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T-essential and t-closed sub module

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Introduction

Through this paper all rings are associative with unity and all modules are unitary right modules. We recall some relevant notions and results. A submodule N of an R-module M is essential in M (briefly N$\leq_{ess}$M) if N ∩ W = (0), W ≤ M implies W = (O)[2]. A submodule N of M is called closed in M (briefly N $\leq_c$ M) if N has no proper essential extension in M, that is if N $\leq_{ess}$W ≤ M, then N = W[9]. The set {x ∈ M: xI = (0) for some essential ideal I of R} is called the singular submodule of M and denoted by Z(M)[10].Equivalently Z(M) = {x ∈ M: ann(x) $\leq_{ess}$R} and ann(x) = {r ∈ M: xr = 0}. M is called singular (nonsingular) if Z(M) = M(Z(M) = 0). It is known that" a module M is called extending(CS-module or module has C_1 -condition) if for every submodule N of M then there exists a direct summand W(W $\leq^{\oplus}$ M) such that N $\leq_{ess}$ W " Equivalently" M is extending module if every closed submodule is a direct summand", where a submodule C of M is called closed if

C $\leq_{ess}$ C' < M implies that C = C'[1].

in chapter two we study as a generalization of essential submodule , Asgari in [1] , introduced the nition of t-essential submodule , where a submodule of is called t-essential (denoted by if whenever ,implies ,is the second singular submodule and is defined by [8] where for some essential ideal of) Equivalently and is called singular (nonsingular) if .Note that for some t-essential ideal of . is called –torsion if . A submodule is called t-closed (denoted by if has no proper t-essential extension in [1] .it is clear that every t-closed submodule is closed , but the convers is not true . However ,under the class of nonsingular the two concepts are equivalent

Chapter One

Essential and Closed Submodule

In this chapter we recall the definition of essential submodules closed submodules and some of their properties that are relevant to our Work.

Definition (1.1) : Let M be an R – Module, recall , recall that a submodule A of M is called essential in M (denoted by A $\leq_{ess}$ M) if A ∩ $W \neq 0$ for every non zero submodule W of M equivalently A $\leq_{ess}$ M if Whenever A ∩ W = 0 , W ≤ M then

W = 0 .

Find essential submodule in Z_{12} and Z_{24}.

Solution: Z_{12}

< 0 > = W_1

< 2 > = {0, 2, 4, 6, 8, 10} = W_2

< 3 > = {0, 3, 6, 9} = W_3

< 4 > = {0, 4, 8} = W_4

< 6 > = {0, 6} = W_5

Z_{12} = {0 , 1 , 2 , 3 , 4 , 5 , 6 , 7 , 8 , 9 , 10 , 11} = W_6

$W_2 \cap W_2 = W_2 \neq 0$

$W_2 \cap W_3 = 0 = \{0 , 6\}$

$W_2 \cap W_4 \neq 0 = \{4 , 8\}$

$W_2 \cap W_5 \neq 0 = \{0 , 6\}$

$W_2 \cap W_6 \neq 0 = < 2 > \qquad then\ W_2 \leq_{ess} Z_{12}$

$W_3 \cap W_1 = 0$

$W_3 \cap W_2 = (0 , 6)$

$W_3 \cap W_4 = W_1 \; But \; W_4 \neq 0$

$W_3 \leq_{ess} Z_n = Z_n$

$W_4 \cap W_2 \neq 0 \; W_4$

$W_4 \cap W_3 = 0 \; but \; W_3 \neq 0$

$W_4 \nleq_{ess} Z_{12}$

$W_5 \cap W_2 \neq 0 = W_5$

$W_5 \cap W_3 \neq 0 = W_5$

$W_5 \cap W_4 = 0 \; but \; W_4 \neq 0$

$W_5 \nleq_{ess} Z_{12}$

The submodules of Z_{24} are.

