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Extending Modules

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بسم الله الرحمن الرحيم

﴿ ذلـــك الكتـــاب لا ريـــب فيـــه

هدى للمتقين ﴾

صدق الله العلي العظيم

سورة البقرة

الآية (٢)

الإهداء

إلى من جرع الكأس فارغاً ليسقيني قطرة حب

إلى من كلّت أنامله ليقدم لنا لحظة سعادة إلى من حصد

الأشواك عن دربي ليمهد لي طريق العلم

إلى القلب الكبير (والدي العزيز)

إلى من أرضعتني الحب والحنان

إلى رمز الحب وبلسم الشفاء

إلى القلب الناصع بالبياض (والدتي الحبيبة)

إلى القلوب الطاهرة الرقيقة والنفوس البريئة إلى

رياحين حياتي (إخوتي)

الآن تفتح الأشرعة وترفع المرساة لتنطلق السفينة في

عرض بحر واسع مظلم هو بحر الحياة وفي هذه

الظلمة لا يضيء إلا قنديل الذكريات ذكريات الأخوة

البعيدة إلى الذين أحببتهم وأحبوني (أصدقائي)

شكر وتقدير

الحمد لله رب العالمين والصلاة والسلام على اشرف خلق الله سيدنا محمد (صلى الله عليه واله وسلم).

من الصعب البوح بكلمة الشكر لأنـها لا تحد عطاء اساتذتي الذين تتلمذت على ايديهم واخص فيهم بالذكر الجميل والثناء الوفير أستاذي المشرف

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فجزاه الله خير جزاء المحسنين

الباحث

علي فتان غالب

Contents

Introduction

Through this paper all rings are associative with unity and all modules are unitary right modules. We recall some relevant notions and results. A submodule N of an R-module M is essential in M (briefly N$\leq_{ess}$M) if N ∩ W = (0), W ≤ M implies W = (O)[2]. A submodule N of M is called closed in M (briefly N $\leq_c$ M) if N has no proper essential extension in M, that is if N $\leq_{ess}$W ≤ M, then N = W[9]. The set {x ∈ M: xI = (0) for some essential ideal I of R} is called the singular submodule of M and denoted by Z(M)[l0].Equivalently Z(M) = {x ∈ M: ann(x) $\leq_{ess}$R} and ann(x) = {r ∈ M: xr = 0}. M is called singular (nonsingular) if Z(M) = M(Z(M) = 0).

It is known that" a module M is called extending(CS-module or module has C_1 -condition) if for every submodule N of M then there exists a direct summand W(W $\leq^{\oplus}$ M) such that N $\leq_{ess}$ W " Equivalently" M is extending module if every closed submodule is a direct summand", where a submodule C of M is called closed if

C $\leq_{ess}$ C' < M implies that C = C'[1].

This work consists of two chapters. In chapter one we deal with certain knows result which is worthwhile throughout this work.

In chapter two we study type of module namely extending module of some properties abut it also we study a character of extending module.

An R – modules M is extending if and only if every closed submodule of M.

Chapter One

Essential and Closed Submodule

In this chapter we recall the definition of essential submodules closed submodules and some of their properties that are relevant to our Work.

Definition (1.1) : Let M be an R – Module, recall , recall that a submodule A of M is called essential in M (denoted by A $\leq_{ess}$ M) if A $\cap$ $W \neq 0$ for every non zero submodule W of M equivalently A $\leq_{ess}$ M if Whenever A ∩ W = 0 , W ≤ M then W = 0 .

Find essential submodule in Z_{12} and Z_{24}.

