University of Qadisiya

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On the compactness of the Composition Operator

C_ϕ

A search

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Abstract

Let U denote the unit ball in the complex plane, the Hordy space H^2 is the set of functions $f(z)=\sum_{n=0}^{\infty} f^{\wedge}(n) z^n$ holomarphic on U such that $\sum_{n=0}^{\infty} \left|f^{\wedge}(n)\right|^2 < \infty$ with $f^{\wedge}(n)$ denotes then the Taylor coeffecient of f .

Let ψ be a holomarphic self-map of U, the composetion operator C_ψ induced by ψ is defined on H^2 by the equation

$$C_\psi f = f \circ \psi \qquad (f \in H^2)$$

We have studied the compactness of the composetion operator $C_\emptyset$ induced by the bijective map ϕ and descussed the adjoint of the composetion operator $C_\emptyset$ induced by the bijactive map ϕ .We have look also at some known properties on composetion operators and tried to se the analogue propirties in order to show how the resultes are changed by changing the function ψ in U.

In arder to make the work accessible to the reader, we have included some known resultes with the details of the proofs for some cases and proofs for the properties .

المستخلص

ليكن U يرمز إلى كرة الوحدة في المستوى العقدي، إن فضاء هاردي H^2 هو

مجموعة كل الدوال $f(z)=\sum_{n=0}^{\infty} f^{\wedge}(n)\, z^n$ التحليلية على U بحيث أن $\sum_{n=0}^{\infty} |f^{\wedge}(n)|^2 < \infty$ ، $f^{\wedge}(n)$ يرمز

إلى معاملات تيلر

لتكن $\psi : U \to U$ دالة تحليلية على U ، المؤثر التركيبي المتولد من ψ يعرف على

فضاء هاردي H^2 بواسطة :

$$C_\psi f = f \, o \psi \quad (f \in H^2).$$

درسنا في هذا البحث تراص المؤثر التركيبي المتولد من الدالة المتقابلة ϕ حيث ناقشنا

المؤثر المرافق للمؤثر التركيبي المتولد من الدالة المتقابلة ϕ . بالإضافة إلى ذلك نظرنا إلى بعض

النتائج المعروفة وحاولنا الحصول على نتائج مناظرة لنتمكن من ملاحظة كيفية تغير النتائج عندما

تتغير الدالة التحليلية ψ .

ومن أجل جعل مهمة القارئ أكثر سهولة ، عرضنا بعض النتائج المعروفة عن المؤثرات

التركيبية وعرضنا براهين مفصلة وكذلك برهنا بعض النتائج .

المقدمة

هذا البحث يشمل فصلين . في الفصل الأول ، سوف نتناول الدالة المتقابلة ϕ وخواصها ، ونناقش النقاط الصامتة الداخلية والخارجية للدالة ϕ أيضا وكذلك نناقش أيضا الدوران المحوري حول الأصل للدالة ϕ وكذلك نناقش أيضا هل الدالة ϕ قطع ناقص ، وكذلك نناقش أيضا هل الدالة ϕ تحويل كسوري خطيّ .

في الفصل الثاني ، سوف نتناول المؤثر التركيبي C_ϕ المتولد بالدالة ϕ وخواصه ، وكذلك نناقش أيضا المرافق للمؤثر التركيبي C_ϕ المتولد بالدالة ϕ ، وكذلك نناقش أيضا هل المؤثر التركيبي C_ϕ مؤثر تركيبي وحدوي وكذلك نناقش أيضا هل المؤثر التركيبي C_ϕ مؤثر مرصوص ام لا .

Contents

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Chapter one

Properties of the Map ϕ

Intraduction

Definition(1.1) :

Lat $U = \{z \in C : |z| < 1\}$ be a unit boll in complex plane C and $\partial U = \{z \in C : |z| = 1\}$

is a boundory of U.

Definition(1.2):

Lat $\phi : U \to U$ be holomarphic on U and define $\phi(z) = \dfrac{\bar{\beta} - z}{\beta z - 1}$ $(z, \beta \in U)$

Proposition (1.3):

ϕ is bijective .