$W_2 = \{0 , 2 , 4 , 6 , 8 , 10 , 13 , 14 , 16 , 18 , 20 , 22\} \; W_1$

$W_3 = \{0 , 3 , 6 , 9 , 12 , 15 , 18 , 21\} \; W_2$

$W_4 = \{0 , 4 , 8 , 13 , 16 , 20\} \; W_3$

$W_5 = \{0 , 6 , 12 , 18\} \; W_4$

$W_6 = \{0 , 8 , 16\} \; W_5$

$W_7 = \{0 , 12\} \; W_6$

$W_2 \cap W_1 \neq 0 = W_1$

$W_2 \cap W_2 \neq 0 = W_4$

$W_2 \cap W_3 \neq 0 = W_3$

$W_2 \cap W_4 \neq 0 = W_4$

$W_2 \cap W_5 \neq 0 = W_5$

$W_2 \cap W_6 \neq 0 = W_6$

$W_2 \leq_{ess} Z_{24}$

$W_3 \cap W_1 \neq 0 = W_4$

$W_3 \cap W_2 \neq 0 = W_2$

$W_3 \cap W_3 \neq 0 = W_6$

$W_3 \cap W_4 \neq 0 = W_4$

$W_3 \cap W_5 = 0 \quad but \quad W_5 \neq 0 \implies W_3 \nleq_{ess} Z_{24}$

$W_4 \cap W_1 \neq 0 = W_3$

$W_4 \cap W_2 \neq 0 = W_6$

$W_4 \cap W_3 \neq 0 = W_3$

$W_4 \cap W_4 \neq 0 = W_6$

$W_4 \cap W_5 \neq 0 = W_5$

$W_4 \cap W_6 \neq 0 = W_6 \qquad then\ W_4 \leq_{ess} Z_{24}$

$W_5 \cap W_1 \neq 0 = W_4$

$W_5 \cap W_2 \neq 0 = W_4$

$W_5 \cap W_3 \neq 0 = W_6$

$W_5 \cap W_4 \neq 0 = W_4$

$W_5 \cap W_5 = 0 \quad but \quad W_5 \neq 0 \quad then\ W_5 \not\leq_{ess} Z_{24}$

Theorem (1.3) [6]**:** Let M be an R - module and A be a submodule of M, then A $\leq_{ess}$ M if and only if every non-zero element of M has a non-zero multiplication in A.

Proposition (1.4) [6]: (1) Let A, A', B and B' be submodules of an R - module M such that A ⊆ B and A' ⊆ B' then,

a. A $\leq_{ess}$ M if and only if A $\leq_{ess}$ B $\leq_{ess}$ M.

b. If A $\leq_{ess}$ B and A' $\leq_{ess}$ B', then A ∩ A' $\leq_{ess}$ B' ∩ B'.

(2) Let M and N be R – modules and let f: M⟶ N be an R-homomorphism, if B $\leq_{ess}$ N, then F^{-1}(B) $\leq_{ess}$ M.

(3) Let M= ⨁ i ∈ J, M_i be an R-module, Where M_i is a submodule of M, ∀ $i \in I$ if Ai $\leq_{ess}$ Mi , for each i ∈ I, then ⨁ $i \in I$ Ai $\leq_{ess}$ M_i, For each i ∈ I , then $\bigoplus_{i \in I}$ Ai $\leq_{ess}$M.

Definition [1.5] [3]**:** Let A be a submodule of an R - module M. Recall that a relative complement of A in M is any submodule B of M Which is maximal with to the property A ∩ B = 0.

Easy application of Zama's lemma gives for every submodule A of an R - module M, there exists a relative complement for A in M.

Proposition (1.6) [3]: Let M be an R - module and A be a submodule of M. If B is any relative complement for A in M, then A $\oplus$ B $\leq_{ess}$ M.

Proof: Let D be a submodule of M such that D ∩ (A $\oplus$ B) =0, we want to show that D = 0. Assume D $\neq$ 0. Now A ∩ (D $\oplus$ B) = 0. But B is a relative complement for A in M, therefore D+B = B and hence D ⊆ B. Then D = D ∩ B = 0. This is a contradiction. Thus $A \oplus B \leq_{ess} M$.

Let M be an R - module. Recall that a submodule A of M is a closed submodule if A has no proper essential extension in M, [3].

Proposition (1.7) [3]: Let M be an R - module If A and B are submodules of M such that M = A $\oplus$ B, then A is closed in M.

Proof: Let A $\leq_{ess}$ D, where D is subniodule of M. $since\ A \cap B = 0$, then D ∩ B = 0.

Let $d \in D$, then d = $a + b, a \in A, b \in B$. Implies that d – a = b ∈ $D \cap B$ = 0, we get $d - a = 0$ and d = a. thus $D = A$, [3].

Proposition (1.8) [3]: Let B be a submodule of an R - module M. Then the following statements are equivalent: -

1- B is a closed sub module of M.

2 – If B ⊆ K $\leq_{ess}$ M. $then \frac{K}{B} \leq_{ess} \frac{M}{B}$.

3- B is a relative complement for some submodule A of M.

Theorem (1.9) [3], [2]: Let A, B and C be submodules of an R-module M with A ⊆ B, then:

1-There exists a closed submodule D of M such that $C \leq_{ess} D$.

