – to – convexity.

Theorem (2.4): Let $f(z) \in MA(\lambda, \alpha, \sigma, \beta, m, p)$. Then $f(z)$ is p – valently starlike of order $\rho(0 \leq \rho < p)$ in the disk $|z| < R_1$, where

$$R_1 = \inf_n \left\{\frac{(p-\rho)\left(1+\frac{(n-p)\sigma}{p+\beta}\right)^m\left(n^2-\alpha(\lambda n^2-\lambda n+1)\right)}{(n-\rho)\left(p^2-\alpha(\lambda p^2-\lambda p+p)\right)}\right\}^{\frac{1}{n-p}} , (n \geq p+1; p \in N) .$$

The result is sharp for the function $f(z)$ given by (2.6).

Proof: It is sufficient to show that

$$\left|\frac{zf^{/}(z)}{f(z)} - p\right| \leq p - \rho \ (0 \leq \rho < p),$$

for $|z| < R_1$, we have

$$\left|\frac{zf^{/}(z)}{f(z)}-p\right| \leq \frac{\sum_{n=p+1}^{\infty}(n-p)a_n\,|z|^{n-p}}{1-\sum_{n=p+1}^{\infty}a_n\,|z|^{n-p}}\,.$$

Thus

$$\left|\frac{zf^{/}(z)}{f(z)}-p\right| \leq p-\rho\,,$$

if

$$\sum_{n=p+1}^{\infty}\frac{(n-\rho)}{(p-\rho)}a_n\,|z|^{n-p} \leq 1\quad . \qquad (2.10)$$

Hence, by Theorem (2.1), (2.10) will be true if

$$\frac{(n-\rho)}{(p-\rho)}|z|^{n-p} \leq \frac{\left(1+\frac{(n-p)\sigma}{p+\beta}\right)^{m}\left(n^2-\alpha(\lambda n^2-\lambda n+1)\right)}{p^2-\alpha(\lambda p^2-\lambda p+p)}$$

and hence

$$|z| \leq \left\{\frac{(p-\rho)\left(1+\frac{(n-p)\sigma}{p+\beta}\right)^{m}\left(n^2-\alpha(\lambda n^2-\lambda n+1)\right)}{(n-\rho)\left(p^2-\alpha(\lambda p^2-\lambda p+p)\right)}\right\}^{\frac{1}{n-p}}, (n \geq p+1; p \in N)\,.$$

Setting $|z| = R_1$, we get the desired result.

Theorem (2.5): Let $f(z) \in MA(\lambda,\alpha,\sigma,\beta,m,p)$. Then f is p – valently convex of order $\rho(0 \leq \rho < p)$ in the disk $|z| < R_2$, where

$$R_2 = \inf_n\left\{\frac{(p-\rho)\left(1+\frac{(n-p)\sigma}{p+\beta}\right)^{m}(n-\lambda\alpha\, n+\lambda\alpha-\alpha)}{(n-\rho)(p-\lambda\alpha\, p+\lambda\alpha-\alpha)}\right\}^{\frac{1}{n-p}}, (n \geq p+1; p \in N)\,.$$

The result is sharp with the extermal function f given by (2.6).

Proof: It is sufficient to show that

$$\left|1+\frac{zf^{//}(z)}{f^{/}(z)}-p\right| \leq p-\rho \quad (0 \leq \rho < p)\,,$$

for $|z| < R_2$, we have

$$\left|1+\frac{zf^{//}(z)}{f^{/}(z)}-p\right| \leq \frac{\sum_{n=p+1}^{\infty} n(n-p)a_n\,|z|^{n-p}}{p-\sum_{n=p+1}^{\infty} na_n\,|z|^{n-p}}$$

Thus

$$\left|1+\frac{zf^{//}(z)}{f^{/}(z)}-p\right| \leq p-\rho \ ,$$

if

$$\sum_{n=p+1}^{\infty} \frac{n(n-\rho)}{p(p-\rho)} a_n\,|z|^{n-p} \leq 1 \quad . \qquad (2.11)$$