Proof:

Since $\phi(z) = \dfrac{\bar{\beta} - z}{\beta z - 1}$ $(z, \beta \in U)$

Suppase $\phi(z_1) = \phi(z_2)$ that is $\dfrac{\bar{\beta} - z_1}{\beta z_1 - 1} = \dfrac{\bar{\beta} - z_2}{\beta z_2 - 1}$, thus $|\beta|^2 z_2 - \bar{\beta} - \beta z_1 z_2 + z_1 =$

$|\beta|^2 z_1 - \bar{\beta} - \beta z_1 z_2 + z_2$, therefare $|\beta|^2 z_2 - z_2 =$ $|\beta|^2 z_1 - z_1$, hance $z_1 = z_2$. Thus ϕ is injective .

Let $y = \phi(z)$, that is $y = \dfrac{\bar{\beta} - z}{\beta z - 1}$, therefare $\bar{\beta} - z = \beta y z - y$, than $\bar{\beta} + y = \beta y z + z$ hance

$$z = \frac{\bar{\beta} + y}{\beta y + 1}, \phi(z) = \phi\left(\frac{\bar{\beta} + y}{\beta y + 1}\right) = \frac{\bar{\beta} - \left(\dfrac{\bar{\beta} + y}{\beta y + 1}\right)}{\beta\left(\dfrac{\bar{\beta} + y}{\beta y + 1}\right) - 1} = \frac{\dfrac{|\beta|^2 y + \bar{\beta} - \bar{\beta} - y}{\beta y + 1}}{\dfrac{|\beta|^2 y + \beta y - \beta y - 1}{\beta y - 1}} = \frac{\left(|\beta|^2 - 1\right) y}{\left(|\beta|^2 - 1\right)} = y$$

, for every $y \in U$, there exists $z \in U$ such that $y = \phi(z)$ .Thus ϕ is surjective . Hance ϕ is bijective.

Definition(1.4) :

For $\beta \in U$, then

$$1-|\phi(z)|^2=\frac{\left(|z|^2-1\right)\left(|\beta|^2-1\right)}{|\beta z-1|^2}$$

Proof:

$$1-|\phi(z)|^2=1-\left|\frac{\bar{\beta}-z}{\beta z-1}\right|^2=1-\frac{|\bar{\beta}-z|^2}{|\beta z-1|^2}=\frac{|\beta z-1|^2-|\bar{\beta}-z|^2}{|\beta z-1|^2}$$

$$=\frac{(\beta z-1)\left(\bar{\beta}\bar{z}-1\right)-\left(\bar{\beta}-z\right)\left(\beta-\bar{z}\right)}{|\beta z-1|^2}$$

$$=\frac{|\beta|^2|z|^2-\beta z-\bar{\beta}\bar{z}+1-|\beta|^2+\bar{\beta}\bar{z}+\beta z-|z|^2}{|\beta z-1|^2}=\frac{\left(|z|^2-1\right)\left(|\beta|^2-1\right)}{|\beta z-1|^2}$$

Proposition (1.5) :

If $\beta \in U$, than ϕ take ∂U into ∂U.

Proof :

Let $z \in \partial U$, than $|z|=1$, hance $|z|^2=1$. By (1.4) $1-|\phi(z)|^2=0$, therefare $|\phi(z)|^2=1$, hance $|\phi(z)|=1$, hance $\phi(z)\in \partial U$, hance ϕ take ∂U into ∂U.

Definition(1.6):

Let $\psi : U \to U$ be holomarphic mop on U, ψ is called an inner mop if $|\psi(z)|=1$.

Proposition (1.7):

ϕ es an inner map .

Proof :

From (1.5) ϕ take ∂U into ∂U., hance $|\phi(z)| = 1$. By (1.6) ϕ es an inner map .

Proposition (1.8) :

$$\phi^{-1}(z) = \frac{\bar{\beta} + z}{\beta z + 1} \ .$$

Proof :

Lot $y = \phi^{-1}(z)$, than $z = \phi(y)$, hance $z = \frac{\bar{\beta} - y}{\beta y - 1}$, thus $\beta yz - z = \bar{\beta} - y$, therefare $\beta yz + y = \bar{\beta} + z$.Thus $y(\beta z + 1) = \bar{\beta} + z$, hance $y = \frac{\bar{\beta} + z}{\beta z + 1}$, than $\phi^{-1}(z) = \frac{\bar{\beta} + z}{\beta z + 1}$.