Solution: Z_{12}

$< 0 > = W_1$

$< 2 > = \{0, 2, 4, 6, 8, 10\} = W_2$

$< 3 > = \{0, 3, 6, 9\} = W_3$

$< 4 > = \{0, 4, 8\} = W_4$

$< 6 > = \{0, 6\} = W_5$

$Z_{12} = \{0 , 1 , 2 , 3 , 4 , 5 , 6 , 7 , 8 , 9 , 10 , 11\} = W_6$

$W_2 \cap W_2 = W_2 \neq 0$

$W_2 \cap W_3 = 0 = \{0 , 6\}$

$W_2 \cap W_4 \neq 0 = \{4 , 8\}$

$W_2 \cap W_5 \neq 0 = \{0 , 6\}$

$W_2 \cap W_6 \neq 0 =< 2 >$ $\quad$ $then\ W_2 \leq_{ess} Z_{12}$

$W_3 \cap W_1 = 0$

$W_3 \cap W_2 = (0, 6)$

$W_3 \cap W_4 = W_1\ But\ W_4 \neq 0$

$W_3 \leq_{ess} Z_n = Z_n$

$W_4 \cap W_2 \neq 0\ W_4$

$W_4 \cap W_3 = 0\ but\ W_3 \neq 0$

$W_4 \nleq_{ess} Z_{12}$

$W_5 \cap W_2 \neq 0 = W_5$

$W_5 \cap W_3 \neq 0 = W_5$

$W_5 \cap W_4 = 0\ but\ W_4 \neq 0$

$W_5 \nleq_{ess} Z_{12}$

The submodules of Z_{24} are.

$W_2 = \{0, 2, 4, 6, 8, 10, 13, 14, 16, 18, 20, 22\}\ W_1$

$W_3 = \{0, 3, 6, 9, 12, 15, 18, 21\}\ W_2$

$W_4 = \{0, 4, 8, 13, 16, 20\}\ W_3$

$W_5 = \{0, 6, 12, 18\}\ W_4$

$W_6 = \{0, 8, 16\}\ W_5$

$W_7 = \{0, 12\}\ W_6$

$W_2 \cap W_1 \neq 0 = W_1$

$W_2 \cap W_2 \neq 0 = W_4$

$W_2 \cap W_3 \neq 0 = W_3$

$W_2 \cap W_4 \neq 0 = W_4$

$W_2 \cap W_5 \neq 0 = W_5$

$W_2 \cap W_6 \neq 0 = W_6$

$W_2 \leq_{ess} Z_{24}$

$W_3 \cap W_1 \neq 0 = W_4$

$W_3 \cap W_2 \neq 0 = W_2$

$W_3 \cap W_3 \neq 0 = W_6$

$W_3 \cap W_4 \neq 0 = W_4$

$W_3 \cap W_5 = 0 \ \ but \ \ W_5 \neq 0 \implies W_3 \nleq_{ess} Z_{24}$

$W_4 \cap W_1 \neq 0 = W_3$

$W_4 \cap W_2 \neq 0 = W_6$

$W_4 \cap W_3 \neq 0 = W_3$

$W_4 \cap W_4 \neq 0 = W_6$

$W_4 \cap W_5 \neq 0 = W_5$

$W_4 \cap W_6 \neq 0 = W_6 \qquad then\ W_4 \leq_{ess} Z_{24}$

$W_5 \cap W_1 \neq 0 = W_4$

$W_5 \cap W_2 \neq 0 = W_4$

$W_5 \cap W_3 \neq 0 = W_6$

$W_5 \cap W_4 \neq 0 = W_4$

$W_5 \cap W_5 = 0 \quad but \quad W_5 \neq 0 \quad then\ W_5 \nleq_{ess} Z_{24}$

Theorem (1.3) [6]**:** Let M be an R - module and A be a submodule of M, then A $\leq_{ess}$ M if and only if every non-zero element of M has a non-zero multiplication in A.

Proposition (1.4) [6]: (1) Let A, A', B and B' be submodules of an R - module M such that A ⊆ B and A' ⊆ B' then,

a. A $\leq_{ess}$ M if and only if A $\leq_{ess}$ B $\leq_{ess}$ M.

b. If A $\leq_{ess}$ B and A' $\leq_{ess}$ B', then A ∩ A' $\leq_{ess}$ B' ∩ B'.

(2) Let M and N be R – modules and let f: M⟶ N be an R-homomorphism, if B $\leq_{ess}$ N, then F^{-1}(B) $\leq_{ess}$ M.

(3) Let M= ⨁ i ∈ J, M_i be an R-module, Where M_i is a submodule of M, ∀ $i \in I$ if Ai $\leq_{ess}$ Mi , for each i ∈ I, then ⨁ $i \in I$ Ai $\leq_{ess}$ M_i, For each i ∈ I , then $\bigoplus_{i \in I}$ Ai $\leq_{ess}$M.

Definition [1.5] [3]**:** Let A be a submodule of an R - module M. Recall that a relative complement of A in M is any submodule B of M Which is maximal with to the property A ∩ B = 0.