Hence, by Theorem (2.1), (2.11) will be true if

$$\frac{(n-\rho)}{(p-\rho)}|z|^{n-p} \leq \frac{\left(1+\frac{(n-p)\sigma}{p+\beta}\right)^{m}(n-\lambda\alpha\, n+\lambda\alpha-\alpha)}{(p-\lambda\alpha\, p+\lambda\alpha-\alpha)} \ ,$$

and hance

$$|z| \leq \left\{\frac{(p-\rho)\left(1+\frac{(n-p)\sigma}{p+\beta}\right)^{m}(n-\lambda\alpha\, n+\lambda\alpha-\alpha)}{(n-\rho)(p-\lambda\alpha\, p+\lambda\alpha-\alpha)}\right\}^{\frac{1}{n-p}} \ ,(n \geq p+1; p \in N\,) \ .$$

Setting $|z| = R_2$, we get the desired result.

Theorem (2.6): Let a function $f(z) \in MA(\lambda, \alpha, \sigma, \beta, m, p)$. Then f is p – valently close – to convex of order ρ $(0 \leq \rho < p)$ in the disk $|z| < R_3$, where

$$R_3 = \inf_n \left\{\frac{(p-\rho)\left(1+\frac{(n-p)\sigma}{p+\beta}\right)^{m}(n-\lambda\alpha\, n+\lambda\alpha-\alpha)}{p(p-\lambda\alpha\, p+\lambda\alpha-\alpha)}\right\}^{\frac{1}{n-p}} \ ,(n \geq p+1; p \in N\,) \ .$$

The result is sharp, with the extermal function f(z) given by (2.6) .

Proof: It is sufficient to show that

$$\left|\frac{f^{/}(z)}{z^{p-1}}-p\right| \leq p-\rho \ (0 \leq \rho < p),$$

for $|z| < R_3$, we have that

$$\left|\frac{f^{/}(z)}{z^{p-1}} - p\right| \leq \sum_{n=p+1}^{\infty} na_n |z|^{n-p} \ .$$

Thus

$$\left|\frac{f^{/}(z)}{z^{p-1}} - p\right| \leq p - \rho \ ,$$

if

$$\sum_{n=p+1}^{\infty} \frac{na_n |z|^{n-p}}{p-\rho} \leq 1 \quad . \qquad (2.12)$$

Hence, by Theorem (2.1) , (2.12) will be true if

$$\frac{1}{(p-\rho)} |z|^{n-p} \leq \frac{\left(1 + \frac{(n-p)\sigma}{p+\beta}\right)^m (n - ʎ\alpha\, n + ʎ\alpha - \alpha)}{p(p - ʎ\alpha\, p + ʎ\alpha - \alpha)} \ ,$$

and hence

$$|z| \leq \left\{\frac{(p-\rho)\left(1 + \frac{(n-p)\sigma}{p+\beta}\right)^m (n - ʎ\alpha\, n + ʎ\alpha - \alpha)}{p(p - ʎ\alpha\, p + ʎ\alpha - \alpha)}\right\}^{\frac{1}{n-p}} , (n \geq p+1; p \in N) \ .$$

Setting $|z| = R_3$, we get the desired result.

2.5: Weighted mean and arithmetic mean

Theorem (2.7): Let f and g be in the class $MA(ʎ, \alpha, \sigma, \beta, m, p)$. Then the weighted mean of f and g is also in the class $MA(ʎ, \alpha, \sigma, \beta, m, p)$.

Proof: By Definition (1.1.17) , we have

$$E_q(z) = \frac{1}{2}[(1-q)f(z) + (1+q)g(z)]$$

$$= \frac{1}{2}\left[(1-q)\left(z^p - \sum_{n=p+1}^{\infty} a_n z^n\right) + (1+q)\left(z^p - \sum_{n=p+1}^{\infty} b_n z^n\right)\right]$$

$$= z^p - \sum_{n=p+1}^{\infty} \frac{1}{2}[(1-q)a_n + (1+q)b_n]z^n .$$

Since f and g are in the class $MA(\lambda, \alpha, \sigma, \beta, m, p)$ so by Theorem (2.1), we get