Remark(1.9) :

If $\beta \in U$, than $\phi'(0) = 1 - |\beta|^2$, $\phi'(\bar{\beta}) = \frac{1}{(1 - |\beta|^2)}$.

Definition(1.10) :

Lat $\psi : U \to U$ be holomarphic map on U. We say that ψ is a rototion round the origen if there exists $\alpha \in \partial U$ such that $\psi(z) = \alpha z \ \ (z \in U)$

Proposition (1.11):

If $\beta = 0$, then ϕ is a rototion a round the origin

Proof:

Since $\phi(z) = \dfrac{\bar{\beta} - z}{\beta z - 1}$, since $\beta = 0$, hance $\phi(z) = z = \alpha z$, $\alpha = 1 \in \partial U$, than by (1.9)

ϕ is a rototion a round the origin .

Definition(1.12):

A linear fractional trancformation is a mapping of the form $\tau(z) = \dfrac{az + b}{cz + d}$,

where a , b , c , and d are complex numbers , and we somatime donete it by $\tau_A(z)$

where A is the non-singulor 2×2 complex matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Proposition (1.13) :

If $\beta \in U$, than ϕ is a linear fractional trancformation .

Proof :

Since $\phi(z) = \dfrac{\bar{\beta} - z}{\beta z - 1} = \dfrac{az + b}{cz + d}$ such that a = -1 , b = $\bar{\beta}$, c = β, d = -1 and a , b , c

, and d are complex numbers and $A = \begin{bmatrix} -1 & \bar{\beta} \\ \beta & -1 \end{bmatrix}$, hance by (1.12) ϕ is a linear

fractional trancformation .

Chapter two

compactness of the C_ϕ

Intraduction

This seoarch consists of two chaptors . In chaptar one ,we ore going to study the bijactive map ϕ and proparties of ϕ, and also descuss the interior and exterior fixed pointes of ϕ and also discuss ϕ is a rotetion around the origin and ϕ is elleptic and ϕ is a linear fractional trancformation .

In chaptor two, we are going to stuidy the Composetion Operator C_ϕ induced by the map ϕ and propirties of C_ϕ, and also discuss the adjoent of Composetion Operator C_ϕ induced by the map ϕ and also discuss the campactness of C_ϕ.

Definition(2.1):

Let U denote the unit boll in the complix plane, the Hordy space H^2 is the set of funcations $f(z)=\sum_{n=0}^{\infty} f^{\wedge}(n)\, z^n$ holomarphic on U such that $\sum_{n=0}^{\infty} \left|f^{\wedge}(n)\right|^2 < \infty$ with $f^{\wedge}(n)$ denoties then the Taylor coeffecient of f and $H^2 : U \to C$.

Remark (2.2) :

We can define an inner praduct of the Hordy space funcations as follows:

f(z) $=\sum_{n=0}^{\infty} f^{\wedge}(n)\, z^n$ and $g(z)=\sum_{n=0}^{\infty} g^{\wedge}(n)\, z^n$, then the inner praduct of f and g is define

$$\langle f, g\rangle = \sum_{n=0}^{\infty} f^{\wedge}(n)\, \overline{g^{\wedge}(n)}$$

Definition (2.3) :

Lat $\alpha \in U$, define $K_\alpha(z)=\dfrac{1}{1-\overline{\alpha} z}$ $(z \in U)$. Sance $\alpha \in U$ then $|\alpha|<1$,hance the geometric series $\sum_{n=0}^{\infty} |\alpha|^{2n}$ is convergent and $K_\alpha(z)=\sum_{n=0}^{\infty} (\overline{\alpha})^n z^n$ thus $k_\alpha \in H^2$

Definition(2.4) :

Let $\psi : U \to U$ be holomarphic map on U, the composetion operator C_ψ induced by ψ is defened on H^2 as follows $C_\psi f = f o \psi$ $(f \in H^2)$

Definition(2.5) :

Let T bo a bounded operator on a Hilbart space H, then the norm of an operotar T is defined by $\|T\| = \sup\{\|Tf\| : f \in H, \|f\| = 1\}$.

Theorem (2.6) :

If $\psi : U \to U$ is holomarphic map on U, than $f\, o\psi \in H^2$ and

$\|f\, o\psi\| \leq \|f\| \sqrt{\frac{1+|\psi(0)|}{1-|\psi(0)|}} \|f\|$ for each $f \in H^2$. The gool of this theorem $C_\psi : H^2 \to H^2$.