Easy application of Zama's lemma gives for every submodule A of an R - module M, there exists a relative complement for A in M.

Proposition (1.6) [3]: Let M be an R - module and A be a submodule of M. If B is any relative complement for A in M, then A ⊕ B $\leq_{ess}$ M.

Proof: Let D be a submodule of M such that D ∩ (A ⊕ B) =0, we want to show that D = 0. Assume D ≠ 0. Now A ∩ (D ⊕ B) = 0. But B is a relative complement for A in M, therefore D+B = B and hence D ⊆ B. Then D = D ∩ B = 0. This is a contradiction. Thus $A \oplus B \leq_{ess} M$.

Let M be an R - module. Recall that a submodule A of M is a closed submodule if A has no proper essential extension in M, [3].

Proposition (1.7) [3]: Let M be an R - module If A and B are submodules of M such that M = A ⊕ B, then A is closed in M.

Proof: Let A $\leq_{ess}$ D, where D is subniodule of M. $since\ A \cap B = 0$, then D ∩ B = 0.

Let $d \in D$, then d = $a + b, a \in A, b \in B$. Implies that d – a = b $\in D \cap B$ = 0, we get $d - a = 0$ and d = a. thus $D = A$, [3].

Proposition (1.8) [3]: Let B be a submodule of an R - module M. Then the following statements are equivalent: -

1- B is a closed sub module of M.

2 – If B ⊆ K $\leq_{ess}$ M. $then\ \frac{K}{B}\ \leq_{ess}\ \frac{M}{B}$.

3- B is a relative complement for some submodule A of M.

Theorem (1.9) [3], [2]: Let A, B and C be submodules of an R-module M with A ⊆ B, then:

1-There exists a closed submodule D of M such that $C \leq_{ess} D$.

2-If A closed in B and B closed in M, then A is closed in M:

3-If Closed in M, then $\frac{B}{A}$ closed in$\frac{M}{A}$.

Definition (1.10) [3]: Let M be an R-module and let x ∈ M Recall that the annihilator of x (denoted by ann (x)) is defined as follows an (x) = { r ∈ R : rx = 0 } Clearly ann (x) is an ideal of R.

Definition (1.11) [3]: Let M be an R-module. Recall that $Z(M) = \{x \in M: ann\ (x) \leq_{ess}$ R} is called singular submodule of M. If $Z(M) = M$, then M is called the singular module .If $Z(M) = 0$ then M is called a nonsingular module.

The following lemma gives some properties of singular submodules which are needed later and can be found in [3].

Lemma (1.12) [3]: Let M and N be an R – modules, then:

1 If $f: M \longrightarrow N$. N is an R – homomorphism, then $f(Z(M)) \subseteq Z(N)$.

2-Epimorphic image of a singular module is, singular.

Proposition (1. 13) [3]: A module C is singular if and only if there exists a shorter exact sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{B} C \longrightarrow 0 \; such\ that\ f(A) \leq_{ess} B.$$

Coro1lar (1.14) [3]: If $A \leq_{ess} B$, then $\frac{B}{A}$ is singular.

Proposition (1.15) [3], [2]: Let B be a nonsingular R - module, and $A \subseteq_e B$. Then $\frac{B}{A}$ is singular if and only if $A \leq_{ess} B$.

Let M be an R - module. Recall that the second singular submodule Z_2 (M) of M is the submodule of M containing $Z(M)$ such that $\frac{Z_2(M)}{Z(M)}$ is the singular submodule of $\frac{M}{Z(M)}$.

Proposition (1.16) [6]: Any direct summand of an R – Module M is closed.

Proof: Let N $\subseteq^{\oplus}$ M, such that M = N ⊕ K for some $K \subseteq K$.