2-If A closed in B and B closed in M, then A is closed in M:

3-If Closed in M, then $\frac{B}{A}$ closed in $\frac{M}{A}$.

Definition (1.10) [3]: Let M be an R-module and let x ∈ M Recall that the annihilator of x (denoted by ann (x)) is defined as follows an (x) = { r ∈ R : rx = 0 } Clearly ann (x) is an ideal of R.

Definition (1.11) [3]: Let M be an R-module. Recall that $Z(M) = \{x \in M\colon ann\ (x)\ \leq_{ess}$ R} is called singular submodule of M. If $Z(M) = M$, then M is called the singular module .If $Z(M) = 0$ then M is called a nonsingular module.

The following lemma gives some properties of singular submodules which are needed later and can be found in [3].

Lemma (1.12) [3]: Let M and N be an R – modules, then:

1 If $f\colon M \longrightarrow N$. N is an R – homomorphism, then $f(Z(M)) \subseteq Z(N)$.

2-Epimorphic image of a singular module is, singular.

Proposition (1. 13) [3]: A module C is singular if and only if there exists a shorter exact sequence

$$0 \longrightarrow A \overset{f}{\rightarrow} B \overset{B}{\rightarrow} C \longrightarrow 0 \; such\ that\ f(A) \leq_{ess} B.$$

Coro1lar (1.14) [3]: If $A \leq_{ess} B$, then $\frac{B}{A}$ is singular.

Proposition (1.15) [3], [2]: Let B be a nonsingular R - module, and $A \subseteq_e B$. Then $\frac{B}{A}$ is singular if and only if $A \leq_{ess} B$.

Let M be an R - module. Recall that the second singular submodule Z_2 (M) of M is the submodule of M containing $Z(M)$ such that $\frac{Z_2(M)}{Z(M)}$ is the singular submodule of $\frac{M}{Z(M)}$.

Proposition (1.16) [6]: Any direct summand of an R – Module M is closed.

Proof: Let N $\subseteq^{\oplus}$ M, such that M = N ⊕ K for some $K \subseteq K$.

To prove:

N is closed in M

Suppose ∃ W ⊆ M such that $N \leq_{ess} W$

We must prove $N = W$

Suppose N ≠ W ⟹ ∃ $x \in W$ and x ∉ N $then$ x ∈ $N = N \oplus k$

$then\ x = n + h\ ,\qquad n \in N\ , R \in K$

Then $0 \neq x - n \in w$

$(for\ if\ x - n = 0 \implies R = 0 \implies x = n + 0 = n \in N)$

(By the N $\leq_{ess}$ W ⟺ ∀ $x \in w\ , x \neq 0\ \exists\ r \neq 0 \implies Ci \in R$

$\exists\ r \neq 0\ x \in N)$

We have: $\exists\, r \in R, r \neq 0 \;\exists\; 0 \neq r\,(x-n) \in N$

Since x = n + k

$rx = rn + rk$

$$\underset{\in N}{\underline{rx - rn}} = \underset{\in k}{\underline{rk}} \quad \in N \cap K = (0)$$

$\therefore$ $rx - rn$ = 0 Which is a c :

Thus $w = n$

Corollary (1.17) [6]: Every Submodule of semi simple R – module is closed:

Remark (1.18): Closed Sub M. $Then$ need not be direct summand for example

Let M = $Z_8 \oplus Z_2$ as a Z – module

Let N = < $\{(\bar{2}, \bar{1}) > = \{(\bar{0}, \bar{0}), (\bar{2}, \bar{1}), (\bar{4}, \bar{0}), (\bar{6}, \bar{1})\}\}$

$N_0 = (\bar{0}) \oplus (\bar{0}) = (\bar{0}, \bar{0}),$

$N_1 = < (\bar{1}, \bar{0}) > = Z_8 \oplus (\bar{0}) = [(a, 0), a \in Z_8]$

$N_2 = < (\bar{2}, \bar{0}) > = (\bar{2}) \oplus (\bar{0}) = [(a, 0), a \in (\bar{2}) \leq Z_8]$

Proof: $N_3 = (\bar{4}) \oplus (\bar{0}) = [(a, 0), a \in (\bar{4}) \leq Z_8]$

$< a > = < -a >\; N_4 = (\bar{4}) \oplus Z_2 = [(a, b), a \in (\bar{4}), b \in Z_2]$

N is closed in M

N is not direct summand of M.