$$\sum_{n=p+1}^{\infty} \left(1 + \frac{(n-p)\sigma}{p+\beta}\right)^m \left(n^2 - \alpha(\lambda n^2 - \lambda n + n)\right)a_n$$
$$\leq p^2 - \alpha(\lambda p^2 - \lambda p + p)$$

and

$$\sum_{n=p+1}^{\infty} \left(1 + \frac{(n-p)\sigma}{p+\beta}\right)^m \left(n^2 - \alpha(\lambda n^2 - \lambda n + n)\right)b_n$$
$$\leq p^2 - \alpha(\lambda p^2 - \lambda p + p) ,$$

hence

$$\sum_{n=p+1}^{\infty} \left(1 + \frac{(n-p)\sigma}{p+\beta}\right)^m \left(n^2 - \alpha(\lambda n^2 - \lambda n + n)\right)\left(\frac{1}{2}(1-q)a_n + \frac{1}{2}(1+q)b_n\right)$$
$$= \frac{1}{2}(1-q) \sum_{n=p+1}^{\infty} \left(1 + \frac{(n-p)\sigma}{p+\beta}\right)^m \left(n^2 - \alpha(\lambda n^2 - \lambda n + n)\right) a_n + \frac{1}{2}(1 + q) \sum_{n=p+1}^{\infty} \left(1 + \frac{(n-p)\sigma}{p+\beta}\right)^m \left(n^2 - \alpha(\lambda n^2 - \lambda n + n)\right) b_n$$
$$\leq \frac{1}{2}(1-q)\left(p^2 - \alpha(\lambda p^2 - \lambda p + p)\right) + \frac{1}{2}(1+q)\left(p^2 - \alpha(\lambda p^2 - \lambda p + p)\right)$$
$$= \left(p^2 - \alpha(\lambda p^2 - \lambda p + p)\right) .$$

This shows $E_q \in MA(\lambda, \alpha, \sigma, \beta, m, p)$.

In the following theorem , we shall prove that the class $MA(\lambda, \alpha, \sigma, \beta, m, p)$ is closed under arithmetic mean.

Theorem (2.8): let $f_1(z), f_2(z), \dots, f_s(z)$ defined by

$f_k(z) = z^p - \sum_{n=p+1}^{\infty} a_n z^n$, $\left(a_{n,k} \geq 0 , k = 1,2, \dots, s, \quad n \geq p + 1\right)$ (2.13)

be in the class $MA(\lambda, \alpha, \sigma, \beta, m, p)$. Then the arithmetic mean of $f_k(z)$ $(k = 1,2, \dots, s)$ defined by

$$h(z) = \frac{1}{s}\sum_{k=1}^{s} f_k(z) ,$$

is also in the class $MA(\lambda, \alpha, \sigma, \beta, m, p)$.

Proof: by (2.13) and (2.14) , we can write

$$h(z) = \frac{1}{s}\sum_{k=1}^{s}(z^p - \sum_{n=p+1}^{\infty} a_{n,k} z^n) = z^p - \sum_{n=p}^{\infty} (\frac{1}{s}\sum_{k=1}^{s} a_{n,k}) z^n .$$

Since $f_k(z) \in MA(\lambda, \alpha, \sigma, \beta, m, p)$ for every $k = 1,2, \dots, s$, So by using theorem (2.1), we prove that

$$\sum_{n=p+1}^{\infty} \left(1 + \frac{(n-p)\sigma}{p+\beta}\right)^m \left(n^2 - \alpha(\lambda n^2 - \lambda n + n)\right) (\frac{1}{s}\sum_{k=1}^{s} a_{n,k})$$

$$= \frac{1}{s}\sum_{k=1}^{\infty}\left(\sum_{n=p+1}^{\infty} \left(1 + \frac{(n-p)\sigma}{p+\beta}\right)^m \left(n^2 - \alpha(\lambda n^2 - \lambda n + n)\right) a_{n,k}\right)$$

$$\leq \frac{1}{s}\sum_{k=1}^{s}\left(p^2 - \alpha(\lambda p^2 - \lambda p + p)\right) = p^2 - \alpha(\lambda p^2 - \lambda p + p) .$$

This ends the proof of theorem (2.8).