To prove:

N is closed in M

Suppose $\exists\, W \subseteq$ M such that $N \leq_{ess} W$

We must prove $N = W$

Suppose N ≠ W ⟹ ∃ $x \in W$ and x ∉ N $then$ x ∈ $N = N \oplus k$

$then\ x = n + h\ ,\qquad n \in N\,, R \in K$

Then $0 \neq x - n \in w$

$(for\ if\ x - n = 0 \Longrightarrow R = 0 \Longrightarrow x = n + 0 = n \in N)$

(By the N $\leq_{ess}$ W ⟺ ∀ $x \in w\,, x \neq 0\, \exists\, r \neq 0 \Longrightarrow Ci \in R$

$\exists\, r \neq 0\, x \in N)$

We have: $\exists\, r \in R\,, r \neq 0\, \exists\, 0 \neq r\,(x - n) \in N$

Since x = n + k

$rx = rn + rk$

$$\frac{rx - rn}{\in N} = \frac{rk}{\in k} \quad \in N \cap K = (0)$$

∴ $rx - rn$ = 0 Which is a c :

Thus $w = n$

Corollary (1.17) [6]: Every Submodule of semi simple R – module is closed:

Remark (1.18): Closed Sub M. $Then$ need not be direct summand for example

Let M = $Z_8 \oplus Z_2$ as a Z – module

Let N = < $\{(\bar{2}, \bar{1}) > = \{(\bar{0}, \bar{0}), (\bar{2}, \bar{1}), (\bar{4}, \bar{0}), (\bar{6}, \bar{1})\}\}$

$N_0 = (\bar{0}) \oplus (\bar{0}) = (\bar{0}, \bar{0}),$

$N_1 = <(\bar{1}, \bar{0})> = Z_8 \oplus (\bar{0}) = [(a, 0), a \in Z_8]$

$N_2 = <(\bar{2}, \bar{0})> = (\bar{2}) \oplus (\bar{0}) = [(a, 0), a \in (\bar{2}) \leq Z_8]$

Proof: $N_3 = (\bar{4}) \oplus (\bar{0}) = [(a, 0), a \in (\bar{4}) \leq Z_8]$

$< a > = < -a > N_4 = (\bar{4}) \oplus Z_2 = [(a, b), a \in (\bar{4}), b \in Z_2]$

N is closed in M

N is not direct summand of M.

Definition (1.19)[6]: Let $B \leq M$, $A \leq M$, A is called a relative complement of B if A is the largest submodule of M With property $A \cap B = (0)$

Such that if $\exists\, A \supseteq A, A' \cap B = 0$ $Then\ A = A'$

A relative complement A of B exists by Zero's Lemma.

Example (1.20) [6]: F is any field, M = f $\oplus$ F

Let A = F $\oplus$ (0)

$\forall\, x \in f, let\ B = <(x, 1)> is\ a\ relative\ complement\ .for\ A$

Special case:

M = $Z_3 \oplus Z_3$, A = $Z_3 \oplus (\bar{0}) = <((\bar{1}, 0))>$

$= \{(\bar{1}, \bar{0}), (\bar{2}, \bar{0}), (\bar{0}, \bar{0})\}$

Let $x = \bar{0}$, $B_1 = <(\bar{1}, \bar{1})> = \{(\bar{1}, \bar{1}), (\bar{2}, \bar{2}), (\bar{0}, \bar{0})\}$

$B_1 \cap A = \{(\bar{0}, \bar{0})\}$

$X = \bar{2}$, $B_2 = <(\bar{2}, \bar{1})> = \{(\bar{2}, \bar{1}), (\bar{1}, \bar{1}), (\bar{0}, \bar{0})\}$

$B_2 \cap A = \{(\bar{0}, \bar{0})\}$

$X = \bar{0}$, $B_3 = <(\bar{0}, \bar{1})> = \{(\bar{0}, \bar{1}), (\bar{0}, \bar{2}), (\bar{0}, \bar{0})\}$

$B_3 \cap A = \{(\bar{0}, \bar{0})\}$

B_1, B_2, B_3 Are relative complement of A in case F is an in finite to field, A has an in finite relative complement.

Proposition (1.21) [6]: Let $A \leq M$ if B is any relative complement of A, then $A \oplus \leq_{ess} M$.

Proof: Let N ≤ M suppose N ∩ (A ⊕ B) = 0

To prove $N = (0)$

Then $N \oplus (A \oplus B) = (A \oplus B) \oplus N$

$= A \oplus (B \oplus N)$

Notice that $A \cap (B \oplus N) = 0$

To prove that:

$q = b + n$ For some $b \in B, n \in N$

$Then$ (a – b) = n ∈ N ∩ (A⨁) = (0)

∴ n = 0 & a – b = 0

$Hence\ a = b \in A \cap B = (0)$ (Since B is a relative complement of A)

$a = 0$, So $A \cap (B \oplus N) = 0$

But B is relative complement of A

And $B \oplus N \supseteq B$

$then$ B ⨁ N = B ⟹ N = (0) [Since N ∩ B = (0) and A ≤ M].

