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On a Certain Subclass of Univalent Functions

A Research Submitted by

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بسم الله الرحمن الرحيم

(وَيَسْأَلُونَكَ عَنِ الرُّوحِ قُلِ الرُّوحُ مِنْ أَمْرِ رَبِّي وَمَا أُوتِيتُم مِّنَ الْعِلْمِ إِلَّا قَلِيلًا)

صدق الله العلي العظيم

سورة الأسراء آية (٨٥)

الأهداء

الى النور الذي ينير لي درب النجاح ويا من علماني الصمود مهما

تبدلت الظروف .... أبي وأمي

الى كل من علمني حرفا .... أساتذتي الأعزاء

وبالخصوص أستاذي العزيز الدكتور نجاح علي الزيادي الذي ساعدني

ووقف معي في العمل

Abstract

In this work we presented a certain class MH (α,β,m,η, σ_1 , σ_2, Λ) of univalent analytic function with generalized operator $I_{\alpha,B}^{m}$ in the open unit disk U.We obtained many geometric properties , like , coefficient inequality , distortion and growth theorems , radii of starlikeness , convexity and close - to - convexity , extreme points , closure theorems .

CONTAINS

Chapter One

Basic Definitions and Standard Results

In this chapter, we list out all the definitions of the family of functions from analytic, univalent and multivalent (p–valent) and all related terms used during the investigation. We also include in this chapter all the standard theorems and lemmas used in the work.

Section 1

1.1 Basic Definitions

Definition (1.1.1)[7]: A function f of the complex variable is analytic at a point z_0 if its derivative exists not only at z_0 but each point z in some neighborhoods of z_0. It is analytic in region $\mathbb{U}$ if it is analytic at every point in $\mathbb{U}$.

Definition (1.1.2)[7]: A function f is said to be univalent if it does not take the same value twice i.e. $f(z_1) \neq f(z_2)$ for all pairs of distinct points $z_1, z_2 \in U$. In other words, f is one – to – one (or injective) mapping of U onto another domain.

If f assumes the same value more than one, then f is said to be multivalent (p-valent) in U.

Definition (1.1.3)[7]: Let $\mathcal{A}$ denotes the class of functions f of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad n \in \mathbb{N} \tag{1.1}$$

which are analytic and univalent in the open unit disk U.

Definition (1.1.4)[7]: We say that $f \in \mathcal{A}$ is normalized if f satisfies the conditions $f(0) = 0$ and $f'(0) = 1$.

Definition (1.1.5)[7]: A set $E \subseteq \mathbb{C}$ is said to be starlike with respect to $w_0 \in E$ if the linear segment joining w_0 to every other point $w \in E$ lies entirely in E. In a more picturesque language, the requirement is that every point of E is visible from w_0. The set E is said to be convex if it is starlike with respect to each of its points, that is , if the linear segment joining any two points of E lies entirely in E.

Definition (1.1.6)[7]: A function f is said to be conformal at a point z_0 if it preserves the angle between oriented curves passing through z_0 in magnitude as well as in sense. Geometrically, images of any two oriented curves taken with their corresponding orientations make the same angle of intersection as the curves at z_0 both in magnitude and direction. A function $w = f(z)$ is said to be conformal in the domain D, if it is conformal at each point of the domain.

Definition (1.1.7)[7]: A function $f \in \mathcal{A}$ is said to be starlike function of order α if and only if

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, (\, 0 \leq \alpha < 1;\; z \in U). \tag{1.2}$$

Denotes the class of all starlike functions of order α in U by $S^*(\alpha)$ and S^* the class of all starlike functions of order 0, $S^*(0) = S^*$. Geometrically, we can say that a starlike function is conformal mapping of the unit disk onto a domain starlike with respect to the origin. For example, the function

$$f(z) = \frac{z}{(1-z)^{2(1-\alpha)}},$$

is starlike function of order α.

Definition (1.1.8)[7]: A function $f \in \mathcal{A}$ is said to be convex function of order α if and only if

$$Re\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \alpha, (\ 0 \le \alpha < 1;\ z \in U). \tag{1.3}$$

Denotes the class of all convex functions of order α in U by $C(\alpha)$ and C for the convex function $C(0) = C$.

Definition (1.1.9)[7]: A function f analytic in the unit disk U is said to be close – to – convex of order $\alpha(0 \le \alpha < 1)$ if there is a convex function g such that

$$Re\left\{\frac{f'(z)}{g'(z)}\right\} > \alpha, \qquad \forall\ z \in U. \tag{1.4}$$

We denote by $K(\alpha)$, the class of close – to – convex functions of order α, f is normalized by the usual conditions $f(0) = f'(0) - 1 = 0$. By using argument, we can write the condition (1.4) as

$$\left|arg\frac{f'(z)}{g'(z)}\right| < \frac{\alpha\pi}{2}\ , \alpha > 0, \forall\ z \in U. \tag{1.5}$$

We note that $C(\alpha) \subset S^*(\alpha) \subset K(\alpha)$.

Note that the Koebe function is starlike, but not convex where the Koebe function is given by the following:

$$K(z) = \frac{z}{(1-z)^2} = \sum_{n=1}^{\infty} nz^n = \frac{1}{4}\left(\frac{1+z}{1-z}\right)^2 - \frac{1}{4},$$

is the most famous function which maps U onto $\mathbb{C}$ minus a slit along the negative real axis from $-\frac{1}{4}$ to $-\infty$.