Definition (1.19)[6]: Let $B \leq M$, $A \leq M$, A is called a relative complement of B if A is the largest submodule of M With property $A \cap B$ = (0)

Such that if $\exists\ A \supseteq A, A' \cap B = 0\ \ Then\ A = A'$

A relative complement A of B exists by Zero's Lemma.

Example (1.20) [6]: F is any field, M = f $\bigoplus$ F

Let A = F $\bigoplus$ (0)

$\forall\ x\ \in f\ , let\ B = <(x\ ,1)>is\ a\ relative\ complement\ .for\ A$

Special case:

M = $Z_3 \bigoplus Z_3$, A = $Z_3 \bigoplus (\bar{0})$ = $<((\bar{1}\ ,0))>$

= $\{(\bar{1}\ ,\bar{0})\ ,(\bar{2}\ ,\bar{0})\ ,(\bar{0}\ ,\bar{0})\}$

Let $x\ =\bar{0}$, B_1 = $<(\bar{1}\ ,\bar{1})>$ = $\{(\bar{1}\ ,\bar{1})\ ,(\bar{2}\ ,\bar{2}),(\bar{0}\ ,\bar{0})\}$

$B_1 \cap$ A = $\{(\bar{0}\ ,\bar{0})\}$

$X\ =\bar{2}\ , B_2\ \ = <(\bar{2}\ ,\bar{1})> = \{(\bar{2}\ ,\bar{1}),(\bar{1}\ ,\bar{1}),(\bar{0}\ ,\bar{0})\}$

$B_2 \cap$ A = $\{(\bar{0}\ ,\bar{0})\}$

$X\ =\bar{0}\ , B_3$ = $<(\bar{0}\ ,\bar{1})> = \{(\bar{0}\ ,\bar{1}),(\bar{0}\ ,\bar{2}),(\bar{0}\ ,\bar{0})\}$

$B_3\ \cap$ A = $\{(\bar{0}\ ,\bar{0})\}$

B_1, B_2, B_3 Are relative complement of A in case F is an in finite to field, A has an in finite relative complement.

Proposition (1.21) [6]: Let $A \leq M$ if B is any relative complement of A, then $A \oplus$ $\leq_{ess} M$.

Proof: Let N ≤ M suppose N ∩ (A ⊕ B) = 0

To prove $N = (0)$

Then $N \oplus (A \oplus B) = (A \oplus B) \oplus N$

$$= A \oplus (B \oplus N)$$

Notice that $A \cap (B \oplus N) = 0$

To prove that:

$q = b + n$ For some $b \in B, n \in N$

$Then$ (a – b) = n ∈ N ∩ (A⊕) = (0)

∴ n = 0 & a – b = 0

$Hence\ a = b \in A \cap B = (0)$ (Since B is a relative complement of A)

$a = 0$, So $A \cap (B \oplus N) = 0$

But B is relative complement of A

And $B \oplus N \supseteq B$

$then$ B ⊕ N = B ⟹ N = (0) [Since N ∩ B = (0) and A ≤ M].

Theorem (1.22) [6]: Let $B \leq M$ and $A \leq M$, the following statement, are equivalent:

(1) B Is a closed sub M of M.

(2) If B ≤ K $\leq_{ess}$ M, then $\frac{K}{B} \leq_{ess} \frac{\mathrm{M}}{\mathrm{B}}$.

(3) If A is a relative complement of B, then B a relative complement of A.

(4) B is relative complement of $A \leq M$.

Proof: (1) ⟶ (2)

Let B ≤ K $\leq_{ess} M$ to prove $\frac{K}{B} \leq_{ess} \frac{M}{B}$

Let $\frac{N}{B} \leq \frac{M}{B}$ with $\frac{K}{B} \cap \frac{M}{B}$ with $\frac{K}{B} \cap \frac{N}{B} = 0 \frac{M}{B}$ (to prove $\frac{N}{B} = 0 \frac{M}{B}$?)

Then $\frac{K \cap N}{B} = O_{\frac{M}{B}}$

Hence $K \cap N = B$

But K $\leq_{\text{ess}}$ M & N $\leq_{\text{ess}}$ N

N ∩ K $\leq_{\text{ess}}$ M ∩ N = N

= N ∩ K $\leq_{\text{ess}}$ N

$B \leq_{\text{ess}} N$, but B is closed in M (B)

$then$ B = N $\Longrightarrow \frac{N}{B} = O_{\frac{M}{B}}$

Then (3)

If A is a relative complement of B, then A ∩ B = (0)

Then B ∩ A = (0)

To prove B is the largest.