Theorem (1.22) [6]: Let $B \leq M$ and $A \leq M$, the following statement, are equivalent:

(1) B Is a closed sub M of M.

(2) If B ≤ K $\leq_{ess}$ M, then $\frac{K}{B} \leq_{ess} \frac{\mathrm{M}}{\mathrm{B}}$.

(3) If A is a relative complement of B, then B a relative complement of A.

(4) B is relative complement of $A \leq M$.

Proof: (1) ⟶ (2)

Let B ≤ K $\leq_{ess} M$ to prove $\frac{K}{B} \leq_{ess} \frac{M}{B}$

Let $\frac{N}{B} \leq \frac{M}{B}$ with $\frac{K}{B} \cap \frac{M}{B}$ with $\frac{K}{B} \cap \frac{N}{B} = 0\ \frac{M}{B}$ (to prove $\frac{N}{B} = 0\ \frac{M}{B}$?)

Then $\frac{K \cap N}{B} = O_{\frac{M}{B}}$

Hence $K \cap N = B$

But K $\leq_{\text{ess}}$ M & N $\leq_{\text{ess}}$ N

N ∩ K $\leq_{\text{ess}}$ M ∩ N = N

= N ∩ K $\leq_{\text{ess}}$ N

$B \leq_{\text{ess}} N$, but B is closed in M (B)

$then$ B = N $\Longrightarrow \frac{N}{B} = O_{\frac{M}{B}}$

Then (3)

If A is a relative complement of B, then A ∩ B = (0)

Then B ∩ A = (0)

To prove B is the largest.

Let B' ≥ B such that B' ∩ A = (0)

But (A ⊕ B) ∩ B' = B ⊕ (A ∩ B') = B ⊕ (0) = B

$$\frac{(A \oplus B) \cap B'}{B} = \frac{B}{B} = O_{\frac{M}{B}}$$

$$\frac{A \oplus B}{B} \bigcap \frac{B'}{B} = O_{\frac{M}{B}}$$

B ≤ A ⊕ B $\leq_{\text{ess}} O_{\frac{M}{B}}$

$B \leq A \oplus B \leq_{\text{ess}} M$ [Since A relative complement of B]

By (2) $\frac{A \oplus B}{B} \leq_{\text{ess}} \frac{M}{B}$

$\frac{B\prime}{B} = 0_{\frac{M}{B}} \Longrightarrow$ B = B′

B is a relative complement of A

(3) ⟹ (4) it is clear

(4) ⟹ (1) if B is a relative complement of A

To Prove B is closed.

Assume B $\leq_{\text{ess}}$ B′ (T prove B = B′).

(B′ ∩ A) ∩ B = B′ ∩ (A ∩ B) = (0)

But B $\leq_{\text{ess}}$ B′ and B′ ∩ A ≤ B′

$Then$ (B′ ∩ A) ∩ B = (0) implies B′ ∩ A ≤ B′

$Then$ (B′ ∩ A) ∩ B = (0) implies B′ ∩ A ≤ 0

But B is a relative complement of A and B′ ⊇ B

$Hence$ B = B $then$ B is closed.

Proposition (1.23) [6]: If $A \leq$ B $\leq$ M, if A is closed in B and B is closed in M then A $\leq_{ess}$ M. ($A \leq_{ess} B$ *and* $B \leq_{ess} M \implies A \leq_{ess} M$)**.**

Poof: A $\leq_{ess}$ B $\Longrightarrow \exists\, \bar{X} \leq$ B $\ni$ A is a relative complement of $\bar{X}$

Then ($B \cap C$ = (0))

Note that $\bar{X} \cap$ C = (0) (Since $\bar{X} \cap$ C $\subseteq$ B $\cap$ C = (0))

We claim that A is a relative complement of $\bar{X} \oplus$ C

To prove A $\cap$ ($\bar{X} \oplus$ C) = (0).