Definition (1.1.10)[7]: Let $\mathcal{A}(p)$ denote the class of analytic p–valently functions in U of the form:

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n, (z \in U, p \in \mathbb{N} = \{1, 2, \dots\}). \tag{1.6}$$

We say that f is p–valently starlike of order α, p–valently convex of order α, and p–valently close - to - convex of order $\alpha (0 \leq \alpha < p)$, respectively if and only if :

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, \qquad Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha, \qquad Re\left\{\frac{f'(z)}{z^{p-1}}\right\} > \alpha.$$

Definition (1.1.11)[7]: Let us denote by $\mathcal{A}_p^*$ the class of meromorphic function f of the form:

$$f(z) = z^{-p} + \sum_{n=p}^{\infty} a_n z^n, \qquad p \in \mathbb{N} \tag{1.7}$$

which are meromorphic and p–valent in the punctured unit disk $U^* = \{z \in \mathbb{C}: 0 < |z| < 1\} = U - \{0\}$. We say that f is p–valently meromorphic starlike of order $\alpha (0 \leq \alpha < p)$ if and only if

$$Re\left\{-\frac{zf'(z)}{f(z)}\right\} > \alpha \text{ for } \ z \in U^*. \tag{1.8}$$

Also, f is p–valently meromorphic convex of order $\alpha (0 \leq \alpha < p)$ if and only if

$$Re\left\{-\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} > \alpha, \qquad z \in U^*. \tag{1.9}$$

Definition (1.1.12)[7]: Radius of starlikeness of a function f is the largest $r_1, 0 < r_1 < 1$ for which it is starlike in $|z| < r_1$.

Definition (1.1.13)[7]: Radius of convexity of a function f is the largest $r_2, 0 < r_2 < 1$ for which it is convex in $|z| < r_2$.

Definition (1.1.14)[7]: The weighted mean h_j of f and g defined by

$$h_j(z) = \frac{1}{2}[(1-j)f(z) + (1+j)g(z)], \qquad 0 < j < 1.$$

Also,

$$h(z) = \frac{1}{m}\sum_{k=1}^{m} f_k(z),$$

is the arithmetic mean of $f_k(z)$ $(k = 1,2,3,\dots,m)$.

Definition (1.1.15)[10]: The convolution (or Hadamard product) for functions f and g denoted by $f * g$ is defined as following for the functions in $\mathcal{A}(p)$ and $\mathcal{A}_p^*$ respectively:

(i) If

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n, \qquad g(z) = z^p + \sum_{n=p+1}^{\infty} b_n z^n,$$

then

$$(f * g)(z) = z^p + \sum_{n=p+1}^{\infty} a_n b_n z^n. \tag{1.10}$$

(ii) If

$$f(z) = z^{-p} + \sum_{n=p}^{\infty} a_n z^n, \qquad g(z) = z^{-p} + \sum_{n=p}^{\infty} b_n z^n,$$

then

$$(f * g)(z) = z^{-p} + \sum_{n=p}^{\infty} a_n b_n z^n. \tag{1.11}$$

Definition (1.1.16)[9]: Let X be a topological vector space over the field $\mathbb{C}$ and let E be a subset of X. A point $x \in E$ is called an extreme point of E if it has no representation of the form $x = ty + (1-t)z, 0 < t < 1$ as a proper convex combination of two distinct points y and z in E.

Section 2

1.2 Basic Results

In this part, we mention some results which we have used in this research.

Theorem (1.2.4)[7]: (Distortion Theorem)

For each $f \in \mathcal{A}$

$$\frac{1-r}{(1+r)^3} \leq |f'(z)| \leq \frac{1+r}{(1-r)^3}\ , |z| = r < 1. \qquad (1.12)$$

For each $z \in U, z \neq 0$ equality occurs if and only if f is a suitable rotation of the Koebe function.

We say upper and lower bounds for $|f'(z)|$ as Distortion bounds.

Theorem (1.2.5)[7]: (Growth Theorem)

For each $f \in \mathcal{A}$

$$\frac{r}{(1+r)^2} \leq |f(z)| \leq \frac{r}{(1-r)^2}\ , |z| = r < 1. \qquad (1.13)$$

For each $z \in U, z \neq 0$ equality occurs if and only if f is a suitable rotation of the Koebe function.

Theorem (1.2.8)[7]: (Maximum Modulus Theorem)

Suppose that a function f is continuous on boundary of $\mathbb{U}$ ($\mathbb{U}$ any disk or region). Then, the maximum value of $|f(z)|$, which is always reached , occurs somewhere on the boundary of $\mathbb{U}$ and never in the interior.

Chapter Two

On a Certain Subclass of Univalent Functions

2.1: Introduction

Let A be the class of function of the form :

$f(z)$ =z+$\sum_{n=2}^{\infty} a_n z^n$, (2.1)

which are analytic and univalent in the open unit disk

U ={z∈ ℂ :| z| <1}.

Let At be subclass of A consisting of functions of the form :

$f(z)$ =z+$\sum_{n=2}^{\infty} a_n z^n$, (≥ 0) . (2.2)

For the function f ∈ AT given by (2.2) and g ∈ AT defined by

g (z) = z + $\sum_{n=2}^{\infty} b_n z^n$, $(b_n \geq 0)$, (2.3)

define the convolution (or Hadmard product) of f and g by

(f * g) (z) = z +$\sum_{n=2}^{\infty} a_n b_n z^n$= (g * f) (z) . (2.4)

For m ∈ N0 = NU {0} ,$\beta \geq 0$,$\alpha \in R$ with α + B > 0 and f ∈ A .