Let B' ≥ B such that B' ∩ A = (0)

But (A ⊕ B) ∩ B' = B ⊕ (A ∩ B') = B ⊕ (0) = B

$$\frac{(A\ \oplus B)\ \cap B'}{B} = \frac{B}{B} = O_{\frac{M}{B}}$$

$$\frac{A\ \oplus B}{B} \bigcap \frac{B'}{B} = O_{\frac{M}{B}}$$

B ≤ A ⊕ B $\leq_{\text{ess}} O_{\frac{M}{B}}$

$B \leq\ A \oplus B \leq_{\text{ess}} M$ [Since A relative complement of B]

By (2) $\frac{A\ \oplus B}{B} \quad \leq_{\text{ess}} \frac{M}{B}$

$\frac{B\prime}{B} = O_{\frac{M}{B}} \Longrightarrow$ B = B′

B is a relative complement of A

(3) ⟹ (4) it is clear

(4) ⟹ (1) if B is a relative complement of A

To Prove B is closed.

Assume B $\leq_{\text{ess}}$ B′ (T prove B = B′).

$(B' \cap A) \cap B = B' \cap (A \cap B) = (0)$

But $B \leq_{\text{ess}} B'$ and $B' \cap A \leq B'$

$Then$ $(B' \cap A) \cap B = (0)$ implies $B' \cap A \leq B'$

$Then$ $(B' \cap A) \cap B = (0)$ implies $B' \cap A \leq 0$

But B is a relative complement of A and $B' \supseteq B$

$Hence$ B = B $then$ B is closed.

Proposition (1.23) [6]: If $A \leq B \leq M$, if A is closed in B and B is closed in M then A $\leq_{\text{ess}}$ M. ($A \leq_{\text{ess}} B$ and $B \leq_{\text{ess}} M \implies A \leq_{\text{ess}} M$)**.**

Poof: $A \leq_{\text{ess}} B \Longrightarrow \exists\, \bar{X} \leq B \ni A$ is a relative complement of $\bar{X}$

Then ($B \cap C$ = (0))

Note that $\bar{X} \cap C = (0)$ (Since $\bar{X} \cap C \subseteq B \cap C = (0)$)

We claim that A is a relative complement of $\bar{X} \oplus C$

To prove $A \cap (\bar{X} \oplus C) = (0)$.

Let $a \in A$ & $a = X + C$, $X \in \bar{X}$, $C \in C$

$Then$ $a - x = c \in B \cap C = (0)$

$then$ $C = 0$, $a = X \in A \cap \bar{X} = (0)$

$then$ $a = 0$

$then$ $A \cap (\bar{X} \oplus C) = (0)$

Let $A' \supseteq A$ and $A' \cap (\bar{X} \oplus C)$ = (0)

$(A' \cap \bar{X}) \oplus (A' \cap C)$ = (0)

$Then$ A′ ∩ $\bar{X}$ = (0)

But A is a relative complement of $\bar{X}$ and $A' \supseteq A$

$Hence$ A = A′

$then$ A is a relative complement of $\bar{X} \oplus$ C

$then$ A is closed in M.

Proposition (1.24) [6]: If A ≤ B ≤ M, and $A \leq_{\text{ess}}$ M then A $\leq_{\text{ess}}$ B.

Proof: A is closed in M ⟹ ∃ $\bar{X}$ ≤ M ∋ A is relative complement of $\bar{X}$.

$Then$ A ∩ $\bar{X}$ = 0

Let B ∩ $\bar{X}$ ≤ B We claim that A is a relative complement of B ∩ $\bar{X}$

A ∩ $(B \cap \bar{X})$ = $B \cap (A \cap \bar{X})$ = B ∩ (0) = (0)

Suppose (∃A′ ≥ A); A′ ∩ $(B \cap \bar{X})$ = (0)

(∃A′ ⊆ B) ⟹ (A′ ∩ B) ∩ $\bar{X}$ = (0)

$Then$ A′ ∩ $\bar{X}$ = (0)

But A is a relative complement of $\bar{X}$ ⟶ A = A′

$Then$A is a relative complement of B ∩ $\bar{X}$ ⊆ B

$Hence$ A is closed in B

Proposition (1.25) [6]: Let C be a closed in M and let $T \leq M$ such that $C \cap T = (0)$

Then C is a relative complement of T

$then$ $C \oplus T \leq_{ess} M$

If $C \oplus T \leq_{ess} M$, to prove C is relative complement of T.