Let a $\in$ A & a = X + C, X $\in$ $\bar{X}$, C $\in$ C

$Then$ a – x = c $\in$ B $\cap$ C = (0)

$then$ C = 0, a = X $\in$ A $\cap$ $\bar{X}$ = (0)

$then$ a = 0

$then$ A $\cap$ ($\bar{X} \oplus$ C) = (0)

Let $A' \supseteq A$ and $A' \cap (\bar{X} \oplus C)$ = (0)

$(A' \cap \bar{X}) \oplus (A' \cap C)$ = (0)

$Then$ A$'$ $\cap$ $\bar{X}$ = (0)

But A is a relative complement of $\bar{X}$ and $A' \supseteq A$

$Hence$ A = A$'$

$then$ A is a relative complement of $\bar{X} \oplus$ C *then* A is closed in M.

Proposition (1.24) [6]: If $A \leq B \leq M$, and $A \leq_{ess} M$ then $A \leq_{ess} B$.

Proof: A is closed in M $\Longrightarrow \exists\ \bar{X} \leq M \ni$ A is relative complement of $\bar{X}$.

Then $A \cap \bar{X} = 0$

Let $B \cap \bar{X} \leq B$ We claim that A is a relative complement of $B \cap \bar{X}$

$A \cap (B \cap \bar{X}) = B \cap (A \cap \bar{X}) = B \cap (0) = (0)$

Suppose $(\exists A' \geq A)$; $A' \cap (B \cap \bar{X}) = (0)$

$(\exists A' \subseteq B) \Longrightarrow (A' \cap B) \cap \bar{X} = (0)$

Then $A' \cap \bar{X} = (0)$

But A is a relative complement of $\bar{X} \longrightarrow A = A'$

*Then*A is a relative complement of $B \cap \bar{X} \subseteq B$

Hence A is closed in B

Proposition (1.25) [6]: Let C be a closed in M and let $T \leq M$ such that $C \cap T = (0)$

Then C is a relative complement of T

then $C \oplus T \leq_{ess} M$

If $C \oplus T \leq_{\text{ess}} M$, to prove C is relative complement of T.

Since C is closed in M, So C is relative complement of $S \leq M$ (then $C \cap S = (0)$)

To prove C is a relative complement of T

$C \cap T = (0)$

Suppose $\exists$ $D \supseteq C$ such that $D \cap T = (0)$

$(C \oplus T) \cap (D \cap S) = [(C \oplus T) \cap D] \cap S$

But $C \oplus T \leq_{\text{ess}} M$, hence

$D \cap S = (0)$ and $D \supseteq C$, C is a relative complement of S. So D = C

$Then$ C is a max. Sub With property $C \cap T = (0)$

$then$ C is a relative complement of T.

Exercise (1.26) [6]:

(1) Let $A \leq B \leq M$. If $B \leq_{\text{ess}} M$. To prove that $\frac{B}{A} \leq_{\text{ess}} \frac{M}{B}$ is the converse true.

(2) If $A \leq_{\text{ess}} M$, $A_2 \leq_{\text{ess}} M_2$. Prove that $A_1 \oplus A_2 \leq_{\text{ess}} M_1 \oplus M_2$.

(3) $A_1 \leq_{\text{ess}} M_1$, $A_2 \leq_{\text{ess}} M_2$.To prove that $A_1 \oplus A_2 \leq_{\text{ess}} M_1 \oplus M_2$.

(4) Let M be a finitely generated Faith. Multiplication. R–module. Let $N \leq M$ prove that.

$N \leq_{ess} M \Leftrightarrow (N\ R\ I\ M) \leq_{ess} R \Leftrightarrow N = Im$ for Some closed ideal I in R.

Chapter TWO

Extending Modules

Definition (2.1) [2], [6]: Let M be an R – Module recall that M is called an extending module (CS – module) if every sub module of M is essential in a direct summand of M.

The following theorem gives a characterization of extending modules.

Theorem (2.2) [6]: An R – Modules M is extending if and only if every closed sub modules of M is direct summand.

Proof: Suppose m is a CS – modules and let A be closed Sub Modules in M then there exists a direct summand K of M such that $A \leq_{ess} K$ But A is closed in m, therefore A = K .

Conversely, let B be any Submodule of m. So by there exists a closed sub. Module H in M such that $B \leq_{ess} H$ since H is closed in M, and then by our assumption H is a direct Summand.

Definition (2.3) [7]: Recall that an R – Module M is called semi simple, if every sub module of M is a direct summand of M.