The generalized operator $I_{\alpha,\beta}^{m}$ (see [11]) is defined by.

$I_{\alpha,\beta}^{m} f$ (z) = z + $\sum_{n=z}^{\infty} \left(\frac{\alpha+n\beta}{\alpha+\beta}\right)^{m} a_n z^n$ (2.5)

Note that the generalized operator $I_{\alpha,B}^{m}$ unifies many operators of A . In particular :

1- $I_{\alpha,1}^{m}$, f (z) = I_{α}^{m} f (z) , α>-1 (see Cho and Srivastava [6] and Cho and Kim [5]) .

2- $I^{m}_{1-\beta,\beta}\, f$ (z) = $D^{m}_{\beta}\, f$ (z) , $\beta \geq 0$ (see Al- Oboudi [1]) .

3- $I^{m}_{c+1-\beta,\beta}\, f$ (z) = $I^{m}_{c,\beta}\, f$ (z) , C >-1 , B ≥ o(see Catas [4])

Definition (2.1) : Let g be a fixed function defined by (2.3) .

The function f ∈ AT given by (2.2) is said to be in the class MH $(\alpha, \beta, m, \eta, \sigma_1, \sigma_2, \lambda)$ if it satisfies the following condition :

$$\left| \frac{\dfrac{z\,(I^{m}_{\alpha,\beta}(f*g)(z))''}{z\,(I^{m}_{\alpha,\beta}(f*g)(z))'}}{-\eta\,\dfrac{z\,(I^{m}_{\alpha,\beta}(f*g)(z))''}{(I^{m}_{\alpha,\beta}(f*g)(z))'} + (\sigma_1 + \sigma_2)} \right| < \lambda , \qquad (2.6)$$

Where m ∉ N0 = NU{0} , $\beta \geq 0$, $\alpha \in$ R with $\alpha + \beta > 0$, $0 < \eta < 1$

, $0 < \sigma_1 < 1$, $0 \leq \sigma_2 < 1$ and $0 < \lambda < 1$.

The following interesting geometric properties of this function subclass were studied by sveral authors for other classes . like , Aout at el , [2] , Atshan and Al-Ziadi [3] and Jassim [&] .

2.2 Coefficient bounds

Now ,we obtain the necessary and sufficient condition for another function f to be in the calss MH (α , β , m , η , σ_1 , σ_2 , λ) .

Theorem (2.1) : let f ∈ AT , then f ∈ MH (α , β , m , η , σ_1 , σ_2 ,λ) if and only if .

$\sum_{n=2}^{\infty} n\,(\,(n-1\,)(1+\lambda\eta\,) - \lambda\,(\sigma_1 + \sigma_2)\,)\ \left(\frac{\alpha+n\beta}{\alpha+\beta}\right)^{m}\ a_n\ b_n \leq \lambda(\sigma_1 + \sigma_2\,)$

, (2.7) where m ∈ N0 = NU {0} , $\beta \geq 0$, $\alpha\epsilon$R with $\alpha + \beta > 0$,

$0 < \eta < 1$, $0 < \sigma_1 < 1$, $0 \leq \sigma_2 < 1$ and $0 < \lambda < 1$.

The result is sharp with the function f given by :

$$f\,(z) = z + \frac{\lambda(\sigma_1 + \sigma_2\,)}{n\,((\;n-1)(1+\lambda\eta) - \lambda(\sigma_1 + \sigma_2)\,)\left(\frac{\alpha+n\beta}{\alpha+\beta}\right)^{m} b_n}\ z^n\ ,\ n \geq 2\,. \qquad (2.8)$$

Proof : Suppose that (2.7) is true for z∈U and |z | = 1

Then , we have

$\left| z\,(I_{\alpha,\beta}^{m}\,(f*g)(z))'' \right| - \lambda \left| -\eta z\,(I_{\alpha,\beta}^{m}\,(f*g)(z))'' + (\sigma_1 + \sigma_2)\ (I_{\alpha,\beta}^{m}\,(f*g)(z))' \right|$

$= \left| \sum_{\eta=2}^{\infty} n(n-1)\left(\frac{\alpha+n\beta}{\alpha+\beta}\right)^{m} a_n\, b_n z^{n-1} \right|$

$-\lambda \left| -\eta \sum_{\eta=2}^{\infty} n(n-1)\left(\frac{\alpha+n\beta}{\alpha+\beta}\right)^{m} a_n\, b_n z^{n-1} (\sigma_1 + \sigma_2)\,(1+ \sum_{n=2}^{\infty}\left(\frac{\alpha+n\beta}{\alpha+\beta}\right)^{m} a_n\, b_n z^{n-1}) \right|$

$= \left| \sum_{n=2}^{\infty} n(n-1)\left(\frac{\alpha+n\beta}{\alpha+\beta}\right)^{m} a_n\, b_n\, z^{n-1} \right|$

$-\lambda \left| (\sigma_1 + \sigma_2) - \sum_{n=2}^{\infty} \eta n(n-1)\left(\frac{\alpha+n\beta}{\alpha+\beta}\right)^{m} a_n\, b_n\, z^{n-1} + \sum_{n=2}^{\infty} n(\sigma_1 + \sigma_2) \left(\frac{\alpha+n\beta}{\alpha+\beta}\right)^{m} a_n\, b_n\, z^{n-1} \right| = \left| \sum_{n=2}^{\infty} n(n-1)\left(\frac{\alpha+n\beta}{\alpha+\beta}\right)^{m} a_n\, b_n z^{n-1} \right|$