2.6: Extreme points

In the following theorem, we obtain the extreme points of the class $MA(\lambda, \alpha, \sigma, \beta, m, p)$.

Theorem (2.9): Let $f_p(z) = z^p$ and

$$f_n(z) = z^p - \frac{p^2 - \alpha(\lambda p^2 - \lambda p + p)}{\left(1 + \frac{(n-p)\sigma}{p+\beta}\right)^m \left(n^2 - \alpha(\lambda n^2 - \lambda n + n)\right)} z^n , \quad (2.15)$$

where $p \in N$, $0 \le \alpha < \frac{p^2}{\lambda p^2 - \lambda p + p}$, $\sigma \ge 0$, $\beta \ge 0$ and $m \ge 0$.

Then the function f is in the class $MA(\lambda, \alpha, \sigma, \beta, m, p)$ if and only if it can be expressed in the form:

$$f(z) = \theta_p z^p + \sum_{n=p+1}^{\infty} \theta_n f_n(z) , \qquad (2.16)$$

where $\left(\theta_p \ge 0,\ \theta_n \ge 0, n \ge p+1\right)$and $\theta_p + \sum_{n=p+1}^{\infty} \theta_n = 1$.

Proof: Suppose that f is expressed in the form (2.16). then

$$f(z) = \theta_p z^p + \sum_{n=p+1}^{\infty} \theta_n \left[z^p - \frac{p^2 - \alpha(\lambda p^2 - \lambda p + p)}{\left(1 + \frac{(n-p)\sigma}{p+\beta}\right)^m \left(n^2 - \alpha(\lambda n^2 - \lambda n + n)\right)} z^n \right]$$

$$= z^p - \sum_{n=p+1}^{\infty} \frac{p^2 - \alpha(\lambda p^2 - \lambda p + p)}{\left(1 + \frac{(n-p)\sigma}{p+\beta}\right)^m \left(n^2 - \alpha(\lambda n^2 - \lambda n + n)\right)} \theta_n z^n$$

hance

$$\sum_{n=p+1}^{\infty} \frac{\left(1 + \frac{(n-p)\sigma}{p+\beta}\right)^m \left(n^2 - \alpha(\lambda n^2 - \lambda n + n)\right)}{p^2 - \alpha(\lambda p^2 - \lambda p + p)} \times \frac{p^2 - \alpha(\lambda p^2 - \lambda p + p)\theta_n}{\left(1 + \frac{(n-p)\sigma}{p+\beta}\right)^m \left(n^2 - \alpha(\lambda n^2 - \lambda n + n)\right)}$$

$$= \sum_{n=p+1}^{\infty} \theta_n = 1 - \theta_p \le 1 .$$

Then $f \in MA(\lambda, \alpha, \sigma, \beta, m, p)$.

Conversely , suppose that $f \in MA(\lambda, \alpha, \sigma, \beta, m, p)$. We may set

$$\theta_n = \frac{\left(1 + \frac{(n-p)\sigma}{p+\beta}\right)^m \left(n^2 - \alpha(\lambda n^2 - \lambda n + n)\right)}{p^2 - \alpha(\lambda p^2 - \lambda p + p)} a_n ,$$

where a_n is given by (2.7). then

$$f(z) = z^p - \sum_{n=p+1}^{\infty} a_n z^n$$

$$= z^p - \sum_{n=p+1}^{\infty} \frac{p^2 - \alpha(\lambda p^2 - \lambda p + p)}{\left(1 + \frac{(n-p)\sigma}{p+\beta}\right)^m \left(n^2 - \alpha(\lambda n^2 - \lambda n + n)\right)} \theta_n z^n$$

$$= z^p - \sum_{n=p+1}^{\infty} [z^p - f_n(z)]\, \theta_n$$

$$= \left(1 - \sum_{n=p+1}^{\infty} \theta_n\right) z^p + \sum_{n=p+1}^{\infty} \theta_n f_n(z)$$

$$= \theta_p z^p + \sum_{n=p+1}^{\infty} \theta_n f_n(z) \ .$$

This completes the proof of theorem (2.9).

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