Definition (2.4) [2]: Recall that a nonzero R – Module. M is called uniform if every nonzero sub module of m is essential in M.

Remarks and examples (2.5) [9]:

1- Every direct summand of CS - module is CS.

2- The module Q as Z - module is CS - module

3- It is easy to see that every semi simple R - module is CS - module, for example Z_n as Z - module, Where n is square free.

Proposition (2.6) [6]: Let M be an R – module. Then M is uniform if and only if M is an indecomposable and CS - module

Definition (2.7) [6]: Let R is an integral domain. Recall that an R - module M is called a torsion free R - module if $ann\ (m) = 0$, for every nonzero element m in M.

Proposition (2.8) [6]: Let R be a principle ideal domain and M be a finitely generated and torsion free R - module, then M is a CS - module.

Definition (2.9) [2]: Let N is an R-module. Recall that an R - module M is called N - injective if for each monomorphic f: A ⟶ N, Where A is any R - module of N, and any homomorphism g: A ⟶ M, there is a homomorphism h: N ⟶ M such that g = h_0f.

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "g"'] & N \arrow[dl, "h"] \\
 & M &
\end{tikzcd}$$

Definition (2.10) [2]: Recall that an R – module is called-self - injective if M is M -injective. Any family of R - modules $\{M_i$, i ∈ I$\}$ are called relatively injective if $M_i$is M_i – injective, for all distinct i, j ∈ I.

Definition (2.11) [2]: Let M is an R - module and let $E(M)$ the injective hull of M. Then M is self - injective if and only if $f(M) \subseteq$ M, for every endomorphism f of $E(M)$.

Definition (2.12) [2]: Recall that an R - module M is called π - injective if $f(M) \subseteq$ M, for every idempotent f of $E(M)$.

The following proposition appeared in [6].

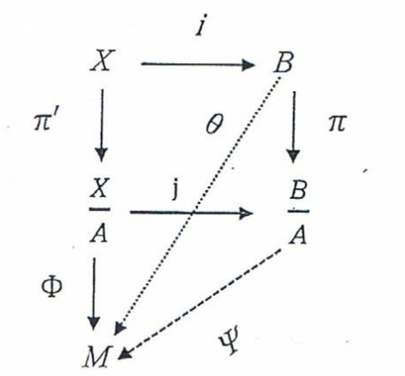
Proposition (2.13) [6]: Let M is a B - injective module. If A ⊆ B, then M is A - injective and M is $\frac{A}{B}$ - injective.

Proof: One can show that M is A - injective to show that M is $\frac{B}{A}$ injective.

Let $\frac{X}{A}$ be a submodule of $\frac{B}{A}$, and $\Phi : \frac{X}{A} \longrightarrow M$ be a homomorphism. Let π : B → $\frac{B}{A}$ be the natural epimorphism and π' = π 1_x . Since M is B - injective, then there exists a homomorphism 0: B → M such that $\theta_0 i = \Phi_0 \pi'$. Now θ (A) = $(\Phi_0 \pi')$ (A) = Φ (0) = 0. Hence $ker\ \pi \subseteq ker\ \theta$, let Ψ : $\frac{B}{A}$ → M be a map defined by Ψ (b +

A) = θ (b) $\forall$ b $\in$ B. One can easily show that Ψ is a homomorphism and $\Psi_0\pi = \theta$. For every x $\in$ X

Ψ (x + A) = $\Psi_0\pi$ (x) = θ (x) = Φ (x + A). Thus $\Psi_0 j = \Phi$ and therefore M is $\frac{B}{A}$ - injective.



Proposition (2.14) [6] : Let M be an R – module , and let {Ai : i $\in$ I } be a family of R - module , then M is $\bigoplus_{i\in I}$ Ai - injective if and only if M is Ai - injective, for every i $\in$ I.

Proposition (2.15) [2]: Let M is a π - injective R -module, then Mis a CS - module.

Lemma (2.16) [2]: let M= A $\oplus$ B. Then A is B - injective if and only if for every submodule N of M such that N $\cap$ A = 0, there exists a submodule M' of M such that M = A $\oplus$ M' and N $\subset$ M'.

It is known that a direct sum of CS - module need not to be CS - module, for example:

Consider the Z - module M = $Z_8 \oplus Z_2$ clearly each of Z_8 and Z_2 is a CS - module. One can show that the submodule A = $((\bar{2},\bar{1}))$ is a closed submodule of M but it is not a direct summand. Thus M is not a CS – module.