$-\lambda \left| (\sigma_1 + \sigma_2) - \sum_{n=2}^{\infty} n(\eta(n-1) - (\sigma_1 + \sigma_2))\left(\frac{\alpha+n\beta}{\alpha+\beta}\right)^{m} a_n\, b_n\, z^{n-1} \right|$

$\leq \sum_{n=2}^{\infty} n(n-1)\left(\frac{\alpha+n\beta}{\alpha+\beta}\right)^{m} a_n\, b_n\, |z|^{n-1}$

$+\sum_{n=2}^{\infty} n\lambda(\eta(n-1) - (\sigma_1 + \sigma_2)\left(\frac{\alpha+n\beta}{\alpha+\beta}\right)^{m} a_n\, b_n\, |z|^{n-1} - \lambda(\sigma_1 + \sigma_2)$

$=\sum_{n=2}^{\infty} n\,((n-1)(1+\lambda\eta)-\lambda(\sigma_1 + \sigma_2))\,(\frac{\alpha+n\beta}{\alpha+\beta})^{m}\, a_n\, b_n -$

$\lambda(\sigma_1 + \sigma_2) \leq 0$

By hypothesis , Hence , by maximum modulus principle , $f \in$ MH $(\alpha , \beta, m , \eta , \sigma_1,\sigma_2, \lambda)$.

Conversely assume that .

$f \in$ MH $(\alpha, \beta, m, \eta, \sigma_1, \sigma_2 , \lambda)$. then from (2.6) , we have .

$$\left| \frac{\frac{z(I^{m}_{\propto,\beta(f*g)(z)})''}{(I^{m}_{\propto,\beta(f*g)(z)})'}}{-\eta\frac{z(I^{m}_{\propto,\beta(f*g)(z)})''}{(I^{m}_{\propto,\beta(f*g)(z)})'} + (\sigma_1+\sigma_2)} \right|$$

$$=\left| \frac{\sum_{n=2}^{\infty} n(n-1)\,(\frac{\alpha+n\beta}{\alpha+\beta})^{m}\, a_n\, b_n\, z^{n-1}}{n \atop -\sum_{n=2}^{\infty} n(\eta(n-1)-(\sigma_1+\sigma_2))\,(\frac{\alpha+n\beta}{\alpha+\beta})^{m} a_n\, b_n\, z^{n-1}+(\sigma_1+\sigma_2)} \right| < \lambda.$$

Since Re (z) $\leq$ $|z|$, we get

$$\text{Re}\left\{ \frac{\sum_{n=2}^{\infty} n\,(n-1)\left(\frac{\alpha+n\beta}{\alpha+\beta}\right) a_n\, b_n\, z^{n-1}}{-\sum_{n=2}^{\infty} n(\eta(n-1)-(\sigma_1+\sigma_2))\left(\frac{\alpha+n\beta}{\alpha+\beta}\right)^{m} a_n\, b_n\, z^{n-1}+(\sigma_1+\sigma_2)} \right\} < \lambda.$$

(2.9)

We choose the ralue of z on the real axis so that .

$\frac{z(I^{m}_{\propto,\beta(f*g)(z)})''}{(I^{m}_{\propto,\beta(f*g)(z)})'}$ $is\ real$

$$\sum_{n=2}^{\infty} n\,(n-1)\left(\frac{\alpha+n\beta}{\alpha+\beta}\right)^{m} \quad a_n\ b_n\ z^{n-1}$$

$$\leq -\sum_{n=2}^{\infty} \lambda n\big(\eta(n-1)-(\sigma_1\ +\ \sigma_2)\big)\left(\frac{\alpha+n\beta}{\alpha+\beta}\right)^{m} \quad a_n\, b_n\, z^{n-1}$$

$$+\lambda(\sigma_1\ +\ \sigma_2)$$

Letting $z \to 1^-$ through real valves ,

$$\sum_{n=2}^{\infty} n(n-1)\left(\frac{\alpha+\eta\beta}{\alpha+\beta}\right)^{m} \quad a_n\, b_n$$

$$\leq -\sum_{n=2}^{\infty} \lambda n\big(\eta\,(n-1)-(\sigma_1\ +\ \sigma_2)\big)\left(\frac{\alpha+n\beta}{\alpha+\beta}\right)^{m} \quad a_n\, b_n$$

$$+\lambda(\sigma_1\ +\ \sigma_2)$$

We obtain inequality (2.7) .

Finally , sharpness follows if we take .

$$f(z) = z + \frac{\lambda(\sigma_1+\sigma_2)}{\lambda\left((n-1)(1+\eta\lambda)-\lambda(\sigma_1+\sigma_2)\right)\left(\frac{\alpha+n\beta}{\alpha+\beta}\right)^{m}\ b_n} \quad z^n \quad , \ (2.10)$$

n = 2,3, …

The proof is complete .

Corollary (2.1) : Let $f \in$ MH $(\alpha, \beta, m, \eta, \sigma_1, \sigma_2, \lambda)$. then

$$a_n \le \frac{\lambda(\sigma_1+\sigma_2)}{n\left((n-1)(1+\eta\lambda)-\lambda(\sigma_1+\sigma_2)\right)\left(\frac{\alpha+\eta\beta}{\alpha+\beta}\right)^m \; b_n} \quad ,n=2,3,\ldots \tag{3.11}$$

2.3: Distortion and growth theorems .

Next, we obtain the growth and distortion bounds for the linear operator $I^m_{\propto,\beta}$.