Since C is closed in M, So C is relative complement of $S \leq M$ (then $C \cap S = (0)$)

To prove C is a relative complement of T

$C \cap T = (0)$

Suppose $\exists\ D \supseteq C$ such that $D \cap T = (0)$

$(C \oplus T) \cap (D \cap S) = [(C \oplus T) \cap D] \cap S$

But $C \oplus T \leq_{ess} M$, hence

$D \cap S = (0)$ and $D \supseteq C$, C is a relative complement of S. So $D = C$

$Then$ C is a max. Sub With property $C \cap T = (0)$

$then$ C is a relative complement of T.

Exercise (1.26) [6]:

(1) Let $A \leq B \leq M$. If $B \leq_{ess} M$. To prove that $\frac{B}{A} \leq_{ess} \frac{M}{B}$ is the converse true.

(2) If $A \leq_{ess} M$, $A_2 \leq_{ess} M_2$. Prove that $A_1 \oplus A_2 \leq_{ess} M_1 \oplus M_2$.

(3) $A_1 \leq_{ess} M_1$, $A_2 \leq_{ess} M_2$.To prove that $A_1 \oplus A_2 \leq_{ess} M_1 \oplus M_2$.

(4) Let M be a finitely generated Faith. Multiplication. R–module. Let $N \leq M$ prove that.

$N \leq_{ess} M \Leftrightarrow (N \, R \, I \, M) \leq_{ess} R \Leftrightarrow N = Im$ for Some closed ideal I in R.

Chapter TWO

Definition(2.1): sup module A of M is said to be t-essential in M (wr=tecn A$\leq_{tess}$M) if for every sup module B of M , $A \cap B \leq Z_2(M)$ implies that $B \leq Z_2(M)$ clearly if A is a Sup module of anon singular module M , then A is t-essentialin M if and omly if is essent ialin M .

The following Proposition is useful

-Proposition(2.2): the following State ements are equivalents for a sup module A of an R –module M ;

1. A is t-essential in M ;
2. $\left(A + Z_2(M)\right)/Z_2$ (M) is essenti al in $M/Z_2\ (M)$
3. $A + Z_2\ (M)$ is essential in M ;
4. $M/A\ is\ Z_2$-torsion

Proof: - $(1) \Rightarrow (2)$ there exists Subodule B of M such that $A \oplus B$ is essential in M . By h_2/po thesis , $B \leq Z_2\ (M)$ hence , $A + Z_2\ (M)$ is essential in M , and since $Z_2(M)$ is a closed Sub module of M , we conclude that $(A + Z_2(M)/Z_2(M)$ is essential in $M/Z_2\ (M)$

$(2) \Rightarrow (3)$ This is obrious

$(3) \Rightarrow (4)$ By hypo thesis $M/(A + Z_2(M))$ is singular ,and hence , Z_2-torsion .on the other hand , $(A + Z_2(M)/A$ is isomorphic to a factor of $Z_2(M)$

Proposition(2.3): The following statements are equir a lent for a sub module A of a module M (the not a tion ≤e denotes an essential Sub module):

1. $A \leq_{tes}\ M$;
2. $(A + Z_2(M))/Z_2(M) \leq_{tes}\ M/Z_2\ (M)$;
3. $A + Z_2(M) \leq_{tes} M$;
4. $M/A\ is\ Z_2$-*torsion ;*

Proof: A shown [1,Proposition 2.2] , $(1) \Leftrightarrow (2) \Leftrightarrow (3) \Leftrightarrow (4)$. The equiralence $(1) \Leftrightarrow (5)$ follows easily from the t-essential property

Corollary(2.4)

$(1) let\ A \leq B \leq M$ be Modules . then $A \leq_{tes} Mi$ fand only if $A \leq_{tes} B\ and\ B \leq_{tes} M$

(2) let $f: M \rightarrow N$ be a homomorph is M of Modules , and $A \leq_{tes} N\ Then\ f^{-1}(A) \leq_{tes} M$

Proof: **(1)** This follows from proposition 1.1(4) and the facts that $B/A \leq M/A\ and\ M/B \cong [M/A]/[B/A]$

Corollary(2.5): let A_λ be a sub module of M_λ For all $\lambda\ in\ a\ set\ \wedge$

1) If ∧ is finite and $A_\lambda \leq_{tes} M_\lambda$ for all $\lambda \in \wedge$ then $\cap_\wedge A_\lambda \leq_{tes} \cap_\wedge M_\lambda$
2) $\oplus_\wedge A_\lambda \leq_{tes} \oplus_\wedge M_\lambda$ if and only if $A_\lambda \leq_{tes} M_\lambda$ For all $\lambda \in \wedge$