Definition(2.7) :

The composetion operator C_ϕ induced by ϕ is defened on H^2 as follows

$C_\phi f = f\, o\phi$, $(f \in H^2)$

Proposition(2.8) :

If $\phi(z) = \frac{\bar{\beta} - z}{\beta z - 1}$, than $f\, o\phi \in H^2$ and $\|f\, o\phi\| \leq \sqrt{\frac{1+|\beta|}{1-|\beta|}} \|f\|$ far each $f \in H^2$.

Proof :

Since $\phi : U \to U$ is holomarphic map on U by (2.6)

$f \in H^2, f\, o\phi \in H^2$ and $\|f\, o\phi\| \leq \sqrt{\frac{1+|\beta|}{1-|\beta|}} \|f\|$ hance $C_\phi : H^2 \to H^2$

Remark (2.9) :

1) One can easely show that $C_\kappa C_\psi = C_{\psi o \kappa}$ and hance $C_\psi^n = C_\psi C_\psi \Lambda\ C_\psi$

$= C_{\psi o \psi o \Lambda\ o \psi} = C_{\psi_n}$

2) C_ψ is the idintity operator on H^2 if and only if ψ is idintity map from U into U and holomarphic on U.

3) It is semple to prove that $C_\kappa = C_\psi$ if end only if $\kappa = \psi$.

Definition(2.10):

Lat T be an operator on a Hilbart space H , The operator T^* es the adjoint of T if $\langle Tx, y\rangle = \langle x, T^* y\rangle$ for each $x, y \in H$.

Theorem (2.11) :

$\{K_\alpha\}_{\alpha\ \in\ U}$ farms a danse subset of H^2 .

Theorem (2.12) :

If $\psi : U \to U$ is holomarphic map on U, then for all $\alpha \in U$

$C_\psi^* K_\alpha = K_{\psi(\alpha)}$

Definition(2.13):

Lat H^{∞} be the set of oll bounded holomarphic map on U .

Definition(2.14):

Lat $g \in H^{\infty}$, the Toeplets operator T_g is the operator on H^2 given by :

$$(T_g f)(z) = g(z)\ f(z) \quad (f \in H^2,\ z \in U)$$

Theorem (2.15) :

If $\psi : U \to U$ is holomarphic map on U, then $C_\psi T_g = T_{g \circ \psi} C_\psi \quad (g \in H^{\infty})$

Remark (2.16) :

Far each $f \in H^2$, it is will- know that $T_h^* f = T_{\bar{h}} f$, such that $h \in H^{\infty}$.

Proposition(2.17) :

If $\beta \in U$, than $C_\phi^* = T_g C_\gamma T_h^*$ where $h(z) = 1 - \beta z$, $g(z) = \dfrac{1}{1+\beta z}$, $\gamma(z) = \dfrac{\bar{\beta} - z}{1 + \beta z}$

Proof :

By (2.16), $T_h^* f = T_{\bar{h}} f$ for each $f \in H^2$. Hance for all $\alpha \in U$,

$$\langle T_h^* f, K_\alpha \rangle = \langle T_{\bar{h}} f, K_\alpha \rangle = \langle f, T_{\bar{h}}^* K_\alpha \rangle \Lambda\ \Lambda\ (2-1)$$

On the other hand ,

$$\left\langle T_h^* f,\ K_\alpha \right\rangle = \langle f, T_h K_\alpha \rangle = \langle f, h(\alpha) K_\alpha \rangle \Lambda\ \Lambda\ (2-2)$$

From (2-1)and (2-2) we can se that $T_{\bar{h}}^* K_\alpha = h(\alpha)\ K_\alpha$. Hance $T_h^* K_\alpha = \overline{h(\alpha)}\ K_\alpha$.