Theorem (2.2): If $f \in$ MH α$,\beta,m,\eta,\sigma_1,\sigma_2,\lambda)$ and $b_n \ge b_2\ (n \ge 3), then$

$$r - \frac{\lambda(\sigma_1+\sigma_2)r^2}{2(1+\lambda\eta)-\lambda(\sigma_1+\sigma_2)} \le \left| I^m_{\propto,\beta}\ (f*g)(Z) \right|$$

$$\le r + \frac{\lambda(\sigma_1+\sigma_2)r^2}{2(1+\lambda\eta)-\lambda(\sigma_1+\sigma_2)}\ ,\ (\,|\,Z\,|\, = r < 1\,)\,. \tag{2.12}$$

Proof : Let $f \in$ MH $(\alpha,\beta,m,\eta,\sigma_1,\sigma_2,\lambda)$. Then by theorem (2.1) , we get

$$2\left((1+\lambda\eta)-\lambda(\sigma_1+\sigma_2)\right)\left(\frac{\alpha+2\beta}{\alpha+\beta}\right)^m b_2 \sum_{n=2}^{\infty} a_n$$

$$\le \sum_{n=2}^{\infty} n\left((n-1)(1+\lambda\eta)-(\sigma_1+\sigma_2)\right)\left(\frac{\alpha+n\beta}{\alpha+\beta}\right)^m a_n b_n \le \lambda(\sigma_{1+}\sigma_2)$$

or

$$\sum_{n=2}^{\infty} a_n \le \frac{\lambda(\sigma_1+\sigma_2)}{2\left((1+\lambda\eta)-\lambda(\sigma_1+\sigma_2)\right)\left(\frac{\alpha+2\beta}{\alpha+\beta}\right)^m \; b_2} \tag{2.13}$$

Hence ,

$$| I^{m}_{\propto,\beta} \quad _{(f*g)(z)} | \leq \ | \ z \ | + \sum_{n=2}^{\infty} \left(\frac{\alpha+n\beta}{\alpha+\beta}\right)^{m} \ a_n \, b_n \ \ | \, z \, |^{\,n}$$

$$\leq \ | \ z \ \ | + \left(\frac{\alpha+2\beta}{\alpha+\beta}\right)^{m} \ b_2 \ \ | \ z \ |^{\,2} \sum_{n=2}^{\infty} a_n$$

$$= \mathrm{r} \ + \left(\frac{\alpha+2\beta}{\alpha+\beta}\right)^{m} \quad b_2 \ r^2 \sum_{n=2}^{\infty} \ a_n$$

$$\leq \ \mathrm{r} \ + \frac{\lambda(\sigma_1+\sigma_2)r^2}{2\big(\ (1+\lambda\eta)-\lambda(\sigma_1+\sigma_2)\big)} \tag{2.14}$$

Similarly,

$$| I^{m}_{\propto,\beta} \quad _{(f*g)(z)} | \geq |z| \sum_{n=2}^{\infty} \left(\frac{\alpha+n\beta}{\alpha+\beta}\right)^{m} \quad a_n \, b_n \, |z|^{\,n}$$

$$\geq |z| - \left(\frac{\alpha+2\beta}{\alpha+\beta}\right)^{m} \quad b_2 \, |z|^{\,2} \sum_{n=2}^{\infty} \quad a_n$$

$$= \mathrm{r} - \left(\frac{\alpha+2\beta}{\alpha+\beta}\right)^{m} \quad b_2 \, r^2 \sum_{n=2}^{\infty} \ a_n$$

$$\geq \ \mathrm{r} - \ \frac{\lambda(\sigma_1+\sigma_2)r^2}{2\big(\ (1+\lambda\eta)-\lambda(\sigma_1+\sigma_2)\big)} \tag{2.15}$$

From (2.14) and (2.15) we get (2.12) and the proof is complete .

Theorem (2.3): If $f \in$ MH $(\alpha,\beta,m,\eta,\sigma_1,\sigma_2,\lambda)$ and $b_n \geq b_2 \, (n \geq 3), then$

$$1- \frac{\lambda(\sigma_1+\sigma_2)r}{\big(\ (1+\lambda\eta)-\lambda(\sigma_1+\sigma_2)\big)} \leq | \, (I^{m}_{\propto,\beta} \quad _{(f*g)(z)})' |$$

$$\leq 1 + \ \frac{\lambda(\sigma_1+\sigma_2)r}{\big(\ (1+\lambda\eta)-\lambda(\sigma_1+\sigma_2)\big)} \ , \quad (\, | \, z \, | \ = \mathrm{r} \ \ < 1) \ . \tag{2.16}$$

Proof , the proof is Similar to that of theorem (2.2).

2.4: Radii of starlinkeness , convexity and close – to – convexity Using the inequalities

(1.2) , (1.3) , (1.4) and

theorem (2.1): we can compute the radii starlikeness , convexity and close – to – convexity .