Proof . (1) clearly , $\cap_\wedge M_\lambda / \cap_\wedge A_\lambda$ embeds in $\prod_\wedge M_\lambda / A_\lambda$. By proposition 1.1 (4) ,$\prod_\wedge M_\lambda / A_\lambda\ is\ Z_2$ to rsion , and so $\cap_\wedge M_\lambda is\ Z_2$ $-$torsion .A gain by $\cap_\wedge A_\lambda \leq_{tes} \cap_\wedge M_\lambda$

(2) This follws from the isomorph is $\oplus_\wedge M_\lambda \cong \oplus_\wedge M_\lambda / A_\lambda$ and proposition 1.1(4)

2.T_{11} $-$TyPEMoDuLES

Recall from [17] that a module M is said to satisfy

C_{11} com dition if every sub module of M has a complent which is a direct sum and . By restricting the C_{11} com dit ionto-closed sub module of M

Definition(2.6): we say that a sub module C of M is t-closed in M and write $C \leq_{tc} M\ if\ \leq_{tec} C` \leq M$

Implies that $C = C`$

Clearly , every t-closed sub module is closed and if C is a sub module of a nonsingular module M , then C is t-closed in M if and only if C is closed in M

Further properties of t-closed sub modules are given below

Lemma (2.7):. let M be a module

1. If $C \leq_{tc} M$, $then\ Z_2(M) \leq C$
2. $0 \leq_{tc} M$ if and only if M is nonsingular
3. $if\ A \leq C$, then $C \leq_{tc} M$ if and only if $C/A \leq_{tc} M/A$

Proof :(1) since $(C + Z_2(M))/C \cong Z_2(M)/(C \cap Z_2(M))$ is Z_2 torsion –by proposition 2.2 , $C \leq_{tes} C + Z_2(M)$ thus $C = C + Z_2(M)$ and so $Z_2(M) \leq C$

(2) let $0 \leq_{tc} M\ since\ 0 \leq_{tes} Z_2(M)$ we conclude that M is nonsingular . the con verse is easy

(3) this following by proposition 2.2(4)

Proposition(2.8): let C be a sub module of a module M the following state mentis are equivalent:

1. There exists a sub module S such that C is with respect to the property that $C \cap S\ is\ Z_2$-torsion;
2. C is t –closed in M;
3. C contains $Z_2(M)$ and C is a closed sub module of M ;
4. C contains Z_2(M) and $C/Z_2(M)$ is a closed sub module of $M/Z_2(M)$;
5. C is a complement to a nonsingular sub module of M ;
6. M/C is nonsingular

Proof: .$(1) \Rightarrow (2) let (1)$ hold and $C \leq_{tes} C` \leq M$ then $C \cap (C` \cap S) \leq Z_2(M)$ implies that $C` \cap S \leq Z_2(M)$ . Hence $C = C`$

$(2) \Rightarrow (3)$ By lemma 2.5 C contains Z , (M) let $C/Z_2(M) \leq C`/Z_2(M)$ By proposition 2.2(2) , $C \leq_{tes} C`$, $hence\ C = C`$

$(3) \Rightarrow (4) let C \leq_e C` \leq M$ every essential sub module is t-essential hence by proposition 2.2(2), $C/Z_2(M) \leq_e C`/Z_2(M)$ Thus $C = C`$

$(4) \Rightarrow (5)$ As C is closed by lam [12,proposition 6.32], $C = X \cap M$ for some direct . sum M and X of the injective hull $E(M)$, say $E(M) = x \oplus y$ and let

$S = M \cap Y$ cleary $C \cap S = 0$ Thus $Z_2(s) = Z_2(M) \cap S \leq C \cap S = 0$ and hence S

Thus it is Z_2- torsion . therefore ,from the is ommorph is

$M\,[M/A]/[(A + Z_2(M))/A] \cong M/(A + Z_2(M))$, we cohclude that $M/A\ is\ Z_2$-torsion

$(4) \Rightarrow (1) since\ M/A\ is\ Z_2$-torsion , $[M/A][Z(M/A)]$ is singular . How every ,the latter is isomorphic to $M/A^*\ where\ A^*/A = Z(M/A)$ thus M/A^* is singular .