The following proposition gives a characterization for CS - module.

Proposition (2. 17) [2]: Let M = A ⊕ B, Where A and B are both CS modules. Then M is a CS module if and only if every closed submodule K of M with K ∩ A = 0 or K ∩ B = 0 is a direct suinmand.

Proposition (2.18) [2]: Let $M = M_1 \oplus M_2 \oplus \ldots M_n$ be a finite direct sum of relatively injective modules M_i, Where i = 1, 2, …, n. Then M is a CS - module if and only if M_i is a CS - module, for every $i \in I$.

Theorem (2.19) [3]: Let M be an R - module. Then the following statements are equivalent:

(1) M is a CS-module.

(2) Every closed submodule of M is a direct summand.

(3) If A is a direct summand of the injective hull $E(M)$ of M, then A ∩ M is a direct summand of M.

Proposition (2. 20) [3]: Let R is a nonsingular ring. Then R is a CS - ring if and only if every cyclic nonsingular R - module is projective. In particular, if R is a nonsingular and CS ring, then every principle ideal in R is projective.

The following proposition can. We give the details of the proof for completeness.

Proposition (2.21) [6]: An R - module M is CS if and only if M = Z_2 (M) $\oplus$ M', for some submodule M' of M, such that M' and Z_2(M) are both CS and Z_2(M) is M' - injective.

Proof: $\Longrightarrow$ suppose that M is CS-module. Because $\frac{M}{Z_2(M)}$ is nonsingular. It easily to show that Z_2(M) is closed in M and hence M = $Z_2(M) \oplus M'$, for some submodule M' of M. By [2], $Z_2(M)$ and M' are both CS. Now let N be a submodule of M such that $N \cap Z_2(M)$ = 0. Then there exists submodules L_1, L_2 of M such that M = $L_1 \oplus L_2$ and $N \leq_{ess} L_1$ clearly $L_2 \cap Z_2(M) = 0$, and hence $Z_2(M) \subset L_2$. It follows that M = $L_2 \oplus L_1$ = $Z_2(M) \oplus (L_2 \cap M') \oplus L_1$ and $N \subset (L_2 \cap M') \oplus L_1$. By (2.16) $Z_2(M)$is M' - injective.

$\Longleftarrow$ Suppose that $M = Z_2(M) \oplus M'$, Where $Z_2(M)$ and M' are both CS and $Z_2(M)$ is M' - injective. Clearly M' is nonsingular, and hence Homomorphism ($Z_2(M)$, M') = 0. Thus M' is $Z_2(M)$- injective. By (2.18), M is CS –module.

Definition (2.22) [5]: Let M be an R - module. Recall that M is called a multiplication R - module if for each submodule N of M, there exists an ideal I of R such that N = IM Equivalently, M is multiplication if for each submodule N of $M, N = [N : M]\, M$, Where $[N : M] = \{r \in R \textit{ such that } rM \subseteq N\}$.

Proposition (2.23) [3]: Let M be a faithful multiplication R - module If R is CS - ring then M is CS - module

Proposition (2.24) [3]: Let M is a finitely generated, faithful and multiplication R - module If M is CS - module, then R is CS – ring.

Definition (2.25) [8]: Recall that a submodule N of an R - module M is called a fully invariant submodule if for every endomorphism

f: M→ M, f(N) ⊆ N.

Proposition (2.26) [3]: Let M = $\underset{i \in I}{\oplus}$ M_i be an R - module, Where each my is a submodule of M. If M is CS - module, then each M is CS - module. The converse is true if each closed submodule of M is fully invariant.

Proposition (2.27) [3]: Let M_1 and M_2 be CS modules such that annM_1 + annM_2 = R, then $M_1 \oplus M_2$ is CS – module.

We end this section by the following two propositions which are appearing in.

Proposition (2.28) [6]: Let R is a ring. The following statements are equivalent:

(1) $\underset{I}{\oplus}$ R is CS - module, for every index set I.

(2) Every projective R - module is CS – module.

Proposition (2.29) [6]: Let R be a ring. The following statements are equivalent:

(1) $\underset{I}{\oplus}$ R is CS - module, for every finite index set I.

(2) Every finitely generated projective R - module is CS – module.

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