Theorem (2.4) : If $f \in$ MH $(\alpha, \beta, m, \eta, \sigma_1, \sigma_2, \lambda)$, then f is univalent starlike of order $\Psi(0 \leq \Psi < 1)$ in the disk $|z| <$ r$_1$, where

r$_1(\alpha, \beta, m, \eta, \sigma_1, \sigma_2, \lambda, \Psi)$=

$$inf_n \left\{ \frac{n(1-\Psi)\left((n-1)(1+\lambda\eta)-\lambda(\sigma_1+\sigma_2)\right)\left(\frac{\alpha+n\beta}{\alpha+\beta}\right)^m \quad b_n}{} \right\}^{\frac{1}{n-1}} \quad , n \geq 2$$

Proof: It is sufficient to show that

$$\left| \frac{z f'(z)}{f(z)} - 1 \right| \leq 1\text{-}\Psi \ , \ (0 \leq \Psi < 1) ,$$

For $|z| <$ r$_1(\alpha, \beta, m, \eta, \sigma_1, \sigma_2, \lambda)$, we have that

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| = \frac{\sum_{n=2}^{\infty} (n-1) \ a_n \ z^{n-1}}{1+\sum_{n=2}^{\infty} \ a_n \ z^{n-1}} \leq \frac{\sum_{n=2}^{\infty} (n-1) \ a_n \ |z|^{\ n-1}}{1+\sum_{n=2}^{\infty} a_n \ |z|^{\ n-1}} \quad .$$

Thus

$$\left| \frac{z f'(z)}{f(z)} - 1 \right| \leq 1\text{-}\Psi ,$$

if

$$\frac{\sum_{n=2}^{\infty} (n-\Psi)a_n|z|^{\,n-1}}{1-\Psi} \leq 1 \tag{2.17}$$

Hence , by Theorem , (2.1) , (2.17) will be true if

$$\frac{n(n-\Psi)|z|^{\,n-1}}{1-\Psi} \leq \frac{n\left((n-1)(1-\lambda\eta)-\lambda(\sigma_1+\sigma_2)\right)\left(\frac{\alpha+n\beta}{\alpha+\beta}\right)^{m} b_n}{\lambda(\sigma_1+\sigma_2)}$$

equivalently if

$$|z| \leq \left\{\frac{(1-\Psi)\left((n-1)(1+\lambda\eta)-\lambda(\sigma_1+\sigma_2)\right)\left(\frac{\alpha+n\beta}{\alpha+\beta}\right)^{m} b_n}{(N-\Psi)\,\lambda\,(\sigma_1+\sigma_2)}\right\}^{\frac{1}{n-1}}, \quad n \geq 2$$

Setting $|z| = r_2$, we get the desired result. The proof is complete .

Theorem (2.6) : If $f \in$ MH $(\alpha,\beta,m,\eta,\sigma_1,\sigma_2,\lambda)$, then f is univalent convex of order $\Psi(0 \leq \Psi < 1)$ in the disk $|z| < r_2$, where

$r_2(\alpha,\beta,m,\eta,\sigma_1,\sigma_2,\lambda,\Psi)=$

$$\inf_n \left\{\frac{(1-\Psi)\left((n-1)(1+\lambda\eta)-\lambda(\sigma_1+\sigma_2)\right)\left(\frac{\alpha+n\beta}{\alpha+\beta}\right)^{m} b_n}{(N-\Psi)\,\lambda\,(\sigma_1+\sigma_2)}\right\}^{\frac{1}{n-1}}, \quad n \geq 2.$$

Proof : It is sufficient to show that

$$\left|\frac{z f''(z)}{f'(z)}\right| \leq 1-\Psi, \quad (0 \leq \Psi < 1)$$

for $|z| < r_2\ (\alpha,\beta,m,\eta,\sigma_1,\sigma_2,\lambda,\Psi)$, we have that

$$\left|\frac{z f''(z)}{f'(z)}\right| = \left|\frac{\sum_{n=2}^{\infty} n(n-1)\, a_n\, z^{\,n-1}}{1+\sum_{n=2}^{\infty} n a_n\, z^{\,n-1}}\right| \leq \frac{\sum_{n=2}^{\infty} n(n-1)\, a_n\, |z|^{\,n-1}}{1-\sum_{n=2}^{\infty} n a_n\, |z|^{\,n-1}}$$

Thus

$$\left| \frac{z f''(z)}{f'(z)} \right| \le 1 - \Psi$$

if

$$\frac{\sum_{n=2}^{\infty} n(n-\Psi)\ a_n\ |z|^{\,n-1}}{1-\Psi} \le 1\ .$$

Hence by theorem (2.1) , (2.18) will by true if

$$\frac{\sum_{n=2}^{\infty}(n-\Psi)\ |z|^{\,n-1}}{(1-\Psi)} \le \frac{n\big((n-1)(1+\lambda\eta)-\lambda(\sigma_{1+}\sigma_2)\big)\left(\frac{\alpha+n\beta}{\alpha+\beta}\right)^m \quad b_n}{\lambda(\sigma_{1+}\sigma_2)}\ .$$

Equivalently if

$$|z| \le \left\{\frac{(1-\Psi)(n-1)(1-\lambda\eta)-\lambda(\sigma_{1+}\sigma_2)\left(\frac{\alpha+n\beta}{\alpha+\beta}\right)^m \quad b_n}{(n-\Psi)\lambda(\sigma_{1+}\sigma_2)}\right\}^{\frac{1}{n-1}} \quad , n \ge 2\ .$$

Setting $|z|$ = r_2 , we get the desired result , the proof is complete .

Theorem (2.6) : Let a function f ∈MH $(\alpha, \beta, m, \eta, \sigma_{1,}\sigma_2\ , \lambda)$ then f is univalent close – to - convex of order $\Psi(0 \le \Psi < 1)$ in the disk

$|\ z\ | < r_3$, where.

$r_3(\alpha, \beta, m, \eta, \sigma_{1,}\sigma_2, \lambda)$=

$$\inf_n\left\{\frac{(1-\Psi)((n-1)(1+\lambda\eta)-\lambda(\sigma_{1+}\sigma_2))\left(\frac{\alpha+n\beta}{\alpha+\beta}\right)^m \quad b_n}{\lambda(\sigma_{1+}\sigma_2)}\right\}^{\frac{1}{n-1}}, n \ge 2$$

Proof : It is sufficient to show that

$\left| f'(z) - 1 \right| \leq 1-\Psi$, $(0 \leq \Psi < 1)$,

for

$|z| < r_3(\alpha, \beta, m, \eta, \sigma_{1,} \sigma_2, \Psi)$,

We have

$\left| f'(z) -1 \right| = \left| \sum_{n=2}^{\infty} n\, a_n\, z^{\,n-1} \right| \leq \sum_{n=2}^{\infty} n\, a_n\, |z|^{\,n-1}$.

Thus

$\left| f'(z) -1 \right| \leq 1-\Psi$,

If

$$\sum_{n=2}^{\infty} \frac{n a_n\, |z|^{n-1}}{1-\Psi} \leq 1. \qquad (2.19)$$

Hence , by theorem (2.1) , (2.19) will be true if .

$$\frac{n\,|\,z\,|^{n-1}}{1-\Psi} \leq \frac{n\big((n-1)(1+\lambda\eta) - \lambda(\sigma_{1+}\sigma_2)\big)\left(\frac{\alpha+n\beta}{\alpha+\beta}\right)^{m} \quad b_n}{\lambda(\sigma_{1+}\sigma_2)}$$

,equivalently . if

$$\left| Z \right| \leq \left\{\frac{(1-\Psi)\,(\,(n-1)(1+\lambda\eta)-\lambda(\sigma_{1+}\sigma_2)\,)\left(\frac{\alpha+n\beta}{\alpha+\beta}\right)^{m} \quad b_n}{\lambda(\sigma_{1+}\sigma_2)}\right\}^{\frac{1}{n-1}} \quad , n \geq 2$$

Setting $|\,z\,| = r_3$, we get the desired result . The proof is complete .

2.5:Extremepoints:

In the following theorem , we obtain the extreme points of the class MH $(\alpha,\beta,m,\eta,\sigma_{1,}\sigma_2,\lambda)$.

We obtain here an extreme points of the class MH $(\alpha,\beta,m,\eta,\sigma_{1,}\sigma_2,\lambda)$.

Theorem (2.7) : let f_1 (z) = z and

$$f n\,(z) = z+\frac{\lambda(\sigma_{1+}\sigma_2)}{n\big((n-1)(1+\lambda\eta)-\lambda(\sigma_{1+}\sigma_2)\big)\left(\frac{\alpha+n\beta}{\alpha+\beta}\right)^m\ b_n} z^n, \qquad (2.20)$$

Where all parameters are constrained as in theorem (2.1) .

Then the function f is in the class MH $(\alpha,\beta,m,\eta,\sigma_1,\sigma_2,\lambda)$ if and only if

$$f(z)=\sum_{n=1}^{\infty}\ \sigma_n f_n(z)\quad , \qquad (2.21)$$

Where $\sigma_n \geq 0$ and $\sum_{n=1}^{\infty}\ \sigma_n= 1$ or $1=\sigma_1+\ \sum_{n=2}^{\infty}\sigma_n$.

Proof : Suppose that f is expressed in the form (2.21). then

$$f\,(z) = \sigma_1 z_1+\sum_{n=2}^{\infty}\sigma_n f_n(z)$$

$$=\sigma_1\, z_1+\sum_{n=2}^{\infty}\sigma_n[z+\frac{\lambda(\sigma_{1+}\sigma_2)}{n((n-1)(1+\lambda\eta)-\lambda(\sigma_{1+}\sigma_2))\left(\frac{\alpha+n\beta}{\alpha+\beta}\right)^m\ b_n} z^n]$$

$$= z+\sum_{n=2}^{\infty}\frac{\lambda(\sigma_{1+}\sigma_2)}{n((n-1)(1+\lambda\eta)-\lambda(\sigma_{1+}\sigma_2))\left(\frac{\alpha+n\beta}{\alpha+\beta}\right)^m\ b_n}\ \sigma_n\, z^n$$

Hence ,

$$\sum_{n=2}^{\infty} \frac{n\big((n-1)(1+\lambda\eta)-\lambda(\sigma_{1+}\sigma_2)\big)\left(\frac{\alpha+n\beta}{\alpha+\beta}\right)^{m} \quad b_n}{\lambda(\sigma_{1+}\sigma_2)}$$

$$\times \frac{\lambda(\sigma_{1+}\sigma_2)\sigma_n}{n((n-1)(1+\lambda\eta)-\lambda(\sigma_{1+}\sigma_2))\left(\frac{\alpha+n\beta}{\alpha+\beta}\right)^{m} \quad b_n}$$

$= \sum_{n=2}^{\infty} \ \sigma_n = 1-\sigma_1 \leq 1$.

Then $f \in$ MH $(\alpha, \beta, m, \eta, \sigma_{1,} \sigma_2, \lambda)$.

Conversely , Suppose that $f \in$ MH $(\alpha, \beta, m, \eta, \sigma_{1,} \sigma_2, \lambda)$.

We may set

$$\sigma_n = \frac{n((n-1)(1+\lambda\eta)-\lambda(\sigma_{1+}\sigma_2))\left(\frac{\alpha+n\beta}{\alpha+\beta}\right)^{m}}{\lambda(\sigma_{1+}\sigma_2)} a_n b_n \ ,$$

Where a_n is given by (2.11) , then

$f(z) = z + \sum_{n=2}^{\infty} \ a_n \ z^n = z + \sum_{n=2}^{\infty} \ \frac{\lambda(\sigma_{1+}\sigma_2)}{n((n-1)(1+\lambda\eta)-\lambda(\sigma_{1+}\sigma_2))\left(\frac{\alpha+n\beta}{\alpha+\beta}\right)^{m}} \ \sigma_n$

z^n

$= z + \sum_{n=2}^{\infty} \ [f_{\ n}(z) - z]\sigma_n$

$= z + \sum_{n=2}^{\infty} \ f_n (z)\ \sigma_n - \sum_{n=2}^{\infty} \sigma_n \ z$

$= (1 - \sum_{n=2}^{\infty} \ \sigma_n)z + \sum_{n=2}^{\infty} f_n(z)\ \sigma_n$

$= \ \sigma_1 \ z + \sum_{n=2}^{\infty} \sigma_n f_n(z)$.

This completes the proof of theorem (2.7).

2.6. Closure theorems

Theorem (2.8) : Let the function f_r defined by

$$f_r(z) = z + \sum_{n=2}^{\infty} a_{n,r} \; z^n, \; (a_{n,r} \geq 0, r = 1, 2, \dots, \ell) \qquad (2.22)$$

be in the class MH $(\alpha, \beta, m, \eta, \sigma_1, \sigma_2, \lambda)$. For every

r= 1,2,3,…, ℓ. Then the function h_1 defined by

$h_1(z) = Z + \sum_{n=2}^{\infty} e_n \; z^n, (e_n, \geq 0)$

Also belongs to the class MH $(\alpha, \beta, m, \eta, \sigma_1, \sigma_2, \lambda)$. Where

$e_n = \frac{1}{\ell} \sum_{r=2}^{\infty} a_{n,r}$, (n=2,3,…..)

Proof : Since $f_r \in$ MH $(\propto, \beta, m, \eta, \sigma_1, \sigma_2, \lambda)$. It follows from theorem (2.1) that

$$\sum_{n=2}^{\infty} n((n-1)(1+\lambda\eta) - \lambda(\sigma_1 + \sigma_2)) \left(\frac{\alpha + n\beta}{\alpha + \beta}\right)^m a_{n,r} b_n \leq \lambda(\sigma_1 + \sigma_2),$$

for every r = 1,2,3, …,ℓ. Hence ,

$$\sum_{n=2}^{\infty} n\big((n-1)(1+\lambda\eta) - \lambda(\sigma_1 + \sigma_2)\big) \left(\frac{\alpha + n\beta}{\alpha + \beta}\right)^m e_{n,r} b_n,$$

$$= \sum_{n=2}^{\infty} n\big((n-1)(1+\lambda\eta) - \lambda(\sigma_1 + \sigma_2)\big) \left(\frac{\alpha+n\beta}{\alpha+\beta}\right)^m b_n \left(\frac{1}{\ell} \sum_{r=2}^{\infty} a_{n,r}\right)$$

$$= \frac{1}{\ell} \sum_{r=2}^{\infty} \left(\sum_{n=2}^{\infty} \big((n-1)(1+\lambda\eta) - \lambda(\sigma_1 + \sigma_2)\big) \left(\frac{\alpha+n\beta}{\alpha+\beta}\right)^m a_{n,r} b_n \right), \leq \lambda(\sigma_1 + \sigma_2).$$

By theorem (2.1) , it follows that $h_1 \in$MH $(\alpha, \beta, m, \eta, \sigma_1, \sigma_2, \lambda)$.

Theorem (2.9) : Let the functions f_r defined by (2.22) by in the class MH$(\alpha,\beta,m,\eta,\sigma_{1,}\sigma_2,\lambda)$. For every

r = 1,2,3,…, . Then the function h_2 defined by

$h_2(z) = \sum_{r=1}^{\infty} C_r f_r(z)$

Is also in the class MH$(\alpha,\beta,m,\eta,\sigma_{1,}\sigma_2,\lambda)$. Where

$\sum_{r=1}^{\infty} C_r = 1$, ($C_r \geq 0$) .

Proof : By theorem (2.1) for every r = 1,2,3, ….., we have

$$\sum_{n=2}^{\infty} n\big((n-1)(1+\lambda\eta)-\lambda(\sigma_{1+}\sigma_2)\big)\left(\frac{\alpha+n\beta}{\alpha+\beta}\right)^{m} a_{n,r} b_n \leq \lambda(\sigma_{1+}\sigma_2),$$

But

$h_2(z) = \sum_{n=2}^{\infty} c_r f_r(z) = \sum_{n=2}^{\infty} c_r(z+\sum_{n=2}^{\infty} a_{n,r} z^n) = z + \sum_{n=2}^{\infty}(\sum_{n=2}^{\infty} c_r\, a_{n,r})\, z^n$.

Therefore

$$\sum_{n=2}^{\infty} n\big((n-1)(1+\lambda\eta)-\lambda(\sigma_{1+}\sigma_2)\big)\left(\frac{\alpha+n\beta}{\alpha+\beta}\right)^{m} b_n(\sum_{n=2}^{\infty} c_r\, a_{n,r})$$

$$=\sum_{n=1}^{\infty} c_r(\sum_{n=2}^{\infty} n\big((n-1)(1+\lambda\eta)\lambda(\sigma_{1+}\sigma_2)\big)\left(\frac{\alpha+n\beta}{\alpha+\beta}\right)^{m} b_n a_{n,r})$$

$$\leq \sum_{n=1}^{\infty} c_r\, \lambda(\sigma_{1+}\sigma_2) = \lambda(\sigma_{1+}\sigma_2)$$

and the proof is complete .

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