Calculotion give:

$$C_\phi^* K_\alpha(z) = K_{\phi(\alpha)}(z)$$

$$= \frac{1}{1-\overline{\phi(\alpha)}\ z} = \frac{1}{1-\dfrac{(\beta-\overline{\alpha})z}{\overline{\beta\alpha}-1}}$$

$$= \frac{1}{\dfrac{\overline{\beta\alpha}-1-\beta z-\overline{\alpha} z}{\overline{\beta\alpha}-1}} = \frac{\overline{\beta\alpha}-1}{-(1+\beta z)+\overline{\alpha}\left(\overline{\beta}-z\right)} = \frac{\overline{(1-\beta\alpha)}}{(1+\beta z)-\overline{\alpha}\left(\overline{\beta}-z\right)}$$

$$= \overline{(1-\beta\alpha)} \cdot \left(\frac{1}{1+\beta z}\right) \cdot \frac{1}{1-\overline{\alpha}\left(\dfrac{\overline{\beta}-z}{1+\beta z}\right)}$$

$$= \overline{h(\alpha)} \cdot g(z)\ .K_\alpha(\gamma(z)) = \overline{h(\alpha)}\ .g(z).(K_\alpha\ o\gamma)(z)$$

$$= \overline{h(\alpha)} \cdot (T_g\ K_\alpha\ o\gamma)(z) = \overline{h(\alpha)}\ T_g C_\gamma K_\alpha(z)$$

$$= T_g\ \overline{h(\alpha)}\ C_\gamma\ K_\alpha(z) = T_g\ C_\gamma\ \overline{h(\alpha)}\ K_\alpha(z)$$

$$= T_g\ C_\gamma\ T_h^*\ K_\alpha(z)$$, therefare

$$C_\phi^*\ K_\alpha(z) = T_g\ C_\gamma\ T_h^*\ K_\alpha(z)\ .$$

But $\overline{\{K_\alpha\}_{\alpha\ \in\ U}} = H^2$, than $C_\phi^* = T_g\ C_\gamma\ T_h^*$

Definition(2.18) :

Let T be an operator on a Hilbart space H , T is called compact if every sequence x_n in H is weekly convorges to x in H , then Tx_n is strangly convorges to Tx.Moreover ($x_n \xrightarrow{W} x$ if $\langle x_n,u\rangle \to \langle x,u\rangle, \forall u \in H$ and $x_n \xrightarrow{S} x$ if $\|x_n - x\| \to 0.$)

Theorem (2.19) :

If $\psi : U \to U$ is holomarphic map on U, than C_ψ is not compact if and only if ψ take ∂U into ∂U

Proposition(2.20) :

If $\lambda \in U$, then C_ϕ is net compact composetion operator

Proof :

From (1.5) ϕ take ∂U into ∂U. By (2.19) C_ϕ is not compact composetion operator .

Theorem (2.21) :

If $\psi : U \to U$ is holomarphic map on U, then $C_\psi C_\phi^*$ is compact if end only if $C_\psi C_\gamma$ is compact , where $C_\phi^* = T_g\ C_\gamma\ T_h^*$, $\gamma(z) = \frac{\bar{\beta} - z}{1 + \beta z}$

Proof:

Suppose that $C_\psi C_\gamma$ is compact . Note that

$$C_\psi C_\phi^* = C_\psi\ T_g\ C_\gamma\ T_h^* \quad (\text{since } C_\phi^* = T_g\ C_\gamma\ T_h^* \text{ by } (2.17))$$

$$= T_{g\circ\psi}\ C_\psi\ C_\gamma\ T_h^* \quad (\text{since } C_\psi\ T_g = T_{g\circ\psi} C_\psi \text{ by } (2.15)).$$

Since $C_\psi C_\gamma$ es compact operator , $T_{g\circ\psi}$ and T_h^* are bounded operators then $C_\psi C_\phi^*$ es compact

Conversaly , suppose that $C_\psi C_\phi^*$ is compact . Note that

$$C_\psi C_\gamma = C_\psi \left(C_\gamma^*\right)^* = C_\psi \left(T_g\, C_\phi\, T_h^*\right)^* = C_\psi\, T_h\, C_\phi^*\, T_g^*$$

$$= T_{h\,o\,\psi}\, C_\psi\, C_\phi^*\, T_g^* \quad \text{(since } C_\psi\, T_h = T_{h\,o\,\psi} C_\psi \text{ by (2.15)).}$$

Since $C_\psi C_\phi^*$ is compact operator , $T_{h\,o\,\psi}$ and T_g^* are bounded operatores then $C_\psi C_\gamma$ is compact .

Corollary (2.22) :

If $\psi : U \to U$ is holomarphic map on U, then $C_\psi C_\sigma^*$ is not compact if and only if there exist points $z_1, z_2 \in \partial U$ such that $(\gamma\, o\, \psi)(z_1) = z_2$.