Now let $A \cap B \leq Z_2(M)$ for some sub module B of M , and $b \in B. As\ M/A^*$ is singular there exists a ness eutial right ideal of R such that bI/A^* then for every $X \in I$. There exists an essential right ideal Z of R such that $bI \leq A^*$. then for every $X \in I$, there exists an essential right ideal $K \circ fR$ Such that $b \times K \leq A \cap B \leq Z_2(M)$ and so $bx + Z_2(m) \in Z(m)/Z(m)) = 0$ thus$bI \leq Z_2(M)$. and this implies that $b + Z_2(M) \in 2(M/Z_2(M)) = 0$ so be$\in Z_2(M)$ consequently , $B \leq Z_2(M)$

Remark(2.3):. every essential sub module of a module M . is t-essential But the converse not true fov example $Z_{12}\ as\ Z$-module

(4) $\leq_{tes}\ Z_{12}\ but\ (4) {}^{\leq}/_{tes} Z_{12}$

Corollary (2.9): let M be a module

1. $Z_2(M)$ is t-closed in M
2. If φ is an endomorph is M of M and C is a t-closed sub module of M , then $\varphi^{-1}(c)$ is t-closed in M

Proof (1) since M/Z_2(M) is nonsingular ,$Z_2(M)$ is t-closed in M by proposition 2.6(2) . there is natural embedding of $M/\varphi^{-1}\,(c)$ in to the nonsingular module M/C thus $M/\varphi^{-1}\,(c)$ is nonsingular ,and hence by propos is it ion 2.6 $\varphi^{-1}(c)$ is t-closed in M

Corollary (2.10) let C be a sub module of a module M

1. If $C \leq_{tc} M$, then $C = Z_2(M)$ if and only if C is Z_2 torsion if and only if there exists a t-essential sub module S of M for which $C \cap S \leq Z_2(M)$
2. Let $C \leq N \leq M\ if\ C \leq_{tc} M\ , then\ C \leq_{tc} N$
3. If $\leq_{tc} N\ and\ N \leq_{tc} M\ then\ C \leq_{tc} M$

Proof .(1) by lemma 2.5(1) it suffices to show that if $C = Z_2(M)$ then there exists a t-essential sub module S of M such that C is maximal with respect to the property that $C \cap S\ is\ Z_2$ –torsion let $S \cap B \leq Z_2(M)$. By Z or n lemma , B can be enlarged in to a t-closed sub module C\` such that $S \cap C\` \leq Z_2(M)$ However by lemma 2.5(1) $C = Z_2(M) \leq C\`$ thus $C\` = C = Z_2(M) hence\ B \leq Z_2(M)$ and so is t-essential

(2) and (3) follow by proposition 2.6[(2)⇔ (6)]

Let $C \leq_C M$ mean that C is a closed sub module of M . we have in general

$$C \leq M, \quad C\` \leq_C M \quad \nRightarrow C \cap C\` \leq_C C;$$

$$C \leq_C M, \quad C\` \leq_C M \quad \nRightarrow \quad C \cap C\` \ \leq_C M\ ;$$

See lam [12, caution 6.27 and proposition 6.32], but these are always true if we replace cbytc

Proposition(2.11) let M be a module then :

1. $C \leq M\ , C\` \leq_{tc} M\ \Rightarrow C \cap C\` \leq_{tc} C$;
2. $C \leq_{tc} M\ , C\` \leq_{tc} M\ \Rightarrow C \cap C\` \leq_{tc} M$

Moreover , an arbitrary in terse action of t-closed sub module is t-closed

Proposition(2.12) [9,proposition 2.4,p.q3]

Let M be a nonsingular R-module and let A be a sub module of M . then A is y-closed in M if

and only if A is closed

Proof : $\Rightarrow By\ (2.1,1.3)$

$\Leftarrow$ Assume that M is a nonsingular R-module and A is a closed sub module of M . let $Z(\frac{M}{A}) = \frac{B}{A}$

where B is a sub module of M with $A \subseteq B$ hence $A \subseteq_e B$ by (1.1.1。) But A is closed in M , there

for $A = B\ and\ Z(\frac{M}{A}) = 0$ thus A is a y-closed sub module of M

Proposition(2.13): let M be a singular R module .then M is the only y-closed sub module of M

Proof: Let A be an y-closed sub module of M

To show that $M = A$, $let\ m \in M$, since M is singular , then an $n(M) \subseteq_e R$

claim that an $n(M) \subseteq ann(M + A)$ trover if y this , let $r \in ann(M)$, then

$rm = 0 \in A$ and hence $r(M + A) = A$

So $r \in ann(M + A)$ .since $ann(m) \subseteq_e R$

But $M + A \in \frac{M}{A}\ and\ \frac{M}{A}$ is nonsingular , there F or $M + A = A$ and hence

$M \in A$ thus $M = A$

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