Proof:

By (2.21) $C_\psi C_\phi^*$ is not compact if and only if $C_\psi C_\gamma = C_{\gamma\,o\,\psi}$ is not compact . Since $\gamma : U \to U$ and $\psi : U \to U$ are holomarphices on U , then also $\gamma\, o\, \psi$. Thus by (2.19) $C_{\gamma\,o\,\psi}$ is not compact if and only if $\gamma\, o\, \psi$ take ∂U into ∂U . So, there exist points $z_1, z_2 \in \partial U$ such that $(\gamma\, o\, \psi)(z_1) = z_2$.

Theorem(2.23) :

If $\psi : U \to U$ is holomarphic on U, then $C_\phi^* C_\psi$ is compact if and only if $C_\gamma C_\psi$ is compact , where $C_\phi^* = T_g\, C_\gamma\, T_h^*$, $\gamma(z) = \dfrac{\bar{\beta} - z}{1 + \beta z}$

Proof:

Suppose that $C_\gamma C_\psi$ is compact . Nate that

$$C_\phi^* C_\psi = T_g\, C_\gamma\, T_h^*\, C_\psi \quad \text{(since } C_\phi^* = T_g\, C_\gamma\, T_h^* \text{ by (2.17))}$$

$$= T_g\, C_\gamma\, T_{\bar{h}}\, C_\psi \quad \text{(by (2.16))}$$

$$= T_g\, T_{\bar{h}\,o\,\gamma}\, C_\lambda\, C_\psi \quad \text{(since } C_\gamma\, T_{\bar{h}} = T_{\bar{h}\,o\,\lambda} C_\gamma \text{ by (2.15)).}$$

Since $C_\gamma C_\psi$ is compact operotar , T_g and $T_{\bar{h}\,o\,\gamma}$ are bounded operatores than $C_\phi^* C_\psi$ is compact

Conversaly ,Suppose that $C_\phi^* C_\psi$ is compact . Note that

$$C_\gamma C_\psi = \left(C_\gamma^*\right)^* C_\psi$$

$$= \left(T_g\ C_\phi\ T_h^*\right)^* C_\psi \quad (\text{since } C_\gamma^* = T_g\ C_\phi\ T_h^*)$$

$$= T_h\ C_\phi^*\ T_g^*\ C_\psi$$

Nate that , by (2.11) it is enaugh to prove the compactness on the family $\{K_\alpha\}_{\alpha \in U}$

Hence for each $z \in U$ we have

$$C_\gamma C_\psi K_\alpha(z) = T_h\ C_\phi^*\ T_g^*\ C_\psi K_\alpha(z)$$

$$= T_h\ C_\phi^*\ T_g^*\ K_\alpha(\psi(z))$$

$$= T_h\ C_\phi^*\ \overline{g(\alpha)}\ K_\alpha(\psi(z)) \qquad (\text{since } T_g^*\ K_\alpha = \overline{g(\alpha)}\ K_\alpha)$$

$$= \overline{g(\alpha)}\ T_h\ C_\phi^*\ K_\alpha(\psi(z))$$

$$= \overline{g(\alpha)}\ T_h\ C_\phi^*\ C_\psi\ K_\alpha(z)$$

Since $C_\phi^* C_\psi$ is compact , T_g is bounded and $g \in H^\infty$, then $C_\gamma C_\psi$ is compact .

Corollary (2.24) :

If $\psi : U \to U$ is holomarphic map on U, then $C_\phi^* C_\psi$ is not compact if end only if there exist points $z_1, z_2 \in \partial U$ such that $(\psi\, o \gamma)(z_1) = z_2$.

Proof:

By (2.23) $C_\phi^* C_\psi$ is not compact if and only if $C_\gamma C_\psi = C_{\psi\, o\, \gamma}$ is not compact . Since $\gamma : U \to U$ and $\psi : U \to U$ are holomarphices on U , then also $\psi\, o \gamma$. Thus by (2.19) $C_{\psi\, o\, \gamma}$ is not compact if and only if $\psi\, o \gamma$ take ∂U into ∂U . So, there exist points $z_1, z_2 \in \partial U$ such that $(\psi\, o \gamma)(z_1) = z_2